2. Relative automorphisms of finite extensions

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1.3. Remark. *a)* Let A and the inclusion map from A to B be complete. Then A is relatively complete in B.

b) Suppose A is relatively complete in B and B is complete. Then A is complete.

2. Relative automorphisms of finite extensions

We first give an internal description of a finite extension (B, A) where $B = A(u_1 \dots u_n)$ and $n \in \omega$. We shall always assume that $u_1, \dots, u_n$ are the atoms of the subalgebra of B generated by $u_1, \dots, u_n$; i.e. that they are non-zero, pairwise disjoint and $u_1 + \dots + u_n = 1$. Let $I_r = \{a \in A \mid a \cdot u_r = 0\}$ for $1 \leqslant r \leqslant n$. Clearly, each I_r is a proper ideal of A and $I_1 \cap \dots \cap I_n = \{0\}$. The family $(I_r \mid 1 \leqslant r \leqslant n)$ completely characterizes the extension (B, A):

2.1. Remark. Suppose $C = A(v_1 \dots v_n)$ is a finite extension of A where $v_1, \dots, v_n$ are pairwise disjoint and $1 = v_1 + \dots + v_n$. Let $B = A(u_1 \dots u_n)$ be as above. There is an isomorphism g from B onto C satisfying $g(a) = a$ for $a \in A$ and $g(u_r) = v_r$ iff, for each r, $\{a \in A \mid a \cdot v_r = 0\} = I_r$.

Proof. By Theorem 12.4 in [7].

2.2. Remark. A is relatively complete in $B = A(u_1 \dots u_n)$ iff, for each r, I_r is a principal ideal.

Proof. The only—if part follows by the definition of relative completeness. Now suppose $\alpha_r \in A$ generates I_r; let $b \in B$ and $I = \{a \in A \mid a \cdot b = 0\}$. There are $a_1, \dots, a_n \in A$ such that $b = a_1 \cdot u_1 + \dots + a_n \cdot u_n$. It follows that I is the principal ideal generated by $\alpha = (-a_1 + \alpha_1) \cdot \dots \cdot (-a_n + \alpha_n)$.

Conversely, given any family $(I_r \mid 1 \leqslant r \leqslant n)$ of proper ideals in A satisfying $I_1 \cap \dots \cap I_n = \{0\}$, there is an extension $A(u_1 \dots u_n)$ of A such that $I_r = \{a \in A \mid a \cdot u_r = 0\}$: let $D = A(x_1 \dots x_n)$ be the free product of A and a finite BA with atoms $x_1, \dots, x_n$. Let

$$K = \{i_1 \cdot x_1 + \dots + i_n \cdot x_n \mid i_1 \in I_1, \dots, i_n \in I_n\} .$$

K is an ideal of D; the canonical epimorphism π from D onto $B = D/K$ is one- one on A, and for $a \in A$, $\pi(a) \cdot u_r = 0$ iff $a \in I_r$ where $u_r = \pi(x_r)$. Now identify A with the subalgebra $\pi(A)$ of B.

For the rest of this section we think, as in section 1, of B as being the set of global sections of a sheaf $\mathcal{S} = (S, \pi, X, \mu)$ of Boolean algebras over a

Boolean space X; we use the abbreviations of section 1. For $p \in X$, $B_p = \{b(p) \mid b \in B\}$. Since $b(p) \in \{0, 1\}$ for $b \in A$ and $B = A(u_1 \dots u_n)$, B_p is a finite BA with atoms $\{u_r(p) \mid 1 \leqslant r \leqslant n\} \setminus \{0\}$.

Let $G = \mathrm{Aut}_A B$ be the group of those automorphisms of B leaving A pointwise fixed, i.e. G is the Galois group of B over A. Suppose $g \in G$ and $p \in X$. Since $g(a) = a$ for $a \in A$, g induces an automorphism of B_p which, in turn, is induced by a permutation of the (at most n) atoms of B_p. This gives rise to the following definitions (S_n is the group of permutations of $\{1, \dots, n\}$).

Let $p \in X$. For $1 \leqslant r, l \leqslant n$, say $u_r \sim u_l$ at p if there is a neighbourhood u of p such that, for $q \in u$, $u_r(q) = 0$ iff $u_l(q) = 0$. $\pi \in S_n$ is said to be compatible with p if $u_r \sim u_{\pi(r)}$ at p for $1 \leqslant r \leqslant n$. $g \in G$ is said to be induced by π at p if $g(u_r)(p) = u_{\pi(r)}(p)$ for $1 \leqslant r \leqslant n$. Note that, if one of these definitions holds (for fixed $u_r, u_l, \pi \in S_n, g \in G$) for some $p \in X$, then it holds (for the same $u_r, u_l, \pi \in S_n, g \in G$) for every q in some neighbourhood of p. And $u_r \sim u_l$ at p means that there is a clopen subset c of X such that $p \in c$ and, for $a \in A$ satisfying $a \leqslant e(c)$, $a \in I_r$ iff $a \in I_l$.

2.3. Lemma. Suppose $p \in X$ and $\pi \in S_n$. Then π is compatible with p iff there is some $g \in G$ which is induced by π at p.

Proof. Suppose π induces g at p and $1 \leqslant r \leqslant n$. Let u be a neighbourhood of p such that $g(u_r)(q) = u_{\pi(r)}(q)$ for $q \in u$. Thus, for $q \in u$, $u_{\pi(r)}(q) = 0$ iff $g(u_r)(q) = 0$ iff $u_r(q) = 0$ since g induces an automorphism of B_q.

Conversely, suppose π is compatible with p. Choose a clopen neighbourhood c of p such that $u_r(q) = 0$ iff $u_{\pi(r)}(q) = 0$ for $1 \leqslant r \leqslant n$ and $q \in u$. Let $a = e(c)$. By 2.1 and the remark preceding this lemma, there is some $g \in G$ such that $g(u_r) = -a \cdot u_r + a \cdot u_{\pi(r)}$ for every r. This g is induced by π at p, since $a(p) = 1$ and hence $g(u_r)(p) = u_{\pi(r)}(p)$.

2.4. Theorem. a) *Let* $X = \cup \{c_\pi \mid \pi \in S_n\}$ *be a partition of* X *into pairwise disjoint clopen subsets such that, for every* $p \in c_\pi$, π *is compatible with* p. *Put* $a_\pi = e(c_\pi)$ *for* $\pi \in S_n$. *Then there is* $g \in G$ *such that, for* $1 \leqslant r \leqslant n$,

$$g(u_r) = \sum \{a_\pi \cdot u_{\pi(r)} \mid \pi \in S_n\}.$$

b) *Conversely, let* $g \in G$. *Then there is a partition* $X = \cup \{c_\pi \mid \pi \in S_n\}$ *of* X *into pairwise disjoint clopen subsets such that, for* $p \in c_\pi$, π *is compatible with* p, *and* $g(u_r) = \sum \{a_\pi \cdot u_{\pi(r)} \mid \pi \in S_n\}$, *where* $a_\pi = e(c_\pi)$.

Proof. First note that $g \in G$, $a_\pi = e(c_\pi)$ where $(c_\pi \mid \pi \in S_n)$ is a partition of X and $g(u_r) = \sum \{a_\pi \cdot u_{\pi(r)} \mid \pi \in S_n\}$ imply that π is compatible with p for $p \in c_\pi$: by $p \in c_\pi$, we get $a_\pi(p) = 1$ and $a_\rho(p) = 0$ for $\rho \in S_n$, $\rho \neq \pi$. So $g(u_r)(p) = u_{\pi(r)}(p)$, g is induced by π at p, and π is compatible with p.

To prove a), note that $\{a_\pi \cdot u_r \mid \pi \in S_n, 1 \leqslant r \leqslant n\}$ is a set of pairwise disjoint elements of B with supremum 1 and generating B over A. The existence of g follows by 2.1 and the remark preceding 2.3.

To prove b), let $g \in G$. For $\pi \in S_n$, put

$$v_\pi = \{p \in X \mid \pi \text{ induces } g \text{ at } p\} .$$

Each v_π is an open subset of X, and $X = \cup \{v_\pi \mid \pi \in S_n\}$: suppose $p \in X$. Define $\pi \in S_n$ as follows: let $1 \leqslant r \leqslant n$. If $u_r(p) = 0$, then $g(u_r)(p) = 0$; put $\pi(r) = r$. If $u_r(p) \neq 0$, $u_r(p)$ and hence $g(u_r)(p)$ is an atom of B_p; let $\pi(r) = l$ where $g(u_r)(p) = u_l(p)$. Clearly, $p \in v_\pi$.

Since X is a Boolean space, there is a family $(c_\pi \mid \pi \in S_n)$ such that c_π is a clopen subset of v_π, $X = \cup \{c_\pi \mid \pi \in S_n\}$ and the c_π are pairwise disjoint. Put $a_\pi = e(c_\pi)$. Suppose $1 \leqslant r \leqslant n$ and $p \in X$, e.g. $p \in c_\pi$. Then $p \in v_\pi$ and

$$(\sum \{a_\pi \cdot u_{\pi(r)} \mid \pi \in S_n\})(p) = g(u_r)(p) .$$

Theorem 2.4 says that the automorphisms of B over A are completely determined by certain partitions $(a_\pi \mid \pi \in S_n)$ of A resp. $(c_\pi \mid \pi \in S_n)$ of C. Unfortunately, for a given $g \in G$, a partition $(c_\pi \mid \pi \in S_n)$ defining g is not uniquely determined, since there may be different possibilities of choosing a clopen disjoint refinement of $(v_\pi \mid \pi \in S_n)$. We conclude this section by illustrating 2.4 by several examples.

If H is any group and A a BA, let X be the Stone space of A and

$$H[A] = \{f : X \to H \mid f \text{ is continuous}\}$$

where H is given the discrete topology. $H[A]$ is a subgroup of H^X and is usually called the bounded Boolean power of H by A. Recall that, for $B = A(u_1 \ldots u_n)$, A and the subalgebra of B generated by $u_1, \ldots, u_n$ are independent iff $a \cdot u_r \neq 0$ for $a \in A \setminus \{0\}$, $1 \leqslant r \leqslant n$. A is then relatively complete in B. Conversely, suppose A is relatively complete in B. Then there is a partition $(a_k \mid 1 \leqslant k \leqslant n)$ of A (some of the a_k may equal zero) such that, for each k, the relative algebra $B \upharpoonright a_k = \{x \in B \mid x \leqslant a_k\}$ is generated over $A \upharpoonright a_k$ by k disjoint elements $v_1, \ldots, v_k$ which are independent from $A \upharpoonright a_k$: for $1 \leqslant r, l \leqslant n$, the set of those $p \in X$ such that $u_r(p) = u_l(p)$ is clopen. Hence, for $1 \leqslant k \leqslant n$, $c_k = \{p \in X \mid B_p \text{ has exactly } k \text{ atoms}\}$ is clopen; put $a_k = e(c_k)$. By a compactness argument, construct $v_1, \ldots, v_k \in B \upharpoonright a_k$ by patching together some of the u_r such that for $p \in c_k$, the atoms of B_p are $v_1(p), \ldots, v_k(p)$.

2.5. Example. Suppose $a \cdot u_r \neq 0$ for $1 \leqslant r \leqslant n$ and $a \in A \setminus \{0\}$. Then $\mathrm{Aut}_A B \cong S_n[A]$.

Proof. Our assumption says that $u_r(p) \neq 0$ for each r and each $p \in X$. Hence each $\pi \in S_n$ is compatible with each $p \in X$ and, for fixed $g \in G$, the open sets v_π in the proof of 2.4 are disjoint, hence $c_\pi = v_\pi$. An isomorphism $\varphi : G \to S_n[A]$ is established by defining $\varphi(g)(p) = \pi$ iff $p \in v_\pi$.

2.6. Example. Suppose A is relatively complete in B. Then there is a partition $(a_k \mid 1 \leqslant k \leqslant n)$ of A such that

$$\mathrm{Aut}_A B \cong S_1[A \upharpoonright a_1] \times \ldots \times S_n[A \upharpoonright a_n].$$

Proof. Choose, for $1 \leqslant k \leqslant n$, $a_k \in A$ as indicated above and let G_k be the Galois group of $B \upharpoonright a_k$ over $A \upharpoonright a_k$. Clearly,

$$\mathrm{Aut}_A B \cong G_1 \times \ldots \times G_n,$$

since $a_k \in A$. By 2.5, $G_k \cong S_k[A \upharpoonright a_k]$.

2.7. Proposition. *The following conditions on* (B, A) *are equivalent:*

a) A *is relatively complete in* B;

b) *there is some* $g \in G$ *such that* $g(b) \neq b$ *for* $b \in B \setminus A$;

c) *there is some finite subgroup* H *of* G *such that, for every* $b \in B \setminus A$, *there is some* $g \in H$ *satisfying* $g(b) \neq b$.

Proof. Assume a). There is a finite partition T of C such that, for $1 \leqslant r \leqslant n$, $t \in T$ and $p, q \in t$, $u_r(p) = 0$ iff $u_r(q) = 0$. For $t \in T$, let $\pi_t \in S_n$ such that, for $p \in t$, $\pi_t(r) = r$ if $u_r(p) = 0$ and $u_r(p) \mapsto u_{\pi_t(r)}(p)$ is a cyclic permutation of the atoms of B_p which moves all these atoms. π_t is compatible with each $p \in t$; hence there is some $g \in G$ such that g is induced by π_t for $p \in t$, $t \in T$. Now let $b \in B \setminus A$. Choose $p \in X$, e.g. $p \in t$ where $t \in T$, such that $b(p) \notin \{0, 1\}$; put $b' = g(b)$. Let $At(B_p)$ be the set of atoms of B_p, $M = \{\alpha \in At(B_p) \mid \alpha \leqslant b(p)\}$, g_p the automorphism of B_p induced by g, $M' = \{g_p(\alpha) \mid \alpha \in M\}$. By the choice of π_t and g,

$$b'(p) = g_p(b(p)) = \sum M' \neq \sum M = b(p)$$

which proves $b' \neq b$ – since, if π is a cyclic permutation of a finite set Y moving every element of Y and $M \subseteq Y$ satisfies $M = \{\pi(m) \mid m \in M\}$, then $M = \phi$ or $M = Y$.

To prove that b) implies c) it is sufficient to know that every finitely generated subgroup of G is finite. We indicate a construction for finite subgroups of G. Let $T \subseteq C$ be a finite partition of C. A function $\varphi : T \to S_n$ is said to be compatible if, for every $t \in T$ and $p \in t$, $\varphi(t)$ is compatible with p. For each compatible $\varphi : T \to S_n$ let g_φ be the element of G mapping u_r to $\sum \{e(t) \cdot u_{\varphi(t)(r)} \mid t \in T\}$. It is easily seen that

$$G_T = \{g_\varphi \mid \varphi : T \to S_n \text{ compatible}\}$$

is a finite subgroup of G and that every finite subset of G is contained in some G_T.

Now suppose c), i.e. there is some finite subgroup H of G moving every $b \in B \setminus A$. We may assume that $H = G_T$ for some finite partition T of C. Assume that A is not relatively complete in B. By 2.2 there is some r such that I_r is not a principal ideal; w.l.o.g., $r = 1$. Let $\sigma = \{p \in X \mid u_1(p) = 0\}$. σ is a subset of X which is open but not closed; choose $p \in X$ which lies in the closure of σ but not in σ. W.l.o.g., for some k satisfying $1 \leqslant k \geqslant n$,

$$\{r \mid 1 \leqslant r \leqslant n \text{ and } u_r \sim u_1 \text{ at } p\} = \{1, \dots, k\}.$$

Let c be a clopen neighbourhood of p such that, for $1 \leqslant r \leqslant k$ and $q \in c$, $u_r(q) = 0$ iff $u_1(q) = 0$. W.l.o.g., $c \in T$. There is some l such that $k < l \leqslant n$ and $u_l(p) \neq 0$; otherwise, let $c' \subseteq c$ a neighbourhood of p such that $u_l(q) = 0$ for $q \in c'$ and $k < l \leqslant n$. Choose $q \in c' \cap \sigma$ (since p lies in the closure of σ). In B_q, which has at least two elements, $1 = u_1(q) + \dots + u_n(q) = 0 + \dots + 0 = 0$, a contradiction. — Put $a = e(c)$ and $b = a \cdot u_1 + \dots + a \cdot u_k$. $b \in B \setminus A$, since $0 < b(p) = u_1(p) + \dots + u_k(p) < 1$ by our preceding claim. We prove that, for $g \in H = G_T$, $g(b) = b$, thus arriving at a final contradiction: there is some compatible $\varphi : T \to S_n$ such that $g = g_\varphi$. Consider $k \leqslant n$, $c \in T$ and $p \in c$ as constructed above. Since φ is compatible, $\pi = \varphi(c)$ is compatible with p; hence π maps the set $\{1, \dots, k\}$ into itself, $g_\varphi(a \cdot u_r) = a \cdot u_{\pi(r)}$ for $1 \leqslant r \leqslant k$ (where $a = e(c)$) and $g(b) = b$.