

### 3. Truth values in for statements about (B, A)

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3. TRUTH VALUES IN  $A$  FOR STATEMENTS ABOUT  $(B, A)$ 

For the rest of this paper, let  $\mathcal{L}_{BA} = \{+, \cdot, -, 0, 1\}$  the language of  $BA$ s and  $\mathcal{L} = \mathcal{L}_{BA} \cup \{U\}$ . Let  $T_{BAU}$  be the theory in  $\mathcal{L}$  such that the models of  $T_{BAU}$  have the form  $(B, +, \cdot, -, 0, 1, A)$  where  $(B, \dots)$  is a  $BA$  and  $A$  is a subalgebra of  $B$ . We abbreviate a model  $(B, \dots, A)$  of  $T_{BAU}$  by  $\mathcal{M} = (B, A)$ . We assume the construction and notations of section 1. For each  $\mathcal{L}$ -formula  $\varphi(x_1 \dots x_n)$  and  $b_1, \dots, b_n \in B$ , we defined

$$\|\varphi[b_1 \dots b_n]\| = \{p \in X \mid B_p \models \varphi[b_1(p) \dots b_n(p)]\}$$

where  $B_p$  abbreviates  $(B_p, 2)$  and  $2$  is the two-element  $BA$ . Our first claim is that if  $c = \|\varphi[b_1 \dots b_n]\|$  is a clopen subset of  $X$  for every  $\varphi$ , then  $e(c) \in A$  is first-order definable in  $\mathcal{M} = (B, A)$  from the parameters  $b_1, \dots, b_n \in B$ :

3.1. LEMMA. There is an effective procedure assigning to each formula  $\varphi(x_1 \dots x_n)$  of  $\mathcal{L}$  a formula  $s_\varphi(yx_1 \dots x_n)$  of  $\mathcal{L}$  (where  $y$  is a variable not occurring in  $\varphi$ ) such that for  $\mathcal{M} \models T_{BAU}$ , properties (i) and (ii) are equivalent and (ii) implies (iii):

- (i)  $\|\varphi[b_1 \dots b_n]\|$  is clopen for every  $\varphi(x_1 \dots x_n)$  in  $\mathcal{L}$  and  $b_1, \dots, b_n \in B$ ;
- (ii)  $\mathcal{M} \models \forall x_1 \dots \forall x_n \exists y s_\varphi(yx_1 \dots x_n)$  for every  $\varphi(x_1 \dots x_n)$  in  $\mathcal{L}$ ;
- (iii) if  $b_1, \dots, b_n \in B$ , then  $a = e(c)$  where  $c = \|\varphi[b_1 \dots b_n]\|$  is the unique element  $b$  of  $B$  such that  $\mathcal{M} \models s_\varphi[bb_1 \dots b_n]$ .

*Proof.* The inductive definition of  $s_\varphi$  will show that (i) is equivalent to (ii) and (i) implies (iii), the interesting cases being  $\varphi$  atomic or  $\varphi$  existential. In both cases the fact that  $\|\varphi[\dots]\|$  is clopen will be expressed by stating " $a (= e(\|\varphi[\dots]\|))$  is the largest element of  $A$  such that  $e^{-1}(a) \subseteq \|\varphi[\dots]\|$ ". This includes, if  $\varphi$  has the form  $\exists x\psi$ , the maximum principle for the Boolean valuation

$$\psi, b_1 \dots b_n \rightarrow \|\psi[b_1 \dots b_n]\|$$

of  $\mathcal{M}$  in  $C$ : there is some  $b \in B$  such that

$$\|\psi[b'b_1 \dots b_n]\| \leq \|\psi[bb_1 \dots b_n]\|$$

for every  $b' \in B$ , and hence  $\|\psi[bb_1 \dots b_n]\| = \|\exists x\psi[xb_1 \dots b_n]\|$ . We now proceed to define the formulas  $s_\varphi$ .

a) Suppose  $\varphi$  is an atomic formula of  $\mathcal{L}_{BA}$ , i.e.  $\varphi$  has the form  $t_1(x_1 \dots x_n) = t_2(x_1 \dots x_n)$  where  $t_1, t_2$  are terms in  $\mathcal{L}_{BA}$ . Let  $s_\varphi(yx_1 \dots x_n)$  be the formula

$$U(y) \wedge y \cdot t_1 = y \cdot t_2 \wedge \forall y' (U(y') \wedge y' \cdot t_1 = y' \cdot t_2 \rightarrow y' \leq y).$$

b) Suppose  $\varphi$  has the form  $U(t(x_1 \dots x_n))$  where  $t$  is a term in  $\mathcal{L}_{BA}$ . Let  $\psi, \chi$  be the atomic  $\mathcal{L}_{BA}$ -formulas " $t = 1$ " resp. " $t = 0$ ". Let  $s_\varphi$  be the formula

$$\exists y_1 \exists y_2 [y = y_1 + y_2 \wedge s_\psi(y_1 x_1 \dots x_n) \wedge s_\chi(y_2 x_1 \dots x_n)].$$

c) Suppose  $\varphi$  has the form  $\neg \psi(x_1 \dots x_n)$ . Let  $s_\varphi$  be the formula

$$\exists y_1 [y = -y_1 \wedge s_\psi(y_1 x_1 \dots x_n)].$$

d) Suppose  $\varphi$  has the form  $\psi(x_1 \dots x_n) \vee \chi(x_1 \dots x_n)$ . Let  $s_\varphi$  be the formula

$$\exists y_1 \exists y_2 [y = y_1 + y_2 \wedge s_\psi(y_1 x_1 \dots x_n) \wedge s_\chi(y_2 x_1 \dots x_n)].$$

e) Suppose  $\varphi$  has the form  $\exists x \psi(x x_1 \dots x_n)$ . Let  $s_\varphi$  be the formula

$$\exists x s_\psi(y x x_1 \dots x_n) \wedge \forall x' \forall y' [s_\psi(y' x' x_1 \dots x_n) \rightarrow y' \leq y].$$

Let  $\sigma$  be the  $\mathcal{L}_{BA}$ -formula stating that the supremum of the atoms of a  $BA$  exists;  $\sigma^U$  is the relativization of  $\sigma$  to the one-place predicate  $U$  of  $\mathcal{L}$ . The models of  $T_{BA} \cup \{\sigma\}$  are called separated  $BA$ s in [3]. Let  $T$  be the  $\mathcal{L}$ -theory

$$T = T_{BAU} \cup \left\{ \forall x_1 \dots \forall x_n \exists y s_\varphi(y x_1 \dots x_n) \mid \varphi(x_1 \dots x_n) \text{ in } \mathcal{L} \right\} \\ \cup \left\{ \sigma^U, s_\sigma(1) \right\}.$$

The last two axioms of  $T$  express, for a model  $\mathcal{M} = (B, A)$  of  $T_{BAU}$ , that  $A$  and each stalk  $B_p$  are separated  $BA$ s. Let  $\mathbf{K}$  be the class of  $\mathcal{L}$ -structures  $\mathcal{M} = (B, A)$  where  $B$  is a  $cBA$  and  $A$  is relatively complete in  $B$ . We shall prove in section 4 that  $T$  is an axiomatization of the first-order theory of  $\mathbf{K}$ . The easy part of this is:

3.2. THEOREM. *Each structure  $\mathcal{M}$  in  $\mathbf{K}$  is a model of  $T$ .*

*Proof.* Let  $\mathcal{M} = (B, A) \in \mathbf{K}$ , i.e.  $B$  is complete and  $A$  is relatively complete in  $B$ . Hence  $\mathcal{M} \models T_{BAU}$  and  $A$  is a separated  $BA$ . By 1.1,  $\|\varphi[b_1 \dots b_n]\|$  is clopen for every atomic formula  $\varphi$  of  $\mathcal{L}$  and arbitrary  $b_1, \dots, b_n \in B$ . If  $\|\varphi[b_1 \dots b_n]\|$  and  $\|\psi[b_1 \dots b_n]\|$  are clopen subsets of  $X$ , so are  $\|\neg \varphi[b_1 \dots b_n]\|$  and  $\|(\varphi \vee \psi)[b_1 \dots b_n]\|$ . Hence we assume that  $\varphi$

has the form  $\exists x \psi (xx_1 \dots x_n)$  and that  $\| \psi [bb_1 \dots b_n] \|$  is clopen for fixed  $b_1, \dots, b_n \in B$  and arbitrary  $b \in B$ . For the rest of the proof, we omit the parameters  $b_1, \dots, b_n$ . Let

$$u = \cup \{ \| \psi [\beta] \| \mid \beta \in B \}.$$

By our inductive assumption,  $u$  is an open subset of  $X$ . Choose, by Zorn's lemma, a maximal family  $F = \{ (b_i, c_i) \mid i \in I \}$  such that  $b_i \in B$ ,  $c_i$  is a clopen subset of  $u$ ,  $c_i \subseteq \| \psi [b_i] \|$ ,  $i \neq j$  implies  $c_i \cap c_j = \phi$ . It follows that  $c$ , the closure of  $\cup_{i \in I} c_i$ , includes  $u$  (by maximality of  $F$ ).  $A$  is a  $cBA$ ,

hence  $X$  is extremally disconnected and  $c$  is clopen. By completeness of  $B$ , there is some  $b \in B$  such that  $b \cdot e(c_i) = b_i$  for  $i \in I$ . Thus, for  $i \in I$ ,  $c_i \subseteq \| \psi [b] \|$ . So, for  $\beta \in B$ ,  $\| \psi [\beta] \| \subseteq u \subseteq c \subseteq \| \psi [b] \| = \| \exists x \psi (x) \|$ .

Finally we show that  $B_p$  is separated for each  $p \in X$ . Let  $\alpha(x)$  be the  $\mathcal{L}_{BA}$ -formula stating that  $x$  is an atom and let  $\beta(x)$ ,  $\gamma(x)$  be the  $\mathcal{L}_{BA}$ -formulas  $\alpha(x) \vee x = 0$  resp.  $\forall y (\alpha(y) \rightarrow y \leq x)$ . Put  $M = \{ f \in B \mid \| \beta [f] \| = 1 \|$  and let  $b$  be the supremum of  $M$  in  $B$ . We show that  $b(p)$  is, for each  $p \in X$ , the supremum of the atoms of  $B_p$ .

First suppose  $s \in B_p$  is an atom of  $B_p$ . There is some  $f \in M$  such that  $f(p) = s$  (note that  $\| \alpha [f] \|$  is clopen for each  $f \in B$ ). So  $f \leq b$  and  $s = f(p) \leq b(p)$ . — On the other hand, suppose  $t \in B_p$  and  $s \leq t$  for every atom  $s$  of  $B_p$ . Choose  $g \in B$  such that  $g(p) = t$ . Then  $p \in c = \| \gamma [g] \|$ . For  $f \in M$ ,  $e(c) \cdot f \leq g$ , since  $q \in c$  implies that  $f(q)$  is zero or an atom of  $B_q$  and thus  $f(q) \leq g(q)$ . By the definition of  $b$ ,  $e(c) \cdot b \leq g$ . This implies (by  $p \in c$ )  $b(p) \leq g(p) = t$ .

#### 4. DECIDABILITY AND COMPLETIONS OF $Th(\mathbf{K})$

Call  $T_{sBA} = T_{BA} \cup \{ \sigma \}$  the theory of separated  $BA$ s, where  $T_{BA}$  is the theory of  $BA$ s and  $\sigma$  was defined in section 3. We give a short review of the completions of  $T_{sBA}$ . Let, for  $n \in \omega$ ,  $\varphi_n$  be the  $\mathcal{L}_{BA}$ -sentence stating that there are exactly  $n$  atoms and  $\psi$  the  $\mathcal{L}_{BA}$ -sentence stating that there is a non-zero atomless element. Let  $\chi_n = \neg (\varphi_0 \vee \dots \vee \varphi_{n-1})$ ; so  $\chi_n$  says that there are at least  $n$  atoms. Define, for  $n \in \omega + 1$  and  $i \in 2 = \{ 0, 1 \}$ , an  $\mathcal{L}_{BA}$ -theory  $T_{ni}$  by

$$\begin{aligned} T_{n0} &= T_{sBA} \cup \{ \varphi_n, \neg \psi \} \\ T_{n1} &= T_{sBA} \cup \{ \varphi_n, \psi \} \end{aligned}$$