Appendix 3 Volume and the Dehn invariant in hyperbolic 3-space

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for $n \geqslant 2$, it follows that the resulting series has the form

$$s_N(x) = \gamma_1(x) + \sum_{N+1}^{\infty} a_n \zeta_n(x)$$

for suitable constants a_n. Setting

$$E(x) = \sum_{N+1}^{\infty} a_n x^{-n},$$

we can write the error term as

$$s_N(x) - \gamma_1(x) = E(x) + E(x+1) + \dots .$$

Note that all of these series converge absolutely for $x > 1$. Evidently

$$E(x) = O(x^{-N-1})$$

as $x \to \infty$, for any fixed N, so

$$s_N(x) - \gamma_1(x) = O(x^{-N})$$

as required. □

This argument yields similar asymptotic series for related functions such as $\zeta_s(x)$, $\gamma_s(x)$, and $\gamma_s'(x)$. Such estimates work also for complex values of x, as long as x stays well away from the negative real axis.

Appendix 3

Volume and the Dehn invariant in hyperbolic 3-space

We will describe some constructions in hyperbolic space involving the dilogarithm function $\mathscr{L}_2(z)$ and its Kubert identity (7). Further details may be found in the paper "Scissors Congruences, II" by J. L. Dupont and C.-H. Sah (*J. Pure Appl. Algebra* 25 (1982), 159-195).

Using the upper half-space model for hyperbolic 3-space, consider a totally asymptotic 3-simplex Δ. In other words, we assume that the vertices a, b, c, d of Δ all lie on the 2-sphere of points at infinity, which we identify with the extended complex plane $\mathbf{C} \cup \infty$. Then Δ is determined up to orientation preserving isometry by the cross ratio

$$z = (a, b; c, d) = (c-a)(d-b)/(c-b)(d-a) .$$

[The semicolon is inserted in our cross ratio symbol as a remainder of its symmetry properties, which are similar to those of the four index symbol R_{hijk} in Riemannian geometry.] In particular, the volume of Δ can be expressed as a function of the cross ratio z.

THEOREM (S. Bloch and D. Wigner). *If* z *belongs to the upper half-plane,* $\mathrm{Im}(z) > 0$, *then this volume* $V = V(z)$ *is equal to the imaginary part of the dilogarithm* $\mathscr{L}_2(z)$ *plus a correction term of* $\log|z|\arg(1-z)$. *The correspondence* $z \mapsto V(z)$ *for* $\mathrm{Im}(z) > 0$ *extends to a function which is single valued and real analytic throughout* $\mathbf{C} - \{0, 1\}$, *and continuous throughout* $\mathbf{C} \cup \infty$.

Here we use the branch $-\pi < \arg(1-z) < \pi$ of the many valued function $\arg(1-z)$ in the region $\mathrm{Im}(z) > 0$.

Proof. For the first assertion, it suffices to consider the simplex Δ with vertices ∞, 0, 1, z; where we assume that $\mathrm{Im}(z) > 0$. The image of Δ under vertical projection from the point ∞ to the boundary plane $\mathbf{C}$ is just the Euclidean triangle with vertices 0, 1, z. Let

$$\theta_1 = \arg(z),\ \theta_2 = \arg(1/(1-z)),\ \theta_3 = \arg((z-1)/z)$$

be the angles at these three vertices, equal to corresponding dihedral angles of the hyperbolic simplex Δ. Note that $\Sigma\theta_k = \pi$. We will assume the volume formula

$$V(z) = \Sigma\Lambda(\theta_k)\,, \tag{28}$$

to be summed from 1 to 3, where

$$\Lambda(\theta) = -\int_0 \log(2\sin\theta)d\theta\,.$$

This is proved for example in [21]. Using the law of sines

$$\sin\theta_1 : \sin\theta_2 : \sin\theta_3 = |1-z| : |z| : 1$$

and the equation $\Sigma\, d\theta_k = 0$, we see that

$$dV(z) = -\Sigma\log(2\sin\theta_k)d\theta_k$$

is equal to $-\log|1-z|\,d\theta_1 - \log|z|\,d\theta_2$; or in other words

$$dV(z) = \log|z|\,d\arg(1-z) - \log|1-z|\,d\arg(z)\,. \tag{29}$$

On the other hand $d\mathscr{L}_2(z) = -\log(1-z)d\log(z)$, hence,

$$d\,\mathrm{Im}\,\mathscr{L}_2(z) = -\log|1-z|\,d\arg(z) - \arg(1-z)d\log|z|\,.$$

The required formula

$$V(z) = \mathrm{Im}\,\mathscr{L}_2(z) + \log|z|\arg(1-z) \tag{30}$$

then follows since both sides of this equation have the same total differential, and since both sides tend to the limit zero as z tends to any point of the real interval (0, 1).

As an example of this formula, note the identity

$$V(e^{2i\theta}) = \operatorname{Im} \mathcal{L}_2(e^{2i\theta}) = 2\Lambda(\theta).$$

Since the right side of (29) is a well defined smooth closed 1-form, everywhere on $\mathbf{C} - \{0, 1\}$, we need only check that its integral in a loop around zero or one vanishes, in order to check that $V(z)$ extends as a single valued function. But the expression (30) shows that $V(z)$ extends to a single valued function near zero, and also that $V(z)$ tends to zero as $z \to 0$. Using the identity

$$V(z) + V(1-z) = 0$$

which follows from (29), we see that the same is true for z near 1.

Now consider the fractional linear automorphism of period three

$$z \mapsto 1/(1-z) \mapsto (z-1)/z \mapsto z$$

which carries the upper half-plane to itself. The expression (28) shows that

$$V(z) = V\big(1/(1-z)\big) = V\big((z-1)/z\big).$$

Since $0 \mapsto 1 \mapsto \infty \mapsto 0$, it follows that $V(z)$ also tends to zero as $z \to \infty$. $\square$

Note that $V(z)$ is strictly positive in the upper half-plane for geometrical reasons. The identity

$$V(\bar{z}) = -V(z)$$

shows that $V(z)$ is negative on the lower half-plane and zero on $\mathbf{R} \cup \infty$. Note also the identities

(31) $$V(1-z) = V(1/z) = -V(z),$$

which are equivalent to the statement that the expression $V(a, b; c, d)$ is skew symmetric as a function of four variables.

This function $V(z)$ satisfies the *multiplicative Kubert identity*

(32) $$V(z^n) = n\Sigma\, V(wz),$$

to be summed over all n-th roots of unity, $w^n = 1$. This follows easily since both $\mathcal{L}_2(z)$ and $\log|z|\arg(1-z)$ satisfy this same identity for z near zero.

Another important property is the *cocycle equation*

(33) $$\Sigma_0^4(-1)^i\, V(a_0, \ldots, \hat{a}_i, \ldots, a_4) = 0,$$

for any five distinct points $a_0, \ldots, a_4$ in $\mathbf{C} \cup \infty$. Geometrically, this is true since the convex body in hyperbolic space spanned by five vertices can be expressed as a union of simplices with disjoint interiors in two different ways. Analytically, it can be proved using the *Abel functional equation*

$$\mathscr{L}_2(xx'yy') = \mathscr{L}_2(xy') + \mathscr{L}_2(yx') + \mathscr{L}_2(-xx') + \mathscr{L}_2(-yy') + \log^2(x'/y')/2\,,$$

where x' stands for $1/(1-x)$. Still another proof will be sketched later.

Dupont and Sah show that the Kubert identity can be proved as a formal consequence of this cocycle equation. Hence it has a geometric interpretation in terms of cutting and pasting of simplices. As a geometric corollary, they prove that the "scissors congruence group" for hyperbolic 3-space is divisible. That is any hyperbolic polyhedron can be cut up and reassembled into n pieces which are isometric to each other, for any n.

Another geometric invariant associated with a hyperbolic simplex is the *Dehn invariant.* For a finite 3-simplex, this is defined to be the six fold sum

$$\Sigma_{\text{edges}} \text{ length} \otimes \text{(dihedral angle)}$$

in the additive group $\mathbf{R} \otimes (\mathbf{R}/2\pi\mathbf{Z})$, taking the tensor product over $\mathbf{Z}$. For a simplex with one or more vertices in $\mathbf{C} \cup \infty$, the definition is the same except that we must first chop off a horospherical neighborhood of each infinite vertex before measuring edge lengths. The result does not depend on the particular choice of horospheres.

LEMMA (Dupont and Sah). *For a totally asymptotic simplex, with dihedral angles* $\theta_1, \theta_2, \theta_3$ *along the edges meeting at a vertex, this Dehn invariant is equal to* $2\Sigma \log(2 \sin \theta_i) \otimes \theta_i$.

If we express this as a function of the associated cross ratio z, using the law of sines as above, the formula becomes

$$\frac{1}{2}\,\text{Dehn}(z) = \log|1 - z| \otimes \arg(z) - \log|z| \otimes \arg(1-z)\,.$$

This function also satisfies the Kubert identity (32), and it is clear from its geometric definition that it satisfies the symmetry condition (31) and the cocycle equation (33).

To prove this lemma, we first choose one particular horospherical neighborhood of each vertex. It is convenient to choose that horosphere which is tangent to the opposite face. Consider, for example, a simplex with vertices ∞, v_1, v_2, v_3. The preferred horosphere through v_i can be described as a Euclidean sphere, tangent to the boundary plane $\mathbf{C}$ at v_i, and tangent to the orthogonal plane which passes through the other two vertices v_j, v_k. The Euclidean radius r_i of this sphere is equal to the distance of v_i from the line through v_j, v_k. In other words r_i is equal to an altitude of the triangle v_1, v_2, v_3. Hence r_i is inversely proportional to the edge length $|v_j - v_k|$, and inversely proportional to $\sin \theta_i$; say $r_i = c/\sin \theta_i$.

This horosphere intersects the line from v_i to ∞ at Euclidean height $h = 2r_i$. On the other hand, the preferred horosphere through the point ∞ intersects each vertical line at some constant height $h = c'$. If we integrate the hyperbolic length element dh/h along the line from v_i to ∞ between these two intersection points, we obtain

$$\text{(34)} \qquad \text{truncated edge length} = \int_{2r_i}^{c'} dh/h = \log(2 \sin \theta_i) + c''$$

where $c'' = \log(c'/4c)$ is constant. (Here negative lengths must be allowed.) The corresponding contribution to the Dehn invariant is

$$\log(2 \sin \theta_i) \otimes \theta_i + c'' \otimes \theta_i .$$

There is an identical contribution from the opposite edge v_j, v_k. *In fact the symmetry property*

$$(a, b; c, d) = (c, d; a, b)$$

of the cross ratio implies that there is an isometry of Δ *carrying any given edge to the opposite edge.* Now, summing over all six edges, since the $c'' \otimes \theta_i$ terms cancel, we obtain the required formula

$$\text{(35)} \qquad \mathrm{Dehn}(\Delta) = 2\Sigma_1^3 \log(2 \sin \theta_i) \otimes \theta_i . \qquad \square$$

Remark. The curious similarity between the two equations (28) and (35) can be explained by a theorem of Schläfli. For a family of simplices Δ in the n-dimensional spherical space of constant curvature $K > 0$, Schläfli's equation can be written as

$$(n-1)K\, dV_n(\Delta) = \Sigma\, V_{n-2}(F) d\theta_F ,$$

to be summed over all $(n-2)$-dimensional faces F, where $V_{n-2}(F)$ is the $(n-2)$-dimensional volume and θ_F is the dihedral angle along F. In other words, we have

$$(n-1)K\, \partial V_n / \partial \theta_F = V_{n-2}(F) .$$

For a proof, also in the case $K < 0$, see Kneser, "Der Simplexinhalt in der nichteuklidischen Geometrie", *Deutsche Math. 1* (1936), 337-340. In the case $n = 3$, $K = -1$, the Schläfli equation takes the form

$$-2\, dV_3(\Delta) = \Sigma_{\text{edges}}\, V_1(E) d\theta_E .$$

For a family of 3-simplices with one or more vertices at infinity, this equation remains valid providing that we cut off a horospherical neighborhood of each infinite vertex before measuring edge lengths. It follows that we can prove equation (34) simply by differentiating (28), or conversely that we can prove (28) by integrating (34), using the identity $\Lambda(0) = \Lambda(\pi) = 0$ to fix the constant of integration.

Although the cocycle equation for the Dehn invariant is an immediate consequence of its geometric definition, it may be of interest to give an analytic proof. Let us introduce the skew-symmetric bimultiplicative symbol

$$(x|y) = \log|x| \otimes \arg(y) - \log|y| \otimes \arg(x),$$

for x and y in the multiplicative group $\mathbf{C}^{\bullet}$, with values in the additive group $\mathbf{R} \otimes (\mathbf{R}/2\pi\mathbf{Z})$. Then

$$\frac{1}{2}\,\mathrm{Dehn}(z) = \frac{1}{2}\,\mathrm{Dehn}(a, b; c, d)$$

is equal to $(1-z|z)$. Expressing z and $1 - z = (a, c; b, d)$ as 4-fold products and using the bimultiplicative property, we can expand $(1-z|z)$ as a sum of sixteen terms, of which four cancel. The remaining twelve can be grouped as

$$(1-z|z) = f(b, c, d) - f(a, c, d) + f(a, b, d) - f(a, b, c),$$

where f stands for the skew function

$$f(a, b, c) = (a-b|b-c) + (b-c|c-a) + (c-a|a-b).$$

This proves that the function $\mathrm{Dehn}(a, b; c, d)$ is a coboundary, and hence a cocycle.

We can define a sharpened Dehn invariant by this same formalism, using the expression

$$\log(x) \wedge \log(y),$$

with values in $\wedge^2_{\mathbf{Z}}(\mathbf{C}/2\pi i\mathbf{Z})$ in place of our symbol $(x|y)$. If we split this exterior power into eigenspaces under the action of complex conjugation, then the component of

$$\log(x) \wedge \log(y)$$

in the -1 eigenspace can be identified with $(x|y)$.

The cocycle equation for the volume function $V(a, b; c, d)$ can also be proved by this formalism. We must simply replace $(x|y)$ by the differential form valued symbol

$$\log|x|\, d\arg(y) - \log|y|\, d\arg(x).$$

Details will be omitted.

Dupont and Sah show that the volume function and the sharpened Dehn invariant can be incorporated into a single function ρ, as follows. Let

$$\rho(z) = 1 \wedge L(z) - 1 \wedge L(1-z) + l(z) \wedge l(1-z),$$

with values in $\wedge^2\mathbf{C}$, where $l(z) = \log(z)/2\pi i$ and

$$L(z) = \mathscr{L}_2(z)/4\pi^2 = \int_0 l(1-z)dl(z).$$

This expression is certainly well defined in the strip $0 < \mathrm{Re}(z) < 1$, and satisfies $\rho(z) + \rho(1-z) = 0$. If we analytically continue each of its constituent functions in a loop around zero or one, then the expression $\rho(z)$ remains unchanged. Hence ρ is well defined as a mapping from $\mathbf{C} - \{0, 1\}$ to $\wedge^2\mathbf{C}$. They show that ρ also satisfies the symmetry condition (31), the Kubert identity (32), and the cocycle equation (33).

REFERENCES

[1] ARTIN, E. *The Gamma Function.* Holt, Rinehart and Winston 1964.

[2] BASS, H. Generators and relations for cyclotomic units. *Nagoya Math. J. 27* (1966), 401-407.

[3] BATEMAN, H., edited by Erdélyi *et al. Higher Transcendental Functions, vol. I.* McGraw-Hill, 1953. (See pp. 24-39.)

[4] BLOCH, S. Applications of the dilogarithm function in algebraic K-theory and algebraic geometry. *Proc. Conf. Algebraic Geom. Kyoto Univ. 1977.* (See also "Algebraic K-theory and zeta functions of elliptic curves", *Proc. Int. Congr. Math. Helsinki 1978*, II, 511-515; and "The dilogarithm and extensions of Lie algebras", preprint, U. Chicago 1981.)

[5] BOREVICH, Z. I. and I. R. SHAFAREVICH. *Number Theory.* Academic Press 1966.

[6] CHUDNOVSKY, G. V. Padé approximations to the generalized hypergeometric functions. I, *J. Math. pures appl. 58* (1979), 445-476.

[7] EWING, J. Spheres as fixed point sets. *Quart. J. Math., Oxford, 27* (1976), 445-455.

[8] HILBERT, D. Die Theorie der algebraischen Zahlkörper. *Jahresb. D. Math.-Ver. 4* (1897), 175-546. (See §120.)

[9] HURWITZ, A. Einige Eigenschaften der Dirichlet'schen Funktionen $F(s) = \sum\left(\frac{D}{n}\right)\frac{1}{n^s}$, die bei der Bestimmung der Klassenanzahlen binärer quadratischer Formen auftreten. *Zeits. Math. Phys. 27* (1882), 86-101.

[10] IWASAWA, K. *Lectures on p-Adic L-Functions.* Princeton University Press, Annals Studies *74* (1972).

[11] JONQUIÈRE, A. Note sur le série $\sum x^n/n^s$. *Bull. Soc. Math. France 17* (1889), 142-152.

[12] KUBERT, D. The universal ordinary distribution. *Bull. Soc. Math. France 107* (1979), 179-202.

[13] KUBERT, D. and S. LANG. Units in the modular function field III. Distribution relations. *Math. Ann. 218* (1975), 273-285.

[14] —— Distributions on toroidal groups. *Math. Zeits. 148* (1976), 33-51.

[15] —— *Modular Units.* Grundlehren math. Wiss. 244, Springer, 1981.