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Artikel: NAIVELY CONSTRUCTIVE APPROACH TO BOUNDEDNESS PRINCIPLES, WITH APPLICATIONS TO HARMONIC ANALYSIS

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Anhang: Appendix Rudin-Shapiro sequences

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are then seen from (10.9), (10.10) and (10.11) to satisfy

$$\left.\begin{aligned} \| Q_j' \| &\leqq 1, \\ \operatorname{sp}(Q_j') &\subseteq [\Delta_j], \\ \left| \int v\, D_{\Delta_j}\, Q_j'\, d\lambda_G \right| &\geqq (1 - 3v^{-\frac{1}{2}}) \| D_{\Delta_j} \|_1 \end{aligned}\right\} \qquad (10.13)$$

provided v is chosen $\geqq 9 \| D_{\Delta_j} \|_1^{-1}$. In view of (7.6), we may choose the integer $v \geqq \max_j (36, 9 \| D_{\Delta_j} \|_1^{-1})$. Then (10.13) shows that there are unimodular complex numbers ξ_j such that the $Q_j = \xi_j Q_j'$ satisfy (7.7).

Appendix

Rudin-Shapiro sequences

A.1 Notations and definitions. As hitherto, all topological groups G are assumed to be Hausdorff; and, for any locally compact group G, λ_G will denote a selected left Haar measure, with respect to which the Lebesgue spaces $L^p(G)$ are to be formed. $C_c(G)$ denotes the set of complex-valued continuous functions on G having compact supports.

If X and Y are topological groups, Hom (X, Y) denotes the set of continuous homomorphisms of X into Y.

Suppose henceforth G to be locally compact. As in 5.1, if $k \in C_c(G)$, T_k will denote the convolution operator

$$f \mapsto f * k$$

with domain $C_c(G)$ and range in $C_c(G)$; and $\| k \|_{p,q}$ will denote the (p, q)-norm of this operator, i.e., the smallest real number $m \geqq 0$ such that

$$\| f * k \|_q \leqq m \| f \|_p \quad (f \in C_c(G)).$$

It is well-known that, if G is Abelian, $\| k \|_{2,2}$ is equal to

$$\| \hat{k} \|_\infty = \sup_{\gamma \in \Gamma} | \hat{k}(\gamma) |,$$

where Γ is the character group of G and $\hat{k}$ is the Fourier transform of k. (Something similar is true whenever G is compact, but we shall not use this.)

U-RS-sequences on G are as defined in 5.4.

In A.2-A.4 we are concerned with conditions on G sufficient to ensure the possibility of constructing U-RS-sequences on G for certain choices of U. In A.5 we use Rudin-Shapiro sequences on infinite compact Abelian groups to support statements made in 7.5.

A.2 THE ABELIAN CASE. If G is Abelian and nondiscrete, the methods of § 2 of [5] show how to construct (reasonably explicitly) a U-RS-sequence (h_n) on G for any preassigned nonvoid open $U \subseteq G$; see also [7], (37.19.b). In addition, we may assume that each $\hat{h}_n$ is integrable on Γ, the character group of G. [To see this, let V be a compact neighbourhood of the origin of G and let W be a nonvoid subset of U such that $V + W \subseteq U$. Let $\{u_i\}$ be an approximate identity on G comprised of functions in $C_c(G)$ with supports in V and Fourier transforms in $L^1(\Gamma)$. Finally, let (k_n) be a W-RS-sequence; then for each $n \in N$ we may select i_n so that $(k_n * u_{i_n})$ is a U-RS-sequence with the further property that $(k_n * u_{i_n})^{\wedge} = \hat{k}_n \hat{u}_{i_n} \in L^1(\Gamma)$, as required.] We take this construction for granted (but see A.5 below) and use it to show how to construct U-RS-sequences on certain non-Abelian groups G. The basis of the extension is a simple technique of passage from a quotient group to the original, the crucial step being A.3.2 below.

A.3 THE NOT-NECESSARILY ABELIAN CASE.

A.3.1 Assume here that K is a compact normal subgroup of G. Let λ_K be normalised so that $\lambda_K(K) = 1$; and let $\pi : x \mapsto \bar{x}$ denote the natural mapping of G onto G/K.

If $f \in C_c(G)$, the function f' on G/K defined by

$$f'(\bar{x}) = \int_K f(xt)\, d\lambda_K(t) \tag{A.1}$$

belongs to $C_c(G/K)$; cf. [7], (15.21). If $g \in C_c(G/K)$, $g \circ \pi \in C_c(G)$ and

$$(g \circ \pi)' = g. \tag{A.2}$$

If τ_a denotes left-translation by amount a, it is verifiable that $(\tau_a f)' = \tau_{\bar{a}} f'$. From this it follows that the disposable factors in λ_G and $\lambda_{G/K}$ can be mutually adjusted so that

$$\int_G f\, d\lambda_G = \int_{G/K} f'\, d\lambda_{G/K} \tag{A.3}$$

for $f \in C_c(G)$. Using (A.3), a direct calculation confirms that

$$(f * (k \circ \pi))' = f' * k \tag{A.4}$$

whenever $f \in C_c(G)$ and $k \in C_c(G/K)$.

Another consequence of (A.3) is that for $1 \leqq p \leqq \infty$

$$\|f\|_p \geqq \|f'\|_p \tag{A.5}$$

for every $f \in C_c(G)$; and that for $0 < p \leqq \infty$

$$\|f\|_p = \|f'\|_p \tag{A.6}$$

for every $f \in C_c(G;K)$, the set of $f \in C_c(G)$ which are constant on cosets modulo K.

A.3.2 Let $k \in C_c(G/K)$. Then

$$\| k \circ \pi \|_{p,q} \leqq \| k \|_{p,q}. \tag{A.7}$$

PROOF. For $f \in C_c(G)$, $f * (k \circ \pi) \in C_c(G;K)$ and (A.6) gives

$$\|f * (k \circ \pi)\|_q = \|(f * (k \circ \pi))'\|_q,$$

which by (A.4)

$$\begin{aligned} &= \|f' * k\|_q \\ &\leqq \|f'\|_p \|k\|_{p,q} \\ &\leqq \|f\|_p \|k\|_{p,q}, \end{aligned}$$

the last step by (A.5). Whence (A.7).

A.3.3 If (h_n) is a V-RS-sequence on G/K and $U = \pi^{-1}(V)$, then $(h_n \circ \pi)$ is a U-RS-sequence on G.

PROOF. In view of A.3.2 it suffices to note that

$$\begin{aligned} \operatorname{supp}(h_n \circ \pi) &= \pi^{-1}(\operatorname{supp} h_n) \\ &\subseteq \pi^{-1}(V), \\ \|h_n \circ \pi\|_\infty &= \|h_n\|_\infty, \\ \|h_n \circ \pi\|_2 &= \|h_n\|_2, \end{aligned}$$

the last two because of (A.6) and (A.2).

A.3.4 Suppose that K is a compact normal subgroup of G and that one can construct V-RS-sequences on G/K for any given nonvoid open $V \subseteq G/K$. Then one can construct U-RS-sequences on G for any given open subset U of G which contains K.

PROOF. Apply A.3.3, taking a nonvoid open subset W of G such that $KW \subseteq U$, and noting that $V = \pi(W)$ is then nonvoid and open in G/K and that $\pi^{-1}(V) = KW \subseteq U$.

A.3.5 Let $\delta(G)$ be the closure in G of the derived (= commutator) subgroup of G, and suppose that $\delta(G)$ is compact and nonopen in G. Then one can construct U-RS-sequences on G for any given open subset U of G containing $\delta(G)$. (Note that, since $\delta(G)$ is a closed subgroup of G, it is nonopen in G if and only if it has empty interior, or if and only if it is locally null for λ_G.)

PROOF. This follows from A.2 and A.3.4 because:
$\delta(G)$ is in any case a normal subgroup of G such that $G/\delta(G)$ is LCA [see [7], (5.22), (5.26), (23.8)]; and $\delta(G)$ is nonopen in G if and only if $G/\delta(G)$ is nondiscrete ([7], (5.21)).

A.3.6 The hypotheses of A.3.5 are satisfied in any one of the following cases (all groups being assumed Hausdorff and locally compact):

(i) $G = G_1 \times G_2$, where $\delta(G_1)$ and $\delta(G_2)$ are compact and $\delta(G_1)$ is nonopen in G_1 (hence in particular if $G = A \times B$, where A is nondiscrete Abelian and $\delta(B)$ is compact);

(ii) $\delta(G)$ is compact and there exists an open connected subset W of G such that $e \in W \nsubseteq \delta(G)$ (hence in particular if G is compact and connected and $\delta(G) \neq G$);

(iii) $\delta(G)$ is compact and, for some Abelian A, some $\varphi \in \mathrm{Hom}\,(G, A)$ and some connected open subset W of G, we have $e \in W$ and $\varphi \mid W$ nonconstant (hence in particular if G is compact and connected and $\mathrm{Hom}\,(G, A)$ is nontrivial);

(iv) $G = \varphi(H)$, where $\varphi \in \mathrm{Hom}\,(G, H)$ is such that $\mathrm{Ker}\,\varphi$ is locally countable (that is, such that $\mathrm{Ker}\,\varphi$ intersects each compact set in a countable set), and where $\delta(H)$ is compact and nonopen in H.

PROOF. (i) It is evident that $\delta(G) \subseteq \delta(G_1) \times \delta(G_2)$, which shows that $\delta(G)$ is compact and nonopen in G [if it were open, $\delta(G_1) = pr_{G_1}(\delta(G_1) \times \delta(G_2))$ would have interior points].

(ii) Were $\delta(G)$ to be open in G, W would be a disjoint union of $W \cap \delta(G)$ and $W \cap (G \setminus \delta(G))$, each relatively open in W. Since

$e \in W \cap \delta(G)$, connectedness of W would imply that $W \cap (G \not\setminus \delta(G)) = \varnothing$, i.e., $W \subseteq \delta(G)$, a contradiction.

(iii) Ker φ is a closed subgroup of G containing $\delta(G)$; since $W \nsubseteq$ Ker φ, it follows that $W \nsubseteq \delta(G)$. Now use (ii).

(iv) Clearly,

$$\delta(G) \subseteq \overline{\varphi\,(\delta(H))} = \varphi\,(\delta(H))$$

is compact. Suppose $\delta(G)$ were open in G. Then $\varphi(\delta(H))$ has interior points, and the same would be true of

$$\varphi^{-1}\,(\varphi\,(\delta(H))) = S\,\delta(H),$$

where $S =$ Ker φ. So there would exist a compact neighbourhood V of the identity in H such that

$$V \subseteq S\delta(H)$$

and so

$$V = V \cap (S\delta(H)).$$

But, if $y \in V \cap (S\delta(H))$, $y = sz$ for some $s \in S$ and $z \in \delta(H)$, hence $s = yz^{-1} \in V\delta(H)^{-1}$, and so $s \in (V\delta(H)^{-1}) \cap S$, which is countable by hypothesis, say $\{s_n : n \in N\}$. But then

$$y \in \bigcup_{n \in N} s_n\,\delta(H).$$

Thus

$$V = V \cap (S\delta(H)) \subseteq \bigcup_{n \in N} s_n\,\delta(H)$$

and so, since $\lambda_H(\delta(H)) = 0$,

$$0 < \lambda_H\,(V) \leqq \sum_{n \in N} \lambda_H\,(\delta(H)) = 0,$$

a contradiction.

A.3.7 Remarks. (i) A.3.6 (iii) suffices to show that any finite-dimensional unitary group $U(n)$ satisfies the hypotheses of A.3.5. [For $U(n)$ is compact and connected (see [7], (7.15)); and we may apply A.3.6 (iii) with $A = T$, the circle group, and $\varphi =$ det.]

On the other hand, it is easy to see (cf. A.3.6 (i) and its proof) that if $G = \prod_{i \in I} G_i$, where the G_i are compact and at least one of them satisfies the hypothesis of A.3.5, then G satisfies the said hypotheses.

So every product of unitary groups satisfies the hypotheses of A.3.5.

(ii) The hypotheses of A.3.5 are also satisfied if $G = G_1 \circledS G_2$, the semidirect product of G_1 and G_2 (see [7], (2.6) and (6.20)), provided G_1 is compact and $\delta(G_2)$ is compact and nonopen in G_2 (hence in particular if $G = A \circledS B$, where A is compact and B is nondiscrete and Abelian). In fact, $\delta(G) \subseteq G_1 \times \delta(G_2)$ and the proof proceeds as for A.3.6 (i).

A.4 THE OPERATORS $f \mapsto k * f$. Retaining the notations introduced in A.3, it turns out that (cf. (A.4))

$$((k \circ \pi) * f)' = k * f^{\vee\prime\vee} \tag{A.8}$$

for every $f \in C_c(G)$ and $k \in C_c(G/K)$, where, for any function g with domain a group X, $\check{g}$ denotes the function $x \mapsto g(x^{-1})$ with domain X. As a consequence, the results of A.3 have direct analogues for the operator $f \mapsto k * f$, provided G/K is unimodular, which is so if and only if G is unimodular.

A.5 CONCERNING 7.5.

A.5.1 Throughout A.5 we suppose G to be infinite compact Abelian. Let Γ_0 be any infinite subsemigroup of the character group Γ of G; $0 \in \Gamma_0$. The construction described in § 2 of [5] may be employed to produce t.p.s f_n $(n \in N)$ on G which, together with their spectra S_n, satisfy the conditions:

$$\left.\begin{aligned} &S_0 = \{0\},\ S_n \subseteq \Gamma_0,\ |S_n| = 2^n \\ &B2^{n/2} \leqq \|f_n\|_s \leqq A2^{n/2} \quad (1 \leqq s \leqq \infty), \\ &\|f_n\|_{2,2} = \|\hat{f}_n\|_\infty \leqq 1, \\ &\hat{f}_n = \varphi \text{ on } S_n,\ 0 \text{ on } \Gamma \setminus S_n, \end{aligned}\right\} \tag{A.9}$$

where A and B are positive absolute constants and φ is a function on Γ with $\operatorname{Ran} \varphi \subseteq \{-1, 0, 1\}$ and $|\varphi(\gamma)| = 1$ if and only if $\gamma \in S_n$. (When $G = T$, these f_n are virtually the original Rudin-Shapiro t.p.s. In the terminology adopted in 5.4 above the $h_n = 2^{-n/2} f_n$ constitute a G-RS-sequence on G.)

If we now choose $\alpha_n \in \Gamma$ inductively so that, on writing $F_n = \alpha_n + S_n$, we have

$$\alpha_{n+1} \in \Gamma_0 \setminus [(F_0 \cup \ldots \cup F_n) - S_{n+1}],$$

then

$$\left.\begin{aligned} &|F_n| = |S_n| = 2^n,\ F_n \subseteq \Gamma_0, \\ &F_n \cap F_m = \varnothing \text{ if } m \neq n, \end{aligned}\right\} \tag{A.10}$$

and the t.p.s

$$w_n = 2^{-n/2} \alpha_n f_n \tag{A.11}$$

satisfy the relations

$$\left.\begin{aligned} \| w_n \|_\infty \leqq A,\ \hat{w}_n = 2^{-n/2} \varphi_n, \\ \operatorname{Ran} \varphi_n \subseteq \{-1, 0, 1\},\ | \varphi_n(\gamma) | = 1 \quad \text{if and only if} \quad \gamma \in F_n. \end{aligned}\right\} \tag{A.12}$$

From (A.10) and (A.12) it follows that at least one of the sets $A_n = \varphi_n^{-1}(\{1\})$, $B_n = \varphi_n^{-1}(\{-1\})$ has not fewer than 2^{n-1} elements. Define $\varepsilon_n = 1$, $C_n = A_n$ if $|A_n| \geqq 2^{n-1}$ and $\varepsilon_n = -1$, $C_n = B_n$ if $|A_n| < 2^{n-1}$. Then

$$\left.\begin{aligned} (\varepsilon_n w_n)^\wedge(\gamma) = 2^{-n/2} \quad \text{if} \quad \gamma \in C_n, \\ C_n \subseteq F_n,\ | C_n | \geqq 2^{n-1}. \end{aligned}\right\} \tag{A.13}$$

A.5.2 In terms of the construction given in A.5.1, it is possible to write down any number of continuous functions f on G and sequences (Δ_j) of finite subsets of Γ_0 such that

$$\left.\begin{aligned} \Delta_j \subseteq \Delta_{j+1}, \\ \operatorname{sp}(f) \subseteq \Gamma_0, \\ S_{\Delta_j} f(0) \quad \text{is real and} \quad \lim_{j \to \infty} S_{\Delta_j} f(0) = \infty, \\ \sum_{\gamma \in \Gamma} | \hat{f}(\gamma) | = \infty; \end{aligned}\right\} \tag{A.14}$$

cf. the statements made in 7.5.

Indeed, if $(c_n)_{n=0}^{\infty}$ is a sequence of real numbers satisfying

$$c_n \geqq 0, \sum_{n=0}^{\infty} c_n < \infty, \sum_{n=0}^{\infty} 2^{n/2} c_n = \infty, \tag{A.15}$$

and if

$$\Delta_j = C_0 \cup \ldots \cup C_j, \tag{A.16}$$

if suffices to take

$$f = \sum_{n=0}^{\infty} c_n \varepsilon_n w_n, \tag{A.17}$$

(A.14) being then a simple consequence of (A.12) and (A.13).

However, it is a consequence of the choice of the γ_n and α_n and of (A.12) [on evaluating the Fourier series of w_n at 0] that $\big| |A_n| - |B_n| \big| \leqq 2^{n/2}$, which implies that C_n contains only about one half the elements of F_n, so that $\bigcup_{j=1}^{\infty} \Delta_j$ falls far short of exhausting Γ_0. In particular, (Δ_j) is not a convergence grouping of the sort described in § 7.

A.5.3 Two further consequences of the construction in A.5.1 are perhaps worth mentioning in passing.

(i) For any complex-valued sequence $(c_n)_{n=1}^{\infty}$ such that

$$\sum_{n=1}^{\infty} |c_n| < \infty, \tag{A.18}$$

the formula

$$g = \sum_{n=1}^{\infty} c_n w_n \tag{A.19}$$

yields a continuous function $g \in C(G)$. It is easy to specify choices of (c_n) in accord with (A.18), and of nonnegative functions η on Γ such that

$$\lim_{\gamma \to \infty} \eta(\gamma) = 0, \tag{A.20}$$

for which

$$\sum_{\gamma \in \Gamma} |\hat{g}(\gamma)|^{2 - 2\eta(\gamma)} = \infty. \tag{A.21}$$

One might, for example, take $c_n = n^{-2}$ and $\eta(\gamma) = 6n^{-1} \log n$ for $\gamma \in F_n$ $(n = 1, 2, \ldots)$ and $\eta(\gamma) = 0$ for $\gamma \in \Gamma \backslash F$, where $F = \bigcup_{n=1}^{\infty} F_n$.

This is an analogue of a well-known result of Banach for the case $G = T$; it provides numerous reasonably constructive counter-examples to conjectural improvements of the Hausdorff-Young theorem.

(ii) Take (c_n), η and g as in (i) immediately above. Let ψ be any nonnegative function on Γ which is bounded away from zero on F. Let further θ be any complex-valued function on Γ such that

$$\theta(\gamma) = \psi(\gamma) |\hat{g}(\gamma)|^{1 - 2\eta(\gamma)} \,.\, \operatorname{sgn} \hat{g}(\gamma) \quad \text{for} \quad \gamma \in F. \tag{A.22}$$

Then (A.21), (A.22) and Bochner's theorem combine to show that θ is not a Fourier-Stieltjes transform. Yet, if ψ is bounded, and if we define $\theta(\gamma) = 0$ for $\gamma \in \Gamma \backslash F$, (A.20) and the fact that $g \in C(G)$ ensure that

$$\theta \in \bigcap_{r > 2} l^r(\Gamma). \tag{A.23}$$

We thus obtain explicit examples of functions θ satisfying (A.23) which are not Fourier-Stieltjes transforms.

Note that, if every c_n is real and nonzero, an (unbounded) ψ can be chosen so as to make Ran $\theta = \{-1, 1\}$; this yields explicit examples of $\pm$ 1-valued functions θ which are not Fourier-Stieltjes transforms. (These are, of course, also obtainable by starting with functions sgn $\hat{h}$, where $h \in C(G)$, $\hat{h}$ is real-valued and $\hat{h} \notin l^1(\Gamma)$.)

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