8. The classification of right equivalence classes

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **25 (1979)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **28.04.2024**

Characterization A7′. $\lim_{r \to 0} r^{-2} \operatorname{vol}(X(r))$ is finite.

Let $\omega = dx \wedge dy \wedge dz$, and note that $\omega \wedge \bar{\omega}$ is $8/i$ times the volume form of $\mathbf{C}^3$. Characterizations A7 and A7′ are equivalent since

$$\lim_{r \to 0} \frac{1}{r^2} \operatorname{vol}(X(r)) = \lim_{r \to 0} \frac{i}{8r^2} \int_{X_r} \omega \wedge \bar{\omega}$$

$$= \lim_{r \to 0} \frac{i}{8r^2} \int_{\Delta(r)} \left(\int_{V_t} \omega_t \wedge \bar{\omega}_t \right) dt \wedge \overline{dt} ,$$

but since

$$\int_{\Delta(r)} \left(\frac{i}{2}\right) dt \wedge \overline{dt} = \operatorname{vol}(\Delta(r)) = 2\pi r^2 ,$$

the above limit equals

$$\frac{\pi}{2} \int_{V_0} \omega_0 \wedge \bar{\omega}_0 .$$

B. Nine characterizations of simple critical points

We switch our attention from the analytic set defined by the zero locus of an analytic function $f(x, y, z)$ to the function itself and the nature of its critical point. We also generalize to functions $f(z_0, \ldots, z_n)$ of an arbitrary number of variables. The characterizations in the following theorem will start in Section 9.

Theorem B. *Let $f(z_0, \ldots, z_n)$ with $n \geqslant 1$ be the germ at the origin* $\mathbf{0}$ *of a complex analytic function, and suppose further that $f(\mathbf{0}) = 0$ and that* $\mathbf{0}$ *is an isolated critical point of f. Then Characterizations B1 through B9 are equivalent.*

8. The classification of right equivalence classes

Let $\mathcal{O}$ be the set of germs f at the origin $\mathbf{0}$ of complex analytic functions on $\mathbf{C}^{n+1}$. (In other words, $\mathcal{O}$ is just the ring $\mathbf{C}\{z_0, \ldots, z_n\}$ of convergent power series.) The ring $\mathcal{O}$ is local with maximal ideal

$$\mathfrak{m} = \{ f \in \mathcal{O} : f(\mathbf{0}) = 0 \} .$$

Let

$$\Delta f = \left(\frac{\partial f}{\partial z_0}, \dots, \frac{\partial f}{\partial z_n}\right)$$

be the ideal in $\mathcal{O}$ generated by the partial derivatives of f.

Lemma 8.1. A germ f in $\mathfrak{m}$ has an isolated critical point at $\mathbf{0}$ if and only if there is a k such that $\mathfrak{m}^k \subset \Delta f \subset \mathfrak{m}$.

Proof. The germ f has a critical point at $\mathbf{0}$ if and only if $f \in \mathfrak{m}^2$, or equivalently, $\Delta f \subset \mathfrak{m}$. If this critical point is isolated, then the origin is an isolated zero of the functions $\partial f/\partial z_0, \dots, \partial f/\partial z_n$. This is equivalent to saying that the set of common zeros of all the functions in the ideal Δf equals the set of common zeros of the ideal $\mathfrak{m}$. By the Nullstellensatz, there exist integers $l_0, \dots, l_n$ such that $z_i^{l_i} \in \Delta f$. Setting $k = (n+1) \max \{l_0, \dots, l_n\}$ gives $\mathfrak{m}^k \subset \Delta f$. Conversely, if $\mathfrak{m}^k \subset \Delta f$ then the origin is an isolated critical point. This proves the lemma.

Let $\mathcal{F}$ be the set of all germs in $\mathcal{O}$ vanishing at the origin and with an isolated critical point there. (This is the set of *finitely-determined* germs.) The *Milnor number* of a germ $f \in \mathcal{F}$ is

$$\mu = \dim_{\mathbf{C}} \mathcal{O}/\Delta f .$$

For a comprehensive discussion of μ, see [Orlik 2]. There are many ways to compute this number, aside from the above formula [Milnor 1, §§7, 10; A'Campo 1; Laufer 6]. The *(right) codimension* of f is $\mu - 1$.

Two germs f and g in $\mathcal{O}$ are *right equivalent* (written $f \sim g$) if there is a germ h of a complex analytic automorphism of $\mathbf{C}^{n+1}$ fixing $\mathbf{0}$ with $f \circ h = g$. The germs f and g are *contact equivalent* if there is an h as above such that the ideal generated by $f \circ h$ in $\mathcal{O}$ is equal to the ideal generated by g. This is equivalent to saying that the analytic sets $f^{-1}(0)$ and $g^{-1}(0)$ are isomorphic. Note that right-equivalent germs are contact equivalent.

Mather, Arnold, and others have classified germs of low Milnor number under right equivalence. The implicit function theorem shows, for example, that if $f(\mathbf{0}) = 0$ but the derivative of f does not vanish at $\mathbf{0}$, then f is right equivalent to the projection $(z_0, \dots, z_n) \mapsto z_0$. If $f(\mathbf{0}) = 0$ and f has a non-degenerate critical point at $\mathbf{0}$, then $f(z_0, \dots, z_n) \sim z_0^2 + \dots + z_n^2$ by the Morse lemma.

Recall that the *k-jet* of a germ f in $\mathcal{O}$ is its power series expansion up to degree k. A germ $f \in \mathcal{O}$ is *k-determined* if any germ with the same k-jet as f is right equivalent to f. In particular, f is right equivalent to its own k-jet. A germ is *finitely determined* if it is k-determined for some $k < \infty$.

The fundamental lemmas used in the classification are as follows:

Lemma 8.2. If $\mathfrak{m}^{k+1} \subset \mathfrak{m}^2 \Delta f$ then f is k-determined.

For the proof, see [Arnold 1, Lemma 3.2; Zeeman, Theorem 2.9; Siersma, p. 8]. Note that $\mathfrak{m}^{k-1} \subset \Delta f$ implies that $\mathfrak{m}^{k+1} \subset \mathfrak{m}^2 \Delta f$. The *corank* of f is defined as $n + 1$ minus the rank of the Hessian matrix $\{(\partial^2 f/\partial z_i \partial z_j)(\mathbf{0})\}$. The proof of part (a) of the following lemma may be found in [Arnold 1, Lemma 4.1; Siersma Lemma 3.2].

Splitting Lemma 8.3. (a) Let $f(z_0, ..., z_n) \in \mathscr{F}$ be of corank $r + 1$. Then there is a $g(z_0, ..., z_r) \in \mathfrak{m}^3$ such that

$$f(z_0, \ldots, z_n) \sim g(z_0, \ldots, z_r) + z_{r+1}^2 + \ldots + z_n^2 .$$

(b) Let $g(z_0, ..., z_r)$ and $g'(z_0, ..., z_r) \in \mathscr{F} \cap \mathfrak{m}^3$. If

$$g(z_0, \ldots, z_r) + z_{r+1}^2 + \ldots + z_n^2 \sim g'(z_0, \ldots, z_r) + z_{r+1}^2 + \ldots + z_n^2$$

then

$$g(z_0, \ldots, z_r) \sim g'(z_0, \ldots, z_r) .$$

The classification proceeds by low corank and low Milnor number. A germ of corank 0 is right equivalent to $z_0^2 + ... + z_n^2$, a germ of corank 1 and Milnor number $k > 1$ is right equivalent to $z_0^{k+1} + z_1^2 + ... + z_n^2$, and so forth. The proofs are not hard [Arnold 1, Zeeman, Siersma]. Table 2, for instance, includes all right-equivalence classes of germs with Milnor number $\mu \leqslant 9$.

9. Characterizations under right and contact equivalence

Characterization B1. The germ f is right equivalent to one of the germs in Table 2a.

Characterization B2. The germ f is contact equivalent to one of the germs in Table 2a.

When $n = 2$, Characterization B2 is the same as Characterization A1. Clearly Characterization B1 implies Characterization B2. Since all the germs in Table 2a are weighted homogeneous (§16), the converse follows from the next lemma.

Lemma 9.1. Let g be a weighted homogeneous polynomial, and suppose that a germ $f \in \mathscr{F}$ is contact equivalent to g. Then f is right equivalent to g.