§2. Notation

There are just $[xp_1^{-k}]$ of these $n \leqslant x$, and since $[z] \geqslant z/2$ for $z \geqslant 1$, we get the result.

It is clear that Theorem 1.24 is essentially best possible in certain extreme cases (for example, if $E = \{p_1\}$, or if $x = p_1^a$ and $y = a - 1$).

When $E = P$ (the set of all primes), we can take $v = \log_2 x$. Then $\Lambda = O(1)$, and we have the following corollary of Theorems 1.21 and 1.24:

Corollary 1.25. *If* $x \geqslant e^e$ *and* $0 \leqslant y \leqslant (\log x)(\log 2)^{-1} - 1$, *then*

$$2^{-y-2} x \leqslant S(x, y; P, \Omega) \leqslant c_6 2^{-y} x (\log x)(\log_2 x)^{1/2}.$$

Corollary 1.25 should be compared with the Erdös-Sárközy result (1.20) and with the asymptotic formula of Selberg mentioned after Theorem 1.18. When $y < 2 \log_2 x$ (roughly), more precise estimates for $S(x, y; P, \Omega)$ can be obtained from [13] and [14].

In a later paper, we shall show that if p_1 is the smallest member of E and $\varepsilon > 0$ is fixed, then the precise order of magnitude of $S(x, y; E, \Omega)$ is

$$p_1^{-y} x \exp\{(p_1 - 1) E(x)\}$$

when $E(x)$ is sufficiently large and

$$p_1 E(x) \leqslant y \leqslant (1 - \varepsilon)(\log x)(\log p_1)^{-1}.$$

This theorem is much more difficult to prove than Theorem 1.21. Its proof depends on Theorem 1.21 and on an extension of Halász's work [4] concerning the local distribution of $\Omega(n; E)$. Theorem 1.21 remains our best upper bound when y is close to $(\log x)(\log p_1)^{-1}$ (cf. Theorem 1.18), and it seems to be the most we can achieve by a fairly simple method.

§2. Notation

The symbols a, m, n always represent integers with $a \geqslant 0, m \geqslant 0, n > 0$. The letter p always denotes a prime, while $v, w, x, y, z, \alpha, \beta, \delta, \varepsilon, \sigma$ are real numbers. $[x]$ means the largest integer $\leqslant x$. The notation $\log_r x$ is defined by (1.5), and the notations $O, O_{\delta, \varepsilon, \ldots}, c_i, c_i(\delta, \varepsilon, \ldots)$ are explained after Theorem 1.6. If a condition such as "$x \geqslant c_i(\delta, \varepsilon, \ldots)$" is used as a hypothesis, it is to be understood that $c_i(\delta, \varepsilon, \ldots)$ is sufficiently large. We shall occasionally use the notations $\ll, \gg$ to imply constants which are *absolute*. (Thus $A = O(B)$ is equivalent to $A \ll B$.)

Empty sums mean 0, empty products 1, and we define $0^0 = 1$. The notation

$$x_1 \cdots x_m/y_1 \cdots y_n$$

is sometimes used instead of

$$(x_1 \cdots x_m)(y_1 \cdots y_n)^{-1}.$$

Throughout this paper, E denotes a nonempty set of primes, to be regarded as quite arbitrary unless further assumptions are stated. $E(x)$ is always defined by (1.2). p_1 always means the smallest member of E, and if

$$E - \{p_1\} = \{p : p \in E \quad \text{and} \quad p \neq p_1\}$$

is not empty, then p_2 denotes the smallest member of $E - \{p_1\}$. When x and v are positive, the function $\Lambda = \Lambda(x, v; E)$ is always defined by (1.22).

§3. Proofs of Theorems 1.6 and 1.11, and related results

Before proving (1.8), we observe that a similar but weaker inequality has a very simple proof. For if $y > 1$, then

$$\log n \geqslant \sum_{p|n} \log p \geqslant \sum_{p|n,\, p \geqslant y} \log p \geqslant (\log y) \sum_{p|n,\, p \geqslant y} 1,$$

and hence

$$\omega(n) = \sum_{p|n,\, p<y} 1 + \sum_{p|n,\, p \geqslant y} 1 \leqslant y + (\log n)(\log y)^{-1}.$$

The right-hand side is approximately minimized by taking

$$y = (\log n)(\log_2 n)^{-2},$$

and we obtain

$$\omega(n) \leqslant \frac{\log n}{\log_2 n}\left\{1 + O\left(\frac{\log_3 n}{\log_2 n}\right)\right\} \quad \text{for} \quad n \geqslant 16\,(>e^e). \tag{3.1}$$

Another simple proof of (3.1) can be based on Newman's observation [11, p. 652] that if $\omega(n) = r$, then $n \geqslant r!$.

To get the sharper inequality (1.8), it seems to be necessary to use an assumption such as (1.7) about the distribution of E. First we need a lemma relating $\pi(x; E)$ (defined by (1.4)) and

$$\theta(x; E) = \sum_{p \leqslant x,\, p \in E} \log p. \tag{3.2}$$