

IV. Ramsey-type Theorems

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then we can construct the sequence $\{h_i\}$ and the set $\mathcal{F}$ to satisfy the Stability and Closure Conditions via Theorem 2.

We could now proceed to show that the above condition is indeed satisfied in $\mathbf{N}$ and thus construct a non-standard model of Peano arithmetic. However, our goal is the construction of a mathematically perspicuous model which is independent of the logical formulas. The functions $\{h_i\}$ given by the above condition require the logical calculus in their definition. Accordingly, we shall consider a larger class $\mathcal{F}$ of functions than those defined above, which we shall construct independently of logical formulas. This class will be constructed from combinatorial principles derived from Ramsey's Partition Theorem.

IV. RAMSEY-TYPE THEOREMS

The infinite Ramsey Theorem states that for every partition $P : [\mathbf{N}]^e \rightarrow r$ ($r \geq 1$) there exists an infinite subset X of $\mathbf{N}$ such that $P \upharpoonright [X]^e$ is constant. In these circumstances one says that X is homogeneous for the partition P . This set-theoretic theorem has various combinatorial consequences which are formalizable in elementary arithmetic. One such immediate consequence which we shall prove independent of the Peano axioms is the following.

PROPOSITION 1. *Let $P : [\mathbf{N}]^e \rightarrow r$ be a primitive recursive partition. For every natural number k there exists a finite subset X of $\mathbf{N}$, with $\# X \geq k$ and $\# X \geq 2^{2^{\min X}}$, which is homogeneous for the partition P .*

In order to apply Theorem 2 we require the construction of a set which is simultaneously homogeneous for several partitions. This is easily done by the infinite Ramsey Theorem. Suppose $P_1 : [\mathbf{N}]^{e_1} \rightarrow r_1$ and $P_2 : [\mathbf{N}]^{e_2} \rightarrow r_2$ are two partitions. Let X_1 be an infinite subset of $\mathbf{N}$ homogeneous for P_1 . Then $P_2 \upharpoonright [X_1]^{e_2}$ is a partition of $[X_1]^{e_2}$, and hence there is an infinite subset X_2 of X_1 which is homogeneous for P_2 (as well as P_1). This proof extends immediately to finitely many partitions. A direct consequence is the following generalization of Proposition 1.

PROPOSITION 2. *Let $P_i : [\mathbf{N}]^{e_i} \rightarrow r_i$, $i \leq i \leq n$ be a set of primitive recursive partitions. For every natural number k there exists a finite subset X of $\mathbf{N}$ with $\# X \geq k$ and $\# X \geq 2^{2^{\min X}}$, which is simultaneously homogeneous for all the partitions $P_1, \dots, P_n$.*

¹⁾ We identify the number r with the set of all natural numbers $< r$.

Proposition 2 may be expressed by a $\prod_2^0$ formula. First it is clear that we can construct a $\sum_1^0$ -formula ϕ_i that expresses the properties that

1. $P_i : [\mathbb{N}]^{e_i} \rightarrow r_i$ is a primitive recursive partition
2. $z_1 < z_2 < \dots < z_{n_k}$
3. $\{z_1, \dots, z_{n_k}\}$ is homogeneous for P_i
4. $k \leq n_k$
5. $2^{2^{z_1}} \leq n_k$

Proposition 2 asserts that for every k

$$\mathbb{N} \models \exists z_1 \dots \exists z_{n_k} \bigwedge_{i \leq k} \phi_i.$$

V. CONSTRUCTION OF THE MODEL

We now have all the ingredients at hand to construct a non-standard model of Peano arithmetic, and we have only to assemble them according to the specifications of Section II.

Let P_i be an effective enumeration of all primitive recursive partitions $P_i : [\mathbb{N}]^{e_i} \rightarrow r_i$. By Proposition 2 we have that for every k

$$\mathbb{N} \models \exists z_1 \dots \exists z_{n_k} \bigwedge_{i \leq k} \phi_i$$

where ϕ_i is the $\sum_1^0$ -formula of Section IV expressing the conditions (1)-(5) satisfied by the partition P_i .

Following the prescription given in Section III we let a_{kn_k} be the smallest number such that $a_{k_1}, \dots, a_{kn_k}$ is an increasing sequence satisfying the formula $\bigwedge_{j \leq k} \phi_j$. Now we define the functions h_j by

$$h_0(k) = n_k \quad \text{for every } k$$

and for $j > 0$

$$h_j(k) = \begin{cases} a_{kj} & \text{for } j \leq n_k \\ h_{j-1}(k)^2 & \text{for } j > n_k. \end{cases}$$

Let $\mathcal{F} = \{f \mid f \leq h_j\}$.

Since $1 \leq h_0$ the function 1 is automatically in $\mathcal{F}$.

By Theorem 2 the sequence $\{h_j\}$ satisfies $\bigwedge_{j < \infty} \phi_j$ in $\mathcal{F}/D$. We now prove that this implies that the sequence $\{h_j\}$ satisfies the Stability and