§6. ACYCLIC MAPS INTO A GIVEN SPACE

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We prove $\tilde{X} \to F$ is a homotopy equivalence with the same argument used in (5.6) to show P_k implies H_k. Since F is also the fibre of $X_N^+ \to [B\pi_1(X)]_N^+$ we have proved the theorem.

(5.8) *Remark.* Using (5.1), we see that for an acyclic map $f : X \to Y$ which is k-simple for all $k \geqq 2$, the homotopy groups $\pi_*(Y)$ can be computed in terms of $\pi_*(X)$ and $\pi_*\left(B\pi_1(X)_N^+\right) \cong \pi_*(BN)^+$ for $i \geqq 2$. Some computations of $\pi_*(BN^+)$ for a certain perfect group N can be found for instance in [H, Chapter 7].

§ 6. Acyclic maps into a given space

In this section we study acyclic maps $f: X \to Y$ into a fixed space Y. Two such map $f: X \to Y$ and $f' : X' \to Y$ are called equivalent provided there is a homotopy equivalence $h : X \to X'$ with $f \simeq f'h$. Let $AC(Y)$ denote the class of equivalence classes of acyclic $f: X \to Y$ over Y where X and Y are CW-spaces.

(6.1) Definition. *An extension data over a space Y is a triple (Φ, i, Φ) where*

(a) *Φ is an extension $1 \to N \to G \to \pi_1(Y) \to 1$ with N perfect,*

(b) *$i : BG \to BG_N^+$ is an acyclic map with* $\ker(\pi_1(i)) = N$ *(whose equivalence class is well defined by* (3.5)), *and*

(c) *$\phi : Y \to BG_N^+$ is a 2-connected map.*

Two triples of extension data (Φ, i, ϕ) and (Φ', i', ϕ') are called equivalent provided there exists an isomorphism $g : G \to G'$ making the following diagrams commutative (up to homotopy for the second one).

$$\begin{tikzcd} G \arrow[rr, "g"] \arrow[dr] & & G' \arrow[dl] \\ & \pi_1(Y) & \end{tikzcd} \qquad \begin{tikzcd} BG \arrow[rr, "Bg"] \arrow[d, "i"'] & & BG' \arrow[d, "i'"] \\ BG_N^+ \arrow[rr, "Bg+"] & & B(G')_N^+ \\ & Y \arrow[ul, "\phi"] \arrow[ur, "\phi'"'] & \end{tikzcd}$$

where $N' = g(N)$ and Bg^+ is the unique homotopy equivalence determined by g with (3.1).

We denote by $ED(Y)$ the class of equivalence classes of extension data.

(6.2) Definition. *The data map ρ is the function $\rho : AC(Y) \to ED(Y)$ which assigns to an acyclic map $f : X \to Y$ the class $\rho(f) = (\Phi, i, \phi)$ of extension data defined as follows:*

(a) *Φ is the extension* $1 \to \ker \pi_1(f) \to \pi_1(X) \to \pi_1(Y) \to 1$.

(b) (c) *With the well defined $j : X \to BG$ for $G = \pi_1(X)$ we form the cocartesian diagram*

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "f"'] & BG \arrow[d, "i"] \\ Y \arrow[r, "\phi"] & Y \cup_X BG \end{tikzcd}$$

Since f is acyclic, i is acyclic, and since $\pi_1(j)$ is an isomorphism, $\ker(\pi_1(i)) = N$. Thus $Y \cup_X BG$ is BG_N^+ up to equivalence.

Now we have to check that the map $\phi : Y \to Y \cup_X BG = BG_N^+$ is 2-connected. Since $\pi_1(j)$ is an isomorphism, $\pi_1(\phi)$ is also an isomorphism. The fact that $\pi_2(\phi)$ is surjective comes from the diagram.

$$\begin{tikzcd} \pi_2(Y) \arrow[d, "\pi_2(\phi)"'] & \pi_2(\tilde{Y}) \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "\pi_2(\tilde{\phi})"] & H_2(\tilde{Y}) \arrow[d] & H_2(\tilde{X}_N) \arrow[l, "\sim"'] \arrow[dl] \\ \pi_2(BG_N^+) & \pi_2(BN^+) \arrow[l, "\sim"'] \arrow[r, "\sim"] & H_2(N) & \end{tikzcd}$$

The surjectivity on the right is a classical result of Hopf which follows easily from the Serre spectral sequence of the fibration $\tilde{X} \to \tilde{X}_N \to BN$.

Now using (2.5) a simple argument, left to the reader, shows that $\rho : AC(Y) \to ED(Y)$ is well defined.

(6.3) Theorem. *Let Y be a CW-space. The map $\rho : AC(Y) \to ED(Y)$ surjective and its restriction to the subclass $AC_S(Y)$ of $AC(Y)$ of $f : X \to Y$ which are k-simple for all $k \geqq 2$ is a bijection.*

Proof. To show ρ is surjective, consider extension data (Φ, i, ϕ) and form the cartesian square

$$\begin{tikzcd} X = Y \times_T BG \arrow[r, "\alpha"] \arrow[d, "f"'] & BG \arrow[d, "i"] \\ Y \arrow[r, "\phi"] & BG_N^+ = T \end{tikzcd}$$

Now f is acyclic by (2.2), and since its fiber is the same as i, we deduce by (5.2) that f is k-simple for all $k \geqq 2$.

Next, let $\rho(f) = (\Phi_0, i_0, \phi_0)$ and we show this extension data is equivalent to (Φ, i, ϕ). Using the homotopy exact sequences for $X \to Y$ and $BG \to BG_N^+$ and the fact that ϕ is 2-connected, we deduce from the five lemma that $\pi_1(\alpha) : \pi_1(X) \to G$ is an isomorphism. The following diagram shows that (Φ_0, i_0, ϕ_0) is equivalent to (Φ, i, ϕ) and ρ is surjective.

$$\begin{tikzcd} & & B\pi_1(X) \arrow[dl, "B\pi_1(\alpha)"'] \arrow[ddd, "i_0"] \\ X \arrow[urr, "j"] \arrow[r, "\alpha"'] \arrow[d, "f"'] & BG \arrow[d, "i"] & \\ Y \arrow[r, "\phi"] \arrow[drr, "\phi_0"'] & BG_N^+ & \\ & & Y \cup_X B\pi_1(X) \arrow[ul, "B\pi_1(\alpha)^+"'] \end{tikzcd}$$

Now, if $f : X \to Y$ is an acyclic map which is k-simple for all $k \geqq 2$ and with $\rho(f) = (\Phi, i, \phi)$, then we form the following commutative diagram.

$$\begin{tikzcd} X \arrow[r, "d"] \arrow[dr, "f"'] & Y \times_T BG \arrow[r] \arrow[d, "f_0"] & BG \arrow[d, "i"] \\ & Y \arrow[r, "\phi"] & BG_N^+ \end{tikzcd} \qquad (G = \pi_1(X))$$

As we have seen in the proof the surjectivity of ρ, the map f_0 is acyclic and k-simple for $k \geqq 2$. The map d induces an isomorphism on the fundamental groups and on homology with $\mathbf{Z}\pi_1(Y)$ twisted coefficients. By (5.3), the map d is a homotopy equivalence. This proves that the acyclic map f is equivalent to the induced map f_0. Thus ρ restricted to $AC_S(U) \to ED(Y)$ is a bijection.

(6.4) *Remark.* This theorem leaves open the question of the fibres of the function.

$$\rho : AC(Y) \to ED(Y).$$

In the next theorem we factor an acyclic map by ones having simplicity properties.

(6.5) *Remark.* In theorem (6.3), if one fixes an extension $\Phi : 1 \to N \to G \to \pi_1(Y) \to 1$, then the same proof permits us to classify acyclic maps $f : X \to Y$ which are k-simple for $k > 2$ together with an identification $d : \pi_1(X) \to G$ such that $\Phi d = \pi_1(f)$. The objects of $ED(Y)$ have to be replaced by couples (i, ϕ) where $i : BG \to BG_N^+$ is as above and $\phi : Y \to BG_N^+$ is 2-connected with the following diagram commuting up to homotopy.

$$\begin{tikzcd}
& B\pi_1(Y) \arrow[rr, "B\Phi"] \arrow[dr, "B\Phi^+"] & & BG \arrow[dl, "i"] \\
Y \arrow[ur] \arrow[rr, "\phi"] & & BG_N^+ &
\end{tikzcd}$$

This is what is done implicitely in [H, Sections 2 and 4]. Observe that we are dealing here with classes which are sets.

(6.6) Lemma. *Let X be a CW-space and N a perfect normal subgroup of $\pi_1(X)$. Let $X \to P_n X$ denote the nth stage of the Postnikov decomposition of X. Then for all $n \geqq 1$ we have that*

(1) $\pi_j(X_N^+) \to \pi_j\left((P_n X)_N^+\right)$ *is an isomorphism for $j \leqq n$ and an epimorphism for $j = n + 1$, and*

(2) $\pi_j(A\tilde{X}_N) \to \pi_j\left(A(P_n\tilde{X}_N)\right)$ *is an isomorphism for $j \leqq n$ and an epimorphism for $j = n + 1$.*

Proof. Consider the following homotopy commutative diagram of fibre sequences

$$\begin{tikzcd}
T \arrow[r] \arrow[d] & A\tilde{X}_N \arrow[r] \arrow[d] & A(P_n\tilde{X}) \arrow[d] \\
F \arrow[r] \arrow[d] & \tilde{X}_N \arrow[r] \arrow[d] & P_n\tilde{X}_N \arrow[d] \\
G \arrow[r] & (\tilde{X}_N)^+ \arrow[r] & (P_n\tilde{X}_N)^+ .
\end{tikzcd}$$

Clearly $\pi_i(F) = 0$ for $i \leqq n+1$. The spaces $\tilde{X}_N$ and $P_n\tilde{X}_N$ have the same $(n+1)$-skeleton and the same can be assumed for $\tilde{X}_N^+$ and $(P_n\tilde{X}_N)^+$. Hence $\pi_i(G) = 0$ for $i \leqq n+1$. Now (1) follows because G is the fibre of $X_N^+ \to (P_nX)^+$.

By comparing Serre spectral sequences, we obtain the surjectivity of

$$H_0(N, H_{n+1}(F)) \to H_0(N, H_{n+1}(G)) = H_{n+1}(G) = \pi_{n+1}(G)\,.$$

Thus $\pi_j(T) = 0$ for $j \leqq n$ and (2) follows.

(6.7) THEOREM. *Let $f: X \to Y$ be a map between CW-spaces. Then there is a factorization*

$$\begin{tikzcd}
 & \vdots \arrow[d] & \\
 & X_2 \arrow[d, "j_1"] \arrow[ddr, "\alpha_2"] & \\
 & X_1 \arrow[d, "j_0"] \arrow[dr, "\alpha_1"] & \\
X \arrow[uur, "\beta_2"] \arrow[ur, "\beta_1"] \arrow[r, "\beta_0"] \arrow[rr, bend right, "f"'] & X_0 \arrow[r, "\mathrm{id}"] & Y
\end{tikzcd}$$

such that β_i is i-connected and α_i is an acyclic map which is k-simple for $k > i$.

Such a decomposition is unique up to a homotopy equivalence.

Proof. The ith stage X_i is defined by the cartesian diagram

$$\begin{tikzcd}
Y \times_T P_i(X) \arrow[r] \arrow[d] & P_iX \arrow[d] \\
Y \arrow[r] & (P_iX)_N^+ = T
\end{tikzcd}$$

where $N = \ker(\pi_1(X) \to \pi_1(Y))$. By (6.6) the map β_i is i-connected since the fiber of the two vertical arrows is $A(P_n\tilde{X})_N$. Now by (5.4) we see that α_i is simple for $k > i$.

For two decompositions (X'_i) and (X''_i) of $f: X \to Y$ satisfying the above conditions, we have $P_i X'_i = P_i X''_i$ and both X'_i and X''_i map into X_i, constructed above, such that the resulting diagrams are homotopy commutative. The connectivity of the β_i and (5.1) shows that these maps are all homotopy equivalences. This proves the theorem.

(6.8) *Remarks.* This theorem (6.7) coincides with the Dror results for Y a point [D1, Theorem 1.3] and $Y = S^n$ [D2]. An interesting problem is to describe the ith stage X_i in terms of invariants of X_{i-1} as in [D1] and [D2]. (See the footnote in the introduction.)

APPENDIX — SIMPLICITY PROPERTIES OF FIBERS

In the proof of (5.4) we used the fact that for a fibration $F \to E \xrightarrow{f} B$ the action of $\pi_1(F)$ on $\mathrm{Im}(\partial : \pi_{k+1}(B) \to \pi_k(F))$ is trivial. This assertion does not seem to be in the literature so we include a proof here.

We extend the mapping sequence of the fibration f to $\Omega B \to F \to E \xrightarrow{f} B$ and study F as the total space of a principal fibration with fibre the H-space ΩB. If G is an H-space, then $\pi_1(G)$ acts trivally on $\pi_*(G)$ because the covering transformations $\tilde{G} \to G$ on the universal covering $\tilde{G}$ of G are homotopic to the identity. This is proved by lifting a loop to a path in $\tilde{G}$ and using the H-space structure on $\tilde{G}$ to deform the identity along this path to the covering transformation defined by the homotopy class of the loop. Recall that a principal fibration is induced from $G \to E_G \to B_G$ up to fibre homotopy equivalence.

(A.1) PROPOSITION. *Let* $G \to X \xrightarrow{\pi} Y$ *be a principal fibration with fibre* G *acting on* X. *Then we have:*

(a) $\mathrm{im}(\pi_1(G) \to \pi_1(X))$ *acts trivially on* $\pi_*(X)$, *and*

(b) $\pi_1(X)$ *acts trivially on* $\mathrm{im}(\pi_*(G) \to \pi_*(X))$.

Proof. For (a) we have the following commutative diagram induced by a covering transformation $T: \tilde{G} \to \tilde{G}$.