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Autor(en): **Singmaster, David**

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ON THE CONCEPT OF DIRECTED DISTANCE

by David SINGMASTER

On the real line, there are concepts of distance and of directed distance. The concept of distance has been generalized to the idea of a metric space. In this note, I shall generalize the concept of directed distance and shall show that the concept is interesting but not fruitful. In particular, it will be shown that any " real directed distance space " is " isometric " with a subset of the real line.

I shall consider a general situation where the values of the directed distance lie in any group, not just the reals. Most proofs will be omitted, being obvious.

Definition 1. Let G be an additively written group (not necessarily commutative). A pair (X, d) is called a G-structure if X is a non-empty set and $d: X \times X \to G$ is a function.

Definition 2. A G-structure (X, d) is called a G-directed distance space if the following axioms hold for any $x, y, z \in X$.

1. $d(x, x) = 0$.
2. $d(x, y) = 0 \Rightarrow x = y$.
3. $d(x, y) = -d(y, x)$.
4. $d(x, z) = d(x, y) + d(y, z)$.

Remarks. It may be possible to give other definitions of directed distance, but I feel that these are the obvious axioms for the concept. Axiom 4 might be called the " Triangle Equality ".

Proposition 3. The following relations hold among the axioms:

A. $4 \Rightarrow 1$.

B. $4 \Rightarrow 3$.

C. If the group has no elements of order 2, then $3 \Rightarrow 1$.

Proof: A: $d(x, x)=d(x, x)+d(x, x) \Rightarrow d(x, x)=0$.

B: Using that $4 \Rightarrow 1$, we have $0=d(x, x)=d(x, y)+d(y, x)$, hence $d(x, y)=-d(y, x)$.

C: $d(x, x)=-d(x, x) \Rightarrow 2\, d(x, x)=0 \Rightarrow d(x, x)=0$, by the hypothesis.

Remark. Some authors use the following form of the triangle inequality in defining a metric space: $d(x, z) \leqq d(x, y)+d(x, z)$. This is stronger than the triangle inequality. However, in the present situation we have that axiom 4 is equivalent to:

5. $d(x, z)=d(x, y)-d(x, z)$,

which is the analogous statement for directed distance.

Proposition 4. Except as indicated in Proposition 3, none of the axioms is implied by the others. (That is, the axioms are independent up to the implications of Proposition 2.)

Proof: We exhibit examples for each situation.

A: -1, 2, 3, -4 (Note that $-1 \Rightarrow -4$. -1 indicates the negation of 1.): Let X be any non-empty set and set $G=Z_2$ (the integers (mod 2)). Set $d(x, y)=1$ for all x, y.

B: 1, -2, 3, 4: Let X be any set of more than one element and let G be any group. Set $d(x, y)=0$ for all x, y.

C: 1, 2, -3, -4 (note that $-3 \Rightarrow -4$.): Let $X=G=R$ (the reals) or Z (the integers). Set $d(x, y)=(x-y)^2$ or $|x-y|$.

D: 1, 2, 3, -4: Let $X=G=R$ or Z. Set

$$d(x, y)=\begin{cases} +1 \text{ if } x>y, \\ 0 \text{ if } x=y, \\ -1 \text{ if } x<y. \end{cases}$$

Remark. Proposition 4 does not rule out all possible relations among the axioms other than those given by Proposition 3. It is conceivable that some relation such as $1 \Rightarrow$ (3 or 4) may hold. To show that no such relations hold, it is necessary and sufficient to exhibit 10 examples of G-structures satisfying the combinations of axioms which are consistent with Proposition 3. (There are, a priori, $16=2^4$ combinations, of which 6 are inconsistent with Proposition 2.) These examples can be easily constructed by trial and error or by use of the following:

Proposition 5. Let (X, d) be a G-structure and let (Y, e) be an H-structure. Then $(X\times Y, d\times e)$ is a $G\times H$-structure, where $(d\times e)\left((x_1, y_1), (x_2, y_2)\right) = \left(d(x_1, x_2), e(y_1, y_2)\right)$. Further Axiom i, $i = 1, 3, 4, 5$ holds in $X\times Y$ if and only if it holds in both X and Y. Axiom 2 holds nonvacuously in $X\times Y$ if and only if it holds nonvacuously in X and Y. It holds vacuously in $X\times Y$ if and only if it holds vacuously in X or Y.

Proposition 6. Any subset X of a group G is a G-directed distance space under either $g(x, y)=x-y$ or $h(x, y)=y-x$.

Definition 7. A pseudo-G-directed distance space is a G-structure satisfying Axiom 4, hence also Axioms 1 and 3.

Definition 8. Let (X, d) and (Y, e) be G-structures. A function f: $X\Rightarrow Y$ is a G-isometry if $d(x, y)=e(f(x), f(y))$. If $d(x, y)=e(f(y), f(x))$, we call f a G-anti-isometry.

Proposition 9. Let f: $(X, d)\Rightarrow(Y, e)$ be a G-isometry or a G-anti-isometry. Suppose (X, d) satisfies Axiom 2 and (Y, e) satisfies Axiom 1. Then f is one to one.

Proof: $f(x)=f(y) \Rightarrow e(f(x), f(y))=0=d(x, y) \Rightarrow x=y$.

Corollary 10. An isometry or anti-isometry from a G-directed distance space to a pseudo-G-directed distance space is one to one.

Proposition 11. Let (X, d) be a G-structure. Set $e(x, y)=d(y, x)$. Then (X, e) is a G-structure and the identity map $I(x)=x$ is an anti-isometry of (X, d) onto (X, e). Further, Axiom i, $i=1, 2, 3, 4, 5$, holds in (X, e) if and only if it holds in (X, d).

Corollary 12. If (X, d) is a G-directed distance space then I is an anti-isometry of (X, d) with $(X, -d)$.

Remark. Isometries and anti-isometries, under composition, behave like positive and negative under mutliplication. E.g. a composition of an isometry and an anti-isometry is an anti-isometry.

Remark. In analogy with pseudo-metric spaces, we have that the relation R determined by xRy *iff* $d(x, y)=0$ is an equivalence relation on any pseudo-G-directed distance space and the mapping $f(x)=[x]$ (the equivalence class of x) is an isometry onto the G-directed distance space of equivalence classes with the induced directed distance. If this mapping is one to one, then the pseudo-G-directed distance space must have been G-directed to start. Hence, we have the following converse of Proposition 9 and Corollary 10.

Proposition 13. If every isometry of the pseudo-G-directed distance space (X, d) to a pseudo-G-directed distance space is one to one, then (X, d) is a G-directed distance space.

Theorem 14. Let (X, d) be a pseudo-G-directed distance space. For each $a \varepsilon X$, the mapping f_a: $X \Rightarrow G$ given by $f_a(x)=d(x, a)$ is an isometry of (X, d) into (G, g), where $g(x, y)=x-y$.

Proof: $g(f_a(x), f_a(y))=f_a(x)-f_a(y)=d(x, a)-d(y, a)=d(x, a)+d(a, y)=$ $=d(x, y)$.

Corollary 15. If (X, d) is a G-directed distance space, then f_a is an injection (or embedding) of (X, d) into (G, g).

Remarks. Hence, the spaces described in Proposition 6 are all the G-directed distance spaces. In particular, a real directed distance space is essentially a subset of the reals.

Note that all the properties of a G-directed distance space must be used to obtain Corollary 15. We have actually shown the equivalence of the statements: "X is a G-directed distance space" *and* "X is a subset of G".

If, in Theorem 14, we replace $f_a(x)=d(x, a)$ by $f_a(x)=d(a, x)$ or $g(x, y)=x-y$ by $h(x, y)=y-x$, but not both, then we get an anti-isometry. If we make both replacements, we again get an isometry.

Proposition 16. Let f: $G \rightarrow H$ be a group homorphism. Then any G-structure (X, d) is also an H-structure (X, e) where $e(x, y)=f(d(x, y))$.

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Further, Axiom, $i=1, 3, 4, 5$, holds in (X, e) if it holds in (X, d). If Axiom 2 holds in (X, d) *and* f is one to one, the Axiom 2 holds in (X, e).

Corollary 17. If (X, d) is a pseudo-directed distance space, then so is (X, e). If (X, d) is a directed distance space and f is one to one, then (X, e) is also a directed distance space.

Remarks. In retrospect, I notice that the associative law of the group G has never been used. Hence, the role of G could be played by a somewhat weaker concept, namely that of a loop with the inverse property. [1, p. 7].

I am indebted to David Makinson for the idea of Proposition 5.

Since first writing this paper, it has been pointed out to me that the axioms for a directed distance space are similar to the axioms for an affine space. [2, p. 420-5, esp. (6) and (7) on p. 421. There is a misprint in (6).] There, the values are assumed to lie in a vector space rather than in a group. In addition to axioms 1, 2 and 4, it is assumed in [2] that

6. $\forall\, x \,\varepsilon\, X, \quad \forall\, g \,\varepsilon\, G, \quad \exists\,!\, y \,\varepsilon\, X : d(y,x) = g.$

(In the notation of [2], this would be written as $g+x=y$). This assumption is equivalent to the assertion that for each $a \,\varepsilon\, x$, the mapping f_a of Theorem 14 is onto. Hence our result, Corollary 15, could be improved to assert that f_a is an isometry onto and we can say that the only G-directed distance space satisfying 6 is G itself.

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Polytechnic of the South Bank,
London S.E.1