B. NINE CHARACTERIZATIONS OF SIMPLE CRITICAL POINTS

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Characterization A7′. $\lim_{r \to 0} r^{-2} \operatorname{vol}(X(r))$ is finite.

Let $\omega = dx \wedge dy \wedge dz$, and note that $\omega \wedge \bar{\omega}$ is $8/i$ times the volume form of $\mathbf{C}^3$. Characterizations A7 and A7′ are equivalent since

$$\lim_{r \to 0} \frac{1}{r^2} \operatorname{vol}(X(r)) = \lim_{r \to 0} \frac{i}{8r^2} \int_{X_r} \omega \wedge \bar{\omega}$$

$$= \lim_{r \to 0} \frac{i}{8r^2} \int_{\Delta(r)} \left(\int_{V_t} \omega_t \wedge \bar{\omega}_t \right) dt \wedge \overline{dt},$$

but since

$$\int_{\Delta(r)} \left(\frac{i}{2}\right) dt \wedge \overline{dt} = \operatorname{vol}(\Delta(r)) = 2\pi r^2 ,$$

the above limit equals

$$\frac{\pi}{2} \int_{V_0} \omega_0 \wedge \bar{\omega}_0 .$$

B. Nine characterizations of simple critical points

We switch our attention from the analytic set defined by the zero locus of an analytic function $f(x, y, z)$ to the function itself and the nature of its critical point. We also generalize to functions $f(z_0, \ldots, z_n)$ of an arbitrary number of variables. The characterizations in the following theorem will start in Section 9.

THEOREM B. *Let $f(z_0, \ldots, z_n)$ with $n \geqslant 1$ be the germ at the origin* $\mathbf{0}$ *of a complex analytic function, and suppose further that $f(\mathbf{0}) = 0$ and that* $\mathbf{0}$ *is an isolated critical point of f. Then Characterizations B1 through B9 are equivalent.*

8. The classification of right equivalence classes

Let $\mathcal{O}$ be the set of germs f at the origin $\mathbf{0}$ of complex analytic functions on $\mathbf{C}^{n+1}$. (In other words, $\mathcal{O}$ is just the ring $\mathbf{C}\{z_0, \ldots, z_n\}$ of convergent power series.) The ring $\mathcal{O}$ is local with maximal ideal

$$\mathfrak{m} = \{ f \in \mathcal{O} : f(\mathbf{0}) = 0 \} .$$

Let

$$\Delta f = \left(\frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n}\right)$$

be the ideal in $\mathcal{O}$ generated by the partial derivatives of f.

Lemma 8.1. A germ f in $\mathfrak{m}$ has an isolated critical point at $\mathbf{0}$ if and only if there is a k such that $\mathfrak{m}^k \subset \Delta f \subset \mathfrak{m}$.

Proof. The germ f has a critical point at $\mathbf{0}$ if and only if $f \in \mathfrak{m}^2$, or equivalently, $\Delta f \subset \mathfrak{m}$. If this critical point is isolated, then the origin is an isolated zero of the functions $\partial f/\partial z_0, \ldots, \partial f/\partial z_n$. This is equivalent to saying that the set of common zeros of all the functions in the ideal Δf equals the set of common zeros of the ideal $\mathfrak{m}$. By the Nullstellensatz, there exist integers $l_0, \ldots, l_n$ such that $z_i^{l_i} \in \Delta f$. Setting $k = (n+1) \max \{l_0, \ldots, l_n\}$ gives $\mathfrak{m}^k \subset \Delta f$. Conversely, if $\mathfrak{m}^k \subset \Delta f$ then the origin is an isolated critical point. This proves the lemma.

Let $\mathcal{F}$ be the set of all germs in $\mathcal{O}$ vanishing at the origin and with an isolated critical point there. (This is the set of *finitely-determined* germs.) The *Milnor number* of a germ $f \in \mathcal{F}$ is

$$\mu = \dim_{\mathbf{C}} \mathcal{O}/\Delta f.$$

For a comprehensive discussion of μ, see [Orlik 2]. There are many ways to compute this number, aside from the above formula [Milnor 1, §§7, 10; A'Campo 1; Laufer 6]. The *(right) codimension* of f is $\mu - 1$.

Two germs f and g in $\mathcal{O}$ are *right equivalent* (written $f \sim g$) if there is a germ h of a complex analytic automorphism of $\mathbf{C}^{n+1}$ fixing $\mathbf{0}$ with $f \circ h = g$. The germs f and g are *contact equivalent* if there is an h as above such that the ideal generated by $f \circ h$ in $\mathcal{O}$ is equal to the ideal generated by g. This is equivalent to saying that the analytic sets $f^{-1}(0)$ and $g^{-1}(0)$ are isomorphic. Note that right-equivalent germs are contact equivalent.

Mather, Arnold, and others have classified germs of low Milnor number under right equivalence. The implicit function theorem shows, for example, that if $f(\mathbf{0}) = 0$ but the derivative of f does not vanish at $\mathbf{0}$, then f is right equivalent to the projection $(z_0, \ldots, z_n) \mapsto z_0$. If $f(\mathbf{0}) = 0$ and f has a non-degenerate critical point at $\mathbf{0}$, then $f(z_0, \ldots, z_n) \sim z_0^2 + \ldots + z_n^2$ by the Morse lemma.

Recall that the *k-jet* of a germ f in $\mathcal{O}$ is its power series expansion up to degree k. A germ $f \in \mathcal{O}$ is *k-determined* if any germ with the same k-jet as f is right equivalent to f. In particular, f is right equivalent to its own k-jet. A germ is *finitely determined* if it is k-determined for some $k < \infty$.

The fundamental lemmas used in the classification are as follows:

Lemma 8.2. If $\mathfrak{m}^{k+1} \subset \mathfrak{m}^2 \Delta f$ then f is k-determined.

For the proof, see [Arnold 1, Lemma 3.2; Zeeman, Theorem 2.9; Siersma, p. 8]. Note that $\mathfrak{m}^{k-1} \subset \Delta f$ implies that $\mathfrak{m}^{k+1} \subset \mathfrak{m}^2 \Delta f$. The *corank* of f is defined as $n + 1$ minus the rank of the Hessian matrix $\{(\partial^2 f / \partial z_i \partial z_j)(\mathbf{0})\}$. The proof of part (a) of the following lemma may be found in [Arnold 1, Lemma 4.1; Siersma Lemma 3.2].

Splitting Lemma 8.3. (a) Let $f(z_0, ..., z_n) \in \mathcal{F}$ be of corank $r + 1$. Then there is a $g(z_0, ..., z_r) \in \mathfrak{m}^3$ such that

$$f(z_0, \ldots, z_n) \sim g(z_0, \ldots, z_r) + z_{r+1}^2 + \ldots + z_n^2 .$$

(b) Let $g(z_0, ..., z_r)$ and $g'(z_0, ..., z_r) \in \mathcal{F} \cap \mathfrak{m}^3$. If

$$g(z_0, \ldots, z_r) + z_{r+1}^2 + \ldots + z_n^2 \sim g'(z_0, \ldots, z_r) + z_{r+1}^2 + \ldots + z_n^2$$

then

$$g(z_0, \ldots, z_r) \sim g'(z_0, \ldots, z_r) .$$

The classification proceeds by low corank and low Milnor number. A germ of corank 0 is right equivalent to $z_0^2 + \ldots + z_n^2$, a germ of corank 1 and Milnor number $k > 1$ is right equivalent to $z_0^{k+1} + z_1^2 + \ldots + z_n^2$, and so forth. The proofs are not hard [Arnold 1, Zeeman, Siersma]. Table 2, for instance, includes all right-equivalence classes of germs with Milnor number $\mu \leqslant 9$.

9. Characterizations under right and contact equivalence

Characterization B1. The germ f is right equivalent to one of the germs in Table 2a.

Characterization B2. The germ f is contact equivalent to one of the germs in Table 2a.

When $n = 2$, Characterization B2 is the same as Characterization A1. Clearly Characterization B1 implies Characterization B2. Since all the germs in Table 2a are weighted homogeneous (§16), the converse follows from the next lemma.

Lemma 9.1. Let g be a weighted homogeneous polynomial, and suppose that a germ $f \in \mathcal{F}$ is contact equivalent to g. Then f is right equivalent to g.

Proof. To say that f is contact equivalent to g means that there is a germ of an analytic isomorphism $h: (\mathbf{C}^{n+1}, \mathbf{0}) \to (\mathbf{C}^{n+1}, \mathbf{0})$ and a function $u: \mathbf{C}^{n+1} \to \mathbf{C}$ with $u(\mathbf{0}) \neq 0$ such that $f = u \cdot (g \circ h)$. Let $h = (h^0, \ldots, h^n)$ be the components of h, and suppose that g is weighted homogeneous with weights $(w_0, \ldots, w_n)$. Then,

$$
\begin{aligned}
f(z_0, \ldots, z_n) &= u(z_0, \ldots, z_n) \cdot g\left(h^0(z_0, \ldots, z_n), \ldots, h^n(z_0, \ldots, z_n)\right) \\
&= g\left((u(z_0, \ldots, z_n))^{1/w_0}\ h^0(z_0, \ldots, z_n), \ldots,\right. \\
&\qquad \left.(u(z_0, \ldots, z_n))^{1/w_n}\ h^n(z_0, \ldots, z_n)\right).
\end{aligned}
$$

Hence f is right equivalent to g.

10. Degeneration

Let J_k be the set of k-jets of germs in $\mathcal{O}$. There is a projection of $\mathcal{O}$ to J_k by mapping germs to their power series expansion truncated in degree k. The ring $\mathcal{O}$ becomes a topological space by letting a basis of open sets be inverse images of open sets in J_k, for all k.

The group of germs of analytic automorphisms fixing $\mathbf{0}$ acts on $\mathcal{O}$, and the orbits of this action *(right-equivalence orbits)* are the right-equivalence classes. Similarly, there is a contact equivalence group which acts on $\mathcal{O}$, and the orbits of this action *(contact-equivalence orbits)* are the contact equivalence classes [Mather, §2]. A right-equivalence orbit is always contained in a contact-equivalence orbit; Lemma 9.1 says that the right-equivalence orbit of a germ f in Table 2a or b equals its contact-equivalence orbit.

A subset A of $\mathcal{O}$ is said to *right* (or *contact*) *degenerate* to a subset B of $\mathcal{O}$ if the closure of the right (or contact) equivalence orbit of A contains B. If A degenerates to B, then B *simplifies* to A (written $A \leftarrow B$). Right degeneracy is also called *adjacency*. For example, when $n = 0$, the germ z_0^k right degenerates to the germ z_0^l for $k < l$, since the one-parameter family $tz_0^k + (1-t)\, z_0^l$ is z_0^l when $t = 0$, and is right-equivalent to z_0^k when $t \neq 0$. All germs of low codimension can be arranged according to right degeneracy in fascinating tables [Arnold 3; Siersma]. Table 3 lists some (but not all) of the simplifications that occur. The following proposition is a principal consequence of the work on degeneration.

Proposition 10.1.

(i) *The germs in Table 2a right simplify only to each other.*

(ii) *The germs in Table 2b right simplify only to the germs in Table 2a.*

(iii) *The germs in Table 2c right simplify only to the germs in Table 2b and 2a.*

(iv) *A germ in $\mathscr{F}$ not right equivalent to a germ in Table 2a, b, or c right simplifies to a germ in Table 2c.*

11. Simple germs and moduli

A germ $f \in \mathfrak{m}$ is said to be *right* (or *contact*) *simple* if there is a neighborhood of f in $\mathfrak{m}$ intersecting only finitely many right (or contact) equivalence orbits. In the language of algebraic geometry, a germ f is contact simple if and only if the versal deformation of $f^{-1}(0)$ contains only finitely many isomorphism classes of analytic spaces.

The germs in Table 2a are right and contact simple by Proposition 10.1. The germs in Table 2b are not contact simple (and hence not right simple): $\tilde{E}_6$ is a family of cones over non-singular elliptic curves in $\mathbf{C}P^2$, $\tilde{E}_7$ is a family of four lines through the origin in $\mathbf{C}^2$, and $\tilde{E}_8$ is a family of three parabolas [Arnold 1; Siersma]. Note that the germs of Table 2c form one-dimensional families under right equivalence, but all members of the family are contact equivalent [Laufer 4; Siersma p. 54]. Clearly if a germ g right simplifies to f and f is not right simple, then g is not right simple; the same applies to contact equivalence.

Characterization B3. The germ f is right simple.

Characterization B4. The germ f is contact simple.

The equivalence of Characterizations B1 and B3 follows from Proposition 10.1 and the above remarks [Arnold 1]. Characterization B3 implies Characterization B4 by definition. Conversely, a contact simple germ f which is not right simple right simplifies to a germ in Table 2b (by Proposition 10.1), but these are not contact simple. Hence f must be right simple.

The classification of simple germs has recently been extended to complete intersections [Giusti]. The *modality* of a germ f is defined in [Arnold 3]. A right-simple germ is zero-modal; all right equivalence classes of 1 and 2-modal germs have been listed [Arnold 2, 3, 5]. Moduli of resolutions of two-dimensional normal singularities are studied in [Laufer 3, 4]. The following result provides a connection between Characterizations A2 and B3.

Theorem 11.1 [Randell]. *For almost all germs $f(x, y, z)$ (in the sense of the Newton diagram), the geometric genus p of $f^{-1}(0)$ is less than or equal to the modality of f.*

12. The quadratic form

Let $f(z_0, \ldots, z_n)$ be a germ with $f(\mathbf{0}) = 0$ and an isolated critical point at $\mathbf{0}$ (that is, a germ in $\mathscr{F}$). There is an $\varepsilon > 0$ such that $f^{-1}(0)$ intersects all spheres of radius ε' about $\mathbf{0}$ transversally for $0 < \varepsilon' \leqslant \varepsilon$. For suitably small $\delta > 0$, $f^{-1}(\delta')$ intersects the closed disk D_ε^{2n+2} of radius ε transversally for all $|\delta'| \leqslant \delta$. Let

$$F = f^{-1}(\delta) \cap D_\varepsilon^{2n+2}$$

be the *Milnor fiber* of f [Milnor 1]. The set F is a smooth real $2n$-manifold with boundary whose diffeomorphism type is independent of the choice of ε and δ. Furthermore, F is $(n-1)$-connected, and the Milnor number μ of §7 is the rank of $H_n(F)$. The Milnor number is zero if and only if the germ f has a regular point at $\mathbf{0}$ [Milnor 1, Corollary 7.3]. The *intersection pairing* (,) of F is the integral bilinear form $H_n(F) \times H_n(F) \to \mathbf{Z}$ defined by sending (x, y) to $(x' \cup y')[F]$, where x' and y' in $H^n(F, \partial F)$ are Lefschetz duals to x and y, and $[F]$ in $H_{2n}(F, \partial F)$ is the orientation class of F given by the underlying complex structure. The intersection pairing is symmetric if n is even, and skew symmetric if n is odd. For example, the germ $f(z_0, \ldots, z_n) = z_0^2 + \ldots + z_n^2$ has $H_n(F)$ a free cyclic group with generator e, and $(e, e) = 2(-1)^{n/2}$ or 0 according as n is even or odd. There are many methods of computing the intersection pairing in special cases.

By a tensor product theorem [Gabrielov 1; Sakamoto], the Milnor numbers of $f(z_0, \ldots, z_n)$ and $f(z_0, \ldots, z_n) + z_{n+1}^2 + \ldots + z_m^2$ are equal. The *quadratic form* of $f(z_0, \ldots, z_n)$ is defined to be the intersection pairing of the germ $f(z_0, \ldots, z_n) + z_{n+1}^2 + \ldots + z_m^2$ where $m \equiv 2 \pmod 4$. This is independent of the choice of m. For example, if $n \equiv 0 \pmod 4$ then the quadratic form of f is the negative of its intersection pairing; all this follows from the tensor product theorem. See also [Kauffmann and Neumann].

A germ f *topologically degenerates* to a germ g if there is an $\eta > 0$ and a family h_t of germs for $\{t \in \mathbf{C}: |t| < 2\eta\}$ with $h_\eta \sim f$, $h_0 \sim g$, and h_t of constant Milnor number for $t \neq 0$. Compare [Lê and Ramanujam]. Clearly right degeneracy implies topological degeneracy.

Lemma 12.1 [Tjurina 1, Theorem 1]. If f topologically degenerates to g, then there is an injection of $H_n(F_f)$ into $H_n(F_g)$ (where F_f is the Milnor fiber of f, and F_g is the Milnor fiber of g), and this injection preserves the intersection pairing. In particular, if g topologically degenerates to f as well, then the intersection pairings of f and g are isomorphic.

Characterization B5. The quadratic form of f is negative definite.

The equivalence of Characterizations B1 and B5 is proved in [Tjurina 1]. By explicit computation the quadratic forms of the germs in Table 2a are shown to be negative definite, and those of Table 2b are shown to be negative semi-definite. (In fact, the quadratic form of a germ in Table 2a is isomorphic to the intersection pairing of its minimal resolution, and the quadratic form of a germ of type $\tilde{E}_k$ in Table 2b is isomorphic to the quadratic form of E_k plus a two-dimensional zero form.) The result then follows from Proposition 10.1 and Lemma 12.1. When $n = 2$, the Milnor fiber F is in fact diffeomorphic to the minimal resolution M of $f^{-1}(0)$, since the singularity of $f^{-1}(0)$ is an absolutely isolated double point [Brieskorn 1, Theorem 4; Tjurina 1, Lemma 1].

When $n = 2$, the equivalence of Characterizations A2 and B5 follows from the following result [Durfee 2, Proposition 3.1].

THEOREM 12.2. *Twice the geometric genus* p *of* $f^{-1}(0)$ *equals the number of positive plus the number of zero diagonal elements in a diagonalization of the intersection pairing over the real numbers.*

The classification of germs according to signature of the quadratic form has been extended in [Arnold 3]; see also [Durfee 2, Proposition 3.3].

13. NEARBY MORSE FUNCTIONS

A *deformation* of a germ $f \in \mathscr{F}$ is a germ $g: \mathbf{C}^{n+1} \times \mathbf{C} \to \mathbf{C}$ with $g(z, 0) = f(z)$. Choose ε and δ for f as in §11. Then choose $\eta > 0$ such that for all $|t| < \eta$ and $|\delta'| \leqslant \delta$, the set $\{z \in \mathbf{C}^{n+1} : g(z, t) = \delta'\}$ intersects S_ε^{2n+1} transversally and the critical values of $g(z, t)$ for fixed t are less than δ in absolute value. A germ $\bar{f}$ is a *nearby Morse function* to f if $\bar{f}$ has only non-degenerate critical points in D_ε^{2n+2} and there is a deformation g and a t_0 with $|t_0| < \eta$ such that $\bar{f}(z) = g(z, t_0)$.

Characterization B6. There is a nearby Morse function to f with one or two critical values.

In fact, the nearby Morse function has one critical value if and only if f is right equivalent to A_2, since the quadratic form diagram is connected (§14). This surprising characterization is in [A'Campo 2II], where it is shown that Characterization B1 implies B6, and B6 implies B7 below.

14. Vanishing cycles

Let f be a germ in $\mathscr{F}$, and let $\bar{f}$ be a nearby Morse function with μ distinct critical values $t_1, ..., t_u$ in the disk D_δ^2 of radius δ about 0 in **C**. A path α_i in $D_\delta^2 - \{t_1, ..., t_u\}$ from δ to t_i determines (up to sign) a *vanishing cycle* δ_i in $H_n(F)$. The self-intersection (δ_i, δ_i) is $2(-1)^{n/2}$ or 0 according as n is even or odd. Choose paths $\alpha_1, ..., \alpha_\mu$ in $D_\delta^2 - \{t_1, ..., t_\mu\}$ from δ to $t_1, ..., t_\mu$ respectively, such that the union of the images of the paths is a deformation retract of D_δ^2; then the corresponding vanishing cycles $\delta_1, ..., \delta_\mu$ are a basis of $H_n(F)$ [Brieskorn 4, Appendix]. The basis $\delta_1, ..., \delta_\mu$ is called an *ordered* (or *distinguished*) *basis of vanishing cycles* if $t_1, ..., t_\mu$ are ordered so that the loop going once counter-clockwise around the boundary of D_δ^2 is homotopic in $\pi_1(D_\delta^2 - \{t_1, ..., t_\mu\}, \delta)$ to the product $\beta_1 * ... * \beta_\mu$, where β_i is the loop going out α_i almost to t_i, around t_i counter-clockwise, and back along α_i. References for this are [Gabrielov 1, Lamotke, Durfee 1].

Choose an ordered basis of vanishing cycles $\delta_1, ..., \delta_\mu$ for the intersection pairing (,) of $f(z_0, ..., z_n) + z_{n+1}^2 + ... + z_m^2$, where $m \equiv 2 \pmod 4$ The *quadratic form diagram* of f with respect to the basis $\delta_1, ..., \delta_\mu$ has vertices $v_1, ..., v_\mu$ and edges from v_i to v_j if $(\delta_i, \delta_j) \neq 0$, weighted by (δ_i, δ_j) if $(\delta_i, \delta_j) \neq 1$. This diagram is connected [Lazzeri; Gabrielov 2]. It determines all the topological information in the singularity if $n \neq 2$ [Durfee 1]. There are a number of methods of computing these diagrams [A'Campo 2I; Gabrielov 3; Gusein-Zade]. The quadratic form diagrams of the germs of Table 2 are listed in column 5. Lemma 12.1 can be strengthened to show that if f topologically degenerates to g, then some quadratic form diagram for f is a subdiagram of some quadratic form diagram for g [Siersma, p. 82].

Characterization B7. There is an ordered basis of vanishing cycles for f such that the quadratic form diagram is a (weighted) tree.

It is shown in [A'Campo 2II] that Characterizations B1 and B7 are equivalent. In fact, the quadratic form diagrams for the germs in Table 2a are the same as the graph of their minimal resolutions (column 3 of Table 1).

15. The monodromy group

Let f be a germ in $\mathscr{F}$, and as above choose an ordered basis $\delta_1, ..., \delta_\mu$ of vanishing cycles for $H_m(F)$, where F is the Milnor fiber of

$$f(z_0, ..., z_n) + z_{n+1}^2 + ... + z_m^2$$

with $m \equiv 2 \pmod 4$. The *Picard-Lefshetz automorphisms* T_i of $H_m(F)$ for $i = 1, ..., \mu$ are defined by

$$T_i(x) = x + (\delta_i, x)\,\delta_i\,.$$

Another way of writing T_i is

$$T_i(x) = x - 2\frac{(\delta_i, x)}{(\delta_i, \delta_i)}\,\delta_i$$

which shows that T_i is a reflection in δ_i [Lamotke].

The *monodromy group* of f is the subgroup of the automorphism group of $H_m(F)$ generated by $T_1, ..., T_\mu$. This group depends only on f, since it may also be defined as a representation of the *braid group* of f, which is the fundamental group of the complement of the bifurcation diagram in the base space of the versal unfolding of f [Arnold 3, §2.8]. (Here is a direct proof that the monodromy group of f is independent of the choice of nearby Morse function $\bar{f}$ and paths $\alpha_1, ..., \alpha_\mu$: The set $D_\delta^2 - \{t_1, ..., t_\mu\}$ is the base space of a fiber bundle with fiber F, so $\pi_1(D_\delta^2 - \{t_1, ..., t_\mu\}, \delta)$ acts on $H_m(F)$. The image of β_i in Aut $H_m(F)$ is T_i; since $\beta_1, ..., \beta_\mu$ generate π_1, the monodromy group is the image of π_1 in Aut $H_m(F)$. Thus the monodromy group is independent of the choice of $\alpha_1, ..., \alpha_\mu$. It is independent of the choice of $\bar{f}$ since any two nearby Morse functions with μ distinct critical values can be joined by a family of such functions.)

Characterization B8. The monodromy group of f is finite.

Characterization B5 implies Characterization B8 since the automorphism group of any positive definite integral lattice is finite. In fact, the monodromy groups are precisely the Coxeter groups of the corresponding quadratic form diagram. Conversely, [Gabrielov 3] shows that if a germ f topologically degenerates to a germ g, then the monodromy group of f is a quotient of a subgroup of the monodromy group of g. Since the monodromy groups of the germs in Table 2b are infinite [Gabrielov 1], Proposition 10.1 shows that Characterization B8 implies Characterization B1.

16. Weighted homogeneous polynomials

A polynomial $g(z_0, ..., z_n)$ is *weighted homogeneous* if there are positive rational numbers $w_0, ..., w_n$ (the *weights*) such that $g(z_0, ..., z_n)$ may be written as a sum of monomials $z_0^{i_0} \dots z_n^{i_n}$ with $i_0/w_0 + \dots + i_n/w_n = 1$

[Milnor 1, p. 75; Orlik and Wagreich]. Another way of saying this is that if the variables z_i are weighted by $1/w_i$, then g is homogeneous of degree one, that is, $g(\lambda^{1/w_0}z_0, \ldots, \lambda^{1/w_n}z_n) = \lambda g(z_0, \ldots, z_n)$ for all complex numbers λ. All the germs in Table 1 are weighted homogeneous with weights as listed in Column 7. These germs remain weighted homogeneous upon adding squares of new variables, each weighted by 2. It is proved in [Saito 1, Lemma 4.3] that the weights of a germ g are uniquely determined (up to permutation) by the analytic isomorphism type of $g^{-1}(0)$.

Characterization B9. The germ $f^{-1}(0)$ is isomorphic to $g^{-1}(0)$, where g is a weighted homogeneous polynomial with weights w_i satisfying $w_0^{-1} + \ldots + w_n^{-1} > n/2$.

The equivalence of Characterizations B2 and B9 is proved in [Saito 2, Satz 2.11]. (The r there is $w_0^{-1} + \ldots + w_n^{-1}$.)

APPENDIX I

NINE CHARACTERIZATIONS OF ALMOST-SIMPLE CRITICAL POINTS (SIMPLE ELLIPTIC SINGULARITIES)

Almost-simple critical points can also be characterized in several ways. The nine characterizations presented in this appendix are analogues of some of those in the main text.

THEOREM C. *Let $f(z_0, \ldots, z_n)$ with $n \geqslant 2$ be the germ at the origin* **0** *of a complex analytic function, and suppose further that $f(\mathbf{0}) = 0$ and that* **0** *is an isolated critical point. Then Characterizations C1 through C9 are equivalent.*

Characterization C1. The germ f is right equivalent to one of the germs in Table 2b.

Characterization C2. The germ f is contact equivalent to one of the germs in Table 2b.

The equivalence of these characterizations follows from Proposition 9.1.