§ 6. (p, q)-multipliers whose transforms are not measures

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$$\inf_{a \in G} \Delta(a)^{1/p - 1/q} = 0,$$

and we infer that $T = 0$.

(ii) In spite of (i) immediately above, there is a partial analogue taking the following form.

Assume that there exists a sequence (h_n) satisfying (5.6), where now $\| h_n \|_{2,2}$ is defined to mean

$$\sup \{ \| h_n * f \|_2 : f \in C_c(G) , \| f \|_2 \leqq 1 \}.$$

Then modification of the proof of Theorem 5.7 will lead to the construction of operators T which are right multipliers of type (p, p) for every $p \in (1, \infty)$, have supports contained in $\bar{U}$, and are not of the form $f \mapsto \mu * f$ for any measure μ.

§ 6. (p, q)-*multipliers whose transforms are not measures*

6.1 INTRODUCTION. Throughout this section we suppose that G is a locally compact Abelian (= LCA) group with dual group Γ, both groups being additively written. We begin by slightly modifying the form of the definition of (p, q)-multipliers, so rendering it possible to make certain statements about their Fourier transforms without attempting a general definition of such transforms. To this end, let F denote the set of functions on G which belong to $\bigcap \{L^p(G) : 1 \leqq p \leqq \infty\}$ and which possess Fourier transforms with compact supports, and denote by $L_p^q(G)$ the set of continuous linear operators from F, equipped with the $L^p(G)$-norm, into $L^q(G)$ which commute with translations. As before, equip $L_p^q(G)$ with the $(L^p(G), L^q(G))$ operator norm. It is easy to specify a natural isometry between $L_p^q(G)$ as defined above and $L_p^q(G)$ as defined in § 5, and so we speak of the elements of $L_p^q(G)$ as (p, q)-multipliers on G.

When T is a (p, q)-multiplier in this sense, we say that its *Fourier transform* $\hat{T}$ *is a measure* μ if and only if there exists a measure μ on Γ such that

$$h * Tg\,(0) = \int_\Gamma \hat{h}\,\hat{g}\,d\mu \tag{6.1}$$

for all $g, h \in F$, where $\hat{u}$ denotes the Fourier transform of u. Similarly, if Ω is an open subset of Γ, we shall write $\hat{T} = \mu$ on Ω if and only if (6.1) holds for all g, $h \in F$ such that supp $\hat{g} \subseteq \Omega$. If Σ is a closed subset of Γ, we shall write supp $\hat{T} \subseteq \Sigma$ if and only if $\hat{T} = 0$ on Γ / Σ.

It is simple to verify that, if $K \in F$ and T_K is the mapping $g \mapsto g * K = K * g$, then $T_K \in L_p^q$ whenever $1 \leqq p \leqq q \leqq \infty$. (In fact, $\| K * g \|_\infty \leqq \| K \|_{p'} \| g \|_p$ and $\| K * g \|_p \leqq \| K \|_1 \| g \|_p$ and the convexity of the function $t \mapsto \log \| K * g \|_{t^{-1}}$, or an appeal to the closed graph theorem, does the rest.) Furthermore, $\hat{T}_K$ is the measure $\hat{K}\lambda_\Gamma$, where λ_Γ is the Haar measure of Γ normalised so that the $L^2(\lambda_\Gamma)$-norm of $\hat{u}$ is equal to $\| u \|_2$ for every $u \in L^2(G)$.

6.2 It has been shown by Gaudry ([5], Theorem 3.1) that, if G is noncompact LCA and $1 \leqq p < 2 < q \leqq \infty$, there exist operators $T \in L_p^q(G)$ such that $\hat{T}$ is not a measure. In 6.3 and its proof we shall indicate how to construct operators T which belong to $L_p^q(G)$ for every pair (p, q) satisfying $1 \leqq p < 2 < q \leqq \infty$ and which are such that supp $\hat{T}$ is contained in a compact subset of Γ and $\hat{T}$ is not a measure. The precise statement of 6.3 requires some prefatory remarks.

Let G be a noncompact LCA group and Ω a relatively compact open subset of the dual group Γ. Since Γ is nondiscrete LCA, an Ω-RS-sequence (h_n) on Γ may be constructed in such a way that the inverse Fourier transform of h_n belongs to $L^1(G)$ for every n; see Appendix A.2. Assuming this to have been done, choose positive integers $m_1 < m_2 < \ldots$ and define $k_n = nh_{m_n}$ exactly as in 5.4, so that (5.7)-(5.9) remain intact (but with Γ, rather than G, as the underlying group). We now consider the functions K_n on G, K_n being defined to be the inverse Fourier transform of k_n.

It is plain that every K_n belongs to F. Moreover, an application of Hölder's inequality yields

$$\| K_n \|_s \leqq \| K_n \|_2^{2/s} \| K_n \|_\infty^{1-2/s} \quad (s > 2). \tag{6.2}$$

By Parseval's formula and (5.8),

$$\| K_n \|_2 = \| k_n \|_2 \leqq A^{\frac{1}{2}} n;$$

also, since G is LCA, (5.9) leads to

$$\| K_n \|_\infty = \| T_{k_n} \|_{2,2} \leqq 2^{-n}.$$

Inserting these last two estimates into (6.2), we obtain

$$\| K_n \|_s = 0\,(n^{2/s} 2^{-n(1-2/s)}) \quad (s > 2). \tag{6.3}$$

We shall need to note also that a construction, similar to that appearing in the proof of Lemma 5.6, shows that for each $n \in N$ we may select and fix u_n, $v_n \in F$ such that

$$\| \hat{u}_n \, \hat{v}_n \|_\infty \leqq 1 \tag{6.4}$$

and

$$| \textstyle\int_\Gamma \hat{u}_n \, \hat{v}_n \hat{K}_n \, d\lambda_\Gamma | \geqq \tfrac{1}{2} \| \hat{K}_n \|_1 = \tfrac{1}{2} \| k_n \|_1 \geqq \tfrac{1}{2} Bn, \tag{6.5}$$

the last link in this chain of inequalities stemming from (5.7).

6.3 THEOREM. Let G be a noncompact LCA group, Ω a relatively compact open subset of the dual group Γ. Suppose the function $K_n (n \in N)$ to be defined as in 6.2. A continuum of sequences $(\omega_n) \in l^1_+(N)$ may be constructed, for each of which the series

$$\textstyle\sum_{n \in N} \omega_n \, T_{K_n} \tag{6.6}$$

converges normally in $L^q_p(G)$ for every pair (p, q) satisfying $1 \leqq p < 2 < q \leqq \infty$, the sum T of the series (6.6) satisfying the conditions

(i) $T \in \bigcap \{ L^q_p(G) : 1 \leqq p < 2 < q \leqq \infty \}$;

(ii) supp $\hat{T} \subseteq \Omega$; and

(iii) $\hat{T}$ is not a measure.

PROOF. Since G is Abelian, (5.4) shows that $L^q_p(G) = L^{p'}_{q'}(G)$ and $\| \cdot \|_{p,q} = \| \cdot \|_{q',p'}$. Accordingly, we may and will restrict attention to those pairs (p, q) such that $1 \leqq p < 2 < q \leqq \infty$ and $1/p + 1/q \geqq 1$; denote by I the set of such pairs.

We propose to appeal to Corollary 3.2, taking therein

$H =$ the space of linear maps from F into $L^1_{loc}(G)$ with the topology of pointwise convergence;

I as defined immediately above;

$E_{(p,q)} = L^q_p(G)$ for every $(p, q) \in I$;

$E =$ the closed linear subspace of $\mathscr{E}$ generated by the T_{K_n} $(n \in N)$;

$f_n : T \mapsto | u_n * T v_n(0) |$;

$x_n = T_{K_n}$.

Regarding the hypotheses of Corollary 3.2, it is clear that 3.2 (i) is satisfied. Also, for any $T \in E$ and any $m \in N$, Hölder's inequality yields

$$f_m(T) \leqq \| u_m \|_{q'} \| T v_m \|_q \leqq \| u_m \|_{q'} \| T \|_{p,q} \| v_m \|_p,$$

which, since u_m and v_m belong to F, shows that f_m is continuous (and therefore certainly bounded) on E.

Next, since (see the remarks at the end of 6.1 above) $\hat{T}_{K_n}$ is the measure $\hat{K}_n \lambda_\Gamma = k_n \lambda_\Gamma$,

$$f_m(T_{K_n}) = | \int_\Gamma \hat{u}_m \hat{v}_m k_n \, d\lambda_\Gamma | \leqq \| k_n \|_1,$$

the inequality coming from (6.4). This makes it clear that $f^* (T_{K_n})$ is finite for every $n \in N$, so that 3.2 (ii) is satisfied.

Turning to 3.2 (iii), note first that by convexity (as in the proof of (5.17)) we have

$$\| T_{K_n} \|_{p,q} \leqq \| T_{K_n} \|_{2,2}^{\alpha} \| T_{K_n} \|_{1,s}^{1-\alpha}, \tag{6.7}$$

where, since $p < 2 < q$, we have $\alpha < 1$ and $s > 2$. Now, by the case $s = \infty$ of (5.8),

$$\| T_{K_n} \|_{2,2} = \| \hat{K}_n \|_\infty = \| k_n \|_\infty \leqq n.$$

Using this in combination with (6.3) and (6.7), it appears that

$$\| T_{K_n} \|_{p,q} = 0 \, (n^\alpha n^{2(1-\alpha)/s} 2^{-\beta n}),$$

where $\beta = (1-\alpha)(1-2/s)$ is positive, and so

$$\lim_{n \to \infty} T_{K_n} = 0 \text{ in } E,$$

which is more than enough to verify 3.2 (iii).

As for 3.2 (iv), the fact that $\hat{T}_{K_n} = \hat{K}_n \lambda_\Gamma$ combines with (6.5) to yield

$$f_n (T_{K_n}) = | \int_\Gamma \hat{u}_n \hat{v}_n \hat{K}_n \, d\lambda_\Gamma | \geqq \tfrac{1}{2} Bn,$$

which confirms 3.2 (iv).

An appeal to Corollary 3.2 is thus justified and assures one of the existence of a continuum of sequences $(\omega_n) \in l^1_+(N)$ for each of which the series (6.6) converges normally to a (unique) sum T in E which satisfies

$$f^* (T) = \infty. \tag{6.8}$$

From this it is evident that (i) is satisfied, and that, for every pair (p, q)

satisfying $1 \leqq p < 2 < q \leqq \infty$, the series (6.6) converges normally in $L_p^q(G)$ to T. Next, T is the limit in E of

$$S_r = \sum_{n=1}^{r} \omega_n T_{K_n}$$

as $r \to \infty$ and, since it is plain that supp $S_r \subseteq \Omega$ for every r, (ii) is easily derived. Finally, if $\hat{T}$ were a measure μ, it would necessarily be the case that supp $\mu \subseteq \bar{\Omega}$ and so, for every $n \in N$, one would have by (6.1) and (6.4)

$$\begin{aligned} f_n(T) &= | u_n * Tv_n(0) | = | \textstyle\int_\Gamma \hat{u}_n \hat{v}_n \, d\mu | \\ &\leqq | \mu | (\bar{\Omega}), \end{aligned}$$

which is finite since Ω is relatively compact. However, this plainly would entail $f^*(T) < \infty$, in conflict with (6.8), so that T cannot be a measure and (iii) is verified. This completes the proof.

6.4 Remark. Theorem 6.3 was proved by Hörmander ([14], Theorem 1.9) for $G = R^n$ and any given pair (p, q) satisfying $1 \leqq p < 2 < q \leqq \infty$, this result being extended to a general noncompact LCA G by Gaudry [5]. The argument given by Hörmander (loc. cit. Theorem 1.6 and the remark immediately following) for the case $G = R^n$ can also be extended to a general LCA G and shows that, if *either* $q \leqq 2$ *or* $p \geqq 2$, then every $T \in L_p^q(G)$ is such that $\hat{T}$ is a measure [and indeed a measure of the form $\psi\lambda_\Gamma$, where $\psi \in L_{loc}^2(\Gamma)$ if $q \leqq 2$ and $\psi \in L_{loc}^p(\Gamma)$ if $p \geqq 2$, and so $\psi \in L_{loc}^2(\Gamma)$ in either case]. Thus the hypotheses made in Theorem 6.3 about p and q are necessary for the validity of the conclusion.

Part 3: Applications to Fourier series

§ 7. *Applications to divergence of Fourier series.*

7.1 Throughout §§ 7-10, G will denote an infinite Hausdorff compact Abelian group with character group Γ, and λ_G the Haar measure on G, normalised so that $\lambda_G(G) = 1$. For any $f \in L^1(G)$, $\hat{f}$ will denote the Fourier transform of f; for any finite subset Δ of Γ,

$$S_\Delta f = \sum_{\gamma \in \Delta} \hat{f}(\gamma)\gamma \tag{7.1}$$

is the Δ-partial sum of the Fourier series of f; and sp (f) will stand for