

Appendix 1 Relations between polylogarithm and Hurwitz function

Objektyp: **Appendix**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **29 (1983)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **28.04.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

is non-zero according to Dirichlet. Thus we obtain a contribution of $-1 + \varphi(m)/2$ to the rank coming from the non-trivial even characters.

On the other hand, for the eigenspace corresponding to the trivial character, using formula (10) of §4 we obtain a contribution equal to the number of primes dividing m . Lemmas 8 and 10 of §5 now complete the proof. $\square$

APPENDIX 1

RELATIONS BETWEEN POLYLOGARITHM AND HURWITZ FUNCTION

For every complex number s , it follows from Theorem 1 that there exists a linear relation between the even [or the odd] part of the function $l_s(x)$ and of the function $\zeta_{1-s}(x)$ or $\beta_s(x) = -s\zeta_{1-s}(x)$. This appendix will work out the precise form of these relations. Compare [3], [19], [27].

For integer values of s , the required relation can be obtained as follows. Recall from formula (9) of §2 that

$$l_0(x) = (-1 + i \cot \pi x)/2$$

hence

$$l_0(x) + l_0(1-x) + \beta_0(x) = 0.$$

Integrating, we see that

$$l_1(x) - l_1(1-x) + 2\pi i \beta_1(x)/1! = 0$$

$$l_2(x) + l_2(1-x) + (2\pi i)^2 \beta_2(x)/2! = 0$$

and so on, for $0 < x < 1$. For even values of the subscript, specializing to $x = 0$ as in §4, this yields Euler's formula

$$2\zeta(2k) + (2\pi i)^{2k} b_{2k}/(2k)! = 0.$$

In particular, it follows that $\zeta(0) = -\frac{1}{2}$, and that the numbers $b_2, -b_4, b_6, -b_8, \dots$ are strictly positive. On the other hand, differentiating the formula for $l_0(x)$, we obtain

$$l_{-1}(x) = -\csc^2(\pi x)/4.$$

This is an even function satisfying $(*_{-1})$, so it must be some multiple of $\zeta_2(x) + \zeta_2(1-x)$. Comparing asymptotic behavior as $x \rightarrow 0$, we obtain the classical formula

$$\zeta_2(x) + \zeta_2(1-x) = \pi^2/\sin^2 \pi x = (2\pi i)^2 l_{-1}(x)/1!.$$

Differentiating, we see that

$$\begin{aligned} -\zeta_3(x) + \zeta_3(1-x) &= (2\pi i)^3 l_{-2}(x)/2! \\ \zeta_4(x) + \zeta_4(1-x) &= (2\pi i)^4 l_{-3}(x)/3! \end{aligned}$$

and so on.

For $s \neq 0, 1, 2, \dots$ we know from §3 that there is some relation of the form

$$(14) \quad l_s(x) = A_s \zeta_{1-s}(x) + B_s \zeta_{1-s}(1-x)$$

for $0 < x < 1$; where A_s and B_s are certain uniquely determined constants. Expressing each of these functions of x as the sum of an even part and an odd part, we see that

$$(15) \quad \begin{cases} l_s^{\text{even}}(x) = (A_s + B_s) \zeta_{1-s}^{\text{even}}(x) \\ l_s^{\text{odd}}(x) = (A_s - B_s) \zeta_{1-s}^{\text{odd}}(x) . \end{cases}$$

Evidently the functions $s \mapsto A_s \pm B_s$ are meromorphic, taking finite non-zero values for all $s \in \mathbf{C} - \mathbf{Z}$. Differentiating with respect to x , we see that

$$(16) \quad A_s \pm B_s = s(A_{s+1} \mp B_{s+1})/(2\pi i) .$$

For integral values of s , using the discussion above, we easily obtain the following table of values, where $0! = 1$.

s	...	-2	-1	0	1	2	3	...
$1, + B_s$	...	0	$\frac{2 \cdot 1!}{(2\pi i)^2}$	0	∞	$\frac{(2\pi i)^2}{2 \cdot 1!}$	∞	...
$1, - B_s$	...	$-\frac{2 \cdot 2!}{(2\pi i)^3}$	0	$-\frac{2 \cdot 0!}{2\pi i}$	$\frac{2\pi i}{2 \cdot 0!}$	∞	$\frac{(2\pi i)^3}{2 \cdot 2!}$	...

Now suppose that we specialize to $x = 0$, by the procedure of §4. Then equation (14) reduces to a form

$$\zeta(s) = (A_s + B_s) \zeta(1-s)$$

of Riemann's functional equation. It follows that

$$(A_s + B_s)(A_{1-s} + B_{1-s}) = 1 ,$$

and hence using (16) that

$$(A_s - B_s)(A_{1-s} - B_{1-s}) = -1 .$$

This discussion gives all of the information about $A_s \pm B_s$ which we will need. However, it is possible to compute precise values as follows. Let $\zeta_{1-s}(e^{2\pi i}x)$ be the result of analytic continuation in a loop circling the origin. Then evidently

$$\zeta_{1-s}(e^{2\pi i}x) - \zeta_{1-s}(x) = (e^{2\pi is} - 1)x^{s-1}.$$

Using the integral formula (6), computation shows that

$$l_s(e^{2\pi i}x) - l_s(x) = -(2\pi i)^s x^{s-1}/\Gamma(s).$$

Comparing these two expressions, and noting that $\zeta_{1-s}(1-x)$ is holomorphic throughout a neighborhood of $x = 0$, we can solve for A_s . The result after some manipulation is

$$A_s = \frac{i(2\pi)^s e^{-\pi is/2}}{2\Gamma(s) \sin(\pi s)}.$$

Now comparing the behavior of l_s and ζ_{1-s} under complex conjugation we see easily that

$$B_s = \overline{A_s} = \frac{-i(2\pi)^s e^{\pi is/2}}{2\Gamma(s) \sin(\pi s)}.$$

In particular, it follows that

$$A_s + B_s = \frac{(2\pi)^s}{2\Gamma(s) \cos(\pi s/2)}, \quad A_s - B_s = \frac{i(2\pi)^s}{2\Gamma(s) \sin(\pi s/2)}.$$

As an application of formula (15), let us prove the corresponding functional equation for a Dirichlet L -function. Recall from Lemma 14 that for any primitive Dirichlet character χ modulo m the function

$$L(s, \chi) = \sum_1^m \chi(k) \zeta_s(k/m)/m^s$$

satisfies

$$L(s, \bar{\chi}) = \sum_1^m \chi(k) l_s(k/m)/\tau.$$

Here we may just as well use either the even or the odd parts of ζ_s and l_s according as $\chi(-1)$ is $+1$ or -1 . Therefore, it follows from (15) that

$$\begin{aligned} L(s, \bar{\chi}) &= (A_s \pm B_s) \sum_1^m \chi(k) \zeta_{1-s}(k/m)/\tau \\ &= m^s (A_s \pm B_s) L(1-s, \chi)/\tau. \end{aligned}$$

Thus we have proved the functional equation

$$(17) \quad L(s, \bar{\chi}) = m^{1-s} (A_s + \chi(-1)B_s) L(1-s, \chi)/\tau(\chi).$$

Here the factor m^{1-s}/τ is never zero or infinite, while $A_s \pm B_s$ is zero or infinite only at certain integer values, as indicated in the table above.

The proof of Lemma 13 can now easily be completed as follows. If $s \leq 0$ is an integer, then $L(1-s, \chi) \neq 0$, so it follows that $L(s, \bar{\chi})$ equals zero if and only if $A_s \pm B_s$ is zero, as indicated in the table. $\square$

APPENDIX 2

SOME RELATIVES OF THE GAMMA FUNCTION

This appendix will describe certain functions $\gamma_1(x)$, $\gamma_2(x)$, ... which satisfy a modified form of the Kubert identities, with a polynomial correction term. (See (22) below.) They are defined as partial derivatives of the Hurwitz function by the formula

$$(18) \quad \gamma_{1-t}(x) = \partial \zeta_t(x) / \partial t.$$

We will show that γ_1 is related to the classical gamma function via Lerch's identity

$$(19) \quad \gamma_1(x) = \log(\Gamma(x)/\sqrt{2\pi}).$$

(Compare [27, p. 60].) As a bonus, we will give a self-contained exposition of the basic properties of the gamma function, based on formulas (18) and (19).

Let us begin with equation (18), which defines $\gamma_s(x)$ as an analytic function of both variables for all $s \neq 0$ and all $x > 0$. Recall that the Hurwitz function $\zeta_t(x) = x^{-t} + (x+1)^{-t} + \dots$ (analytically extended in t for $t \neq 1$) satisfies

$$\zeta_t(x+1) = \zeta_t(x) - x^{-t}.$$

Differentiating with respect to t , and then substituting $t = 1 - s$, we obtain

$$(20) \quad \gamma_s(x+1) = \gamma_s(x) + x^{s-1} \log x.$$

In particular,

$$\gamma_1(x+1) = \gamma_1(x) + \log x.$$

Note that

$$\zeta'_t(x) = -t\zeta_{t+1}(x)$$

hence

$$\zeta''_t(x) = t(t+1)\zeta_{t+2}(x),$$