NOTE ON THE DEGREE OF APPROXIMATION TO CONTINUOUS FUNCTIONS

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To the memory of J. Karamata

1. If f is a continuous function on $[-1, 1]$ and ω_f its modulus of continuity defined by

$$\omega_f(h) = \sup \{ |f(x) - f(y)| : x, y \in [-1,1], \ |x-y| \leq h \},$$

then according to the well known theorem of D. Jackson the sequence $(P_n^* [f])$ of polynomials of best approximation to f satisfies the inequalities

$$\max_{-1 \leq x \leq 1} |P_n^* [f](x) - f(x)| \leq C \omega_f \left(\frac{1}{n} \right), \quad n = 1, 2, \ldots$$

where C is a constant (see [1] and [2], p. 56).

Several authors have constructed explicitly sequences of polynomials $(P_n [f])$ which have essentially the same deviation from f, or the same degree of precision of approximation to f, as $(P_n^* [f])$. V. K. Dzjadik [3] has used polynomials $(P_n [f])$ defined by

$$P_n [f](x) = \frac{1}{3} \int_{-1}^{1} f(4t^2) \left(D_{nk} \left(\frac{2t + x}{3} \right) + D_{nk} \left(\frac{2t - x}{3} \right) \right) dt$$

for the approximation to f on $[0, 1]$. Here

$$D_{nk}(x) = c_{nk} \left(\frac{1 - T_n(1 - \frac{1}{2} x^2)}{x^2} \right)^k,$$

T_n is the Chebyshev polynomial of degree n, and c_{nk} is chosen so that $\int_{-1}^{1} D_{nk}(x)\, dx = 1$ (see [3], p. 339).

R. DeVore [4] has introduced the sequence of polynomials $(L_n [f])$ defined by

$$L_n [f](x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(t) \Lambda (t - x)\, dt$$

where Λ_n is the polynomial

$$\Lambda_n(x) = c_n \left(\frac{P_{2n}(x)}{x^2 - \alpha_{2n}^2} \right)^2 .$$

Here P_{2n} is the Legendre polynomial of degree $2n$, α_{2n} is the smallest positive zero of P_{2n} and c_n is chosen so that $\int_{-1}^{1} \Lambda_n(x)\, dx = 1$.

More complicated sequences of interpolatory polynomials have been obtained by G. Freud [5], R. B. Saxena [6] and M. Sallay [7].

The aim of this note is to give a general result of this type which should illuminate better the nature of approximation processes which are close to the best possible approximation.

We shall consider here approximating polynomials generated by a sequence of orthogonal polynomials (p_n) on $[-1, 1]$ whose weight function w is non-negative, even and L-integrable on $[-1, 1]$ and has the following properties:

$$(1.1) \qquad 0 < m \leqq w(x) \quad \text{for} \quad x \in [-c, c], \quad 0 < c \leqq 1$$

$$(1.2) \qquad w(x) \leqq M \quad \text{for} \quad x \in [-\delta, \delta], \quad 0 < \delta \leqq 1 .$$

We shall denote the zeros of p_n in their increasing order by x_{kn}, $k = 1, \dots, n$:

$$-1 < x_{1n} < \dots\dots < x_{nn} < 1 .$$

Since w is even, the zeros of p_n are symmetrically distributed in $[-1, 1]$.

Our basic result can be stated as follows:

THEOREM. *Let w be a non-negative, even and L-integrable function on* $[-1, 1]$ *satisfying conditions* (1.1) *and* (1.2) *and let* (p_n) *be the sequence of orthogonal polynomials on* $[-1, 1]$ *generated by the weight function w.*

Let (R_n) be either one of the following two sequences of polynomials

$$(i) \quad c_n \left(\frac{p_{2n}(x)}{x^2 - \alpha_{2n}^2} \right)^2 \qquad (ii) \quad c_n \left(\frac{p_{2n+1}(x)}{x(x^2 - \alpha_{2n+1}^2)} \right)^2$$

where α_n is the smallest positive zero of p_n and c_n is chosen so that $\int_{-c}^{c} R_n(x)\, dx = 1$.

For any f continuous on $[-\frac{1}{2} c, \frac{1}{2} c]$ *let the sequence of polynomials* $(K_n[f])$ *be defined by*

$$K_n[f](x) = \int_{-\frac{c}{2}}^{\frac{c}{2}} f(t)\, R_n(x-t)\, dt\,.$$

Then for all n sufficiently large we have the inequality

$$\max_{-\frac{c}{4} \leq x \leq \frac{c}{4}} |K_n[f](x) - f(x)| \leq C\omega_f\left(\frac{1}{n}\right)$$

where C depends only on the choice of the weight function w.

This theorem states essentially that if the weight function w is bounded away from zero and infinity in a neighborhood of 0, and if f is continuous there, then the sequence $(K_n[f])$ converges uniformly to f in a smaller neighborhood of 0 and the rate of convergence is close to the best possible.

The weight functions of all classical orthogonal polynomials clearly satisfy conditions (1.1) and (1.2).

The simplest sequences of approximating polynomials are obtained by choosing $w(x) = (1-x^2)^{-\frac{1}{2}}$, $x \in (-1, 1)$. The corresponding orthogonal polynomials are then the Chebyshev polynomials T_n, with $\alpha_{2n} = \sin \dfrac{\pi}{4n}$ and $\alpha_{2n+1} = \sin \dfrac{\pi}{2n+1}$. The kernels R_n are

$$(i)\ \ R_n(x) = c_n \left(\frac{T_{2n}(x)}{x^2 - sin^2 \dfrac{\pi}{4n}}\right)^2 \qquad (ii)\ \ R_n(x) = c_n \left(\frac{T_{2n+1}(x)}{x\left(x^2 - sin^2 \dfrac{\pi}{2n+1}\right)}\right)^2.$$

Other simple approximating sequences are obtained by choosing $w(x) = (1-x^2)^{1/2}$, $x \in [-1, 1]$. In this case we have the Chebyshev polynomials U_n with $\alpha_{2n} = \sin \dfrac{\pi}{4n+2}$ and $\alpha_{2n+1} = \sin \dfrac{\pi}{2n+2}$. The corresponding kernels are

$$(i)\ R_n(x) = c_n \left(\frac{U_{2n}(x)}{x^2 - sin^2 \dfrac{\pi}{4n+2}}\right)^2 \quad (ii)\ R_n(x) = c_n \left(\frac{U_{2n+1}(x)}{x\left(x^2 - sin^2 \dfrac{\pi}{2n+2}\right)}\right)^2.$$

Finally, the choice $w(x) = 1$, $x \in [-1, 1]$ leads to kernels generated by Legendre polynomials P_n:

$$(i)\quad R_n(x) = c_n\left(\frac{P_{2n}(x)}{x^2 - \alpha_{2n}^2}\right)^2 \qquad (ii)\quad R_n(x) = c_n\left(\frac{P_{2n+1}(x)}{x(x^2 - \alpha_{2n+1}^2)}\right)^2.$$

The first of these kernels was used in R. DeVore's proof of Jackson's theorem.

2. The proof of our theorem is based on certain properties of zeros x_{kn} of p_n and the corresponding Cotes numbers λ_{kn} which appear in the Gauss quadrature formula. As it is well known, the Gauss formula states that for any polynomial P of degree $\leqq 2n-1$ we have

$$\int_{-1}^{1} P(x)\, w(x)\, dx = \sum_{k=1}^{n} \lambda_{kn} P(x_{kn}).$$

In order to simplify the proof of the theorem we shall formulate the most important steps in the proof as lemmas.

Lemma 1. *Let w be a non-negative, even and L-integrable function on* $[-1, 1]$ *satisfying condition* (1.1). *Then the sequence* (α_n) *of smallest positive zeros of* p_n, $n = 1, 2, \ldots$ *converges to zero.*

Proof. Given $0<\varepsilon<c$, choose $[a, b]$, $0<a<b<\varepsilon$. Since $\int_a^b w(x)\, dx \geqq m(b-a) > 0$, for all n sufficiently large the polynomial p_n will have a zero in $[a, b]$ (see [8], pp. 110-111). This means that $0<\alpha_n<\varepsilon$ for all $n>N_\varepsilon$. Hence $\alpha_n \to 0$ $(n\to\infty)$.

Lemma 2. *Let w be a non-negative and L-integrable function on* $[-1, 1]$ *satisfying condition* (1.2). *If* $x_{kn} \in [-\frac{1}{2}\delta, \frac{1}{2}\delta]$, *then the corresponding Cotes number* λ_{kn} *satisfies the inequality* $0<\lambda_{kn}\leqq C_1/n$ *where* $C_1 \leqq 4\pi M + \frac{2}{\delta}\int_{-1}^{1} w(t)\, dt$.

Proof. This is a result of P. Erdös and P. Turán (see [9], Lemma V, p. 530). Another simpler and more direct proof of this inequality was given by G. Freud (see [10], Hilfssatz IV*a*, p. 259).

Lemma 3. *Let w be a non-negative, even and L-integrable weight function on* $[-1, 1]$ *satisfying conditions* (1.1) *and* (1.2). *Then for all n sufficiently large the smallest positive zero α_n of p_n satisfies the inequality* $\alpha_n \leqq C_2/n$.

Proof. Since by Lemma 1 the sequence (α_n) converges to zero, we can find N_δ such that

$$\alpha_n \in [0, \tfrac{1}{2}\delta] \quad \text{for all} \quad n \geqq N_\delta .$$

Let β_n be the largest non-positive zero of p_n. Since w is even, we have $\beta_n = -\alpha_n$ or $\beta_n = 0$. By the separation theorem of Chebyshev, Markov and Stieltjes we have

$$m\alpha_n \leqq m(\alpha_n - \beta_n) \leqq \int_{\beta_n}^{\alpha_n} w(t)\, dt \leqq \lambda(\beta_n) + \lambda(\alpha_n)$$

where $\lambda(\beta_n)$ and $\lambda(\alpha_n)$ are Cotes numbers in the Gauss formula corresponding to the zeros β_n and α_n of p_n. Since $\beta_n, \alpha_n \in [-\frac{1}{2}\delta, \frac{1}{2}\delta]$, we have by Lemma 2 $m\alpha_n \leqq 2C_1/n$ for all $n \geqq N_\delta$ and the Lemma is proved.

LEMMA 4. *If R is a non-negative polynomial of degree $\leqq 4n-1$ such that $\int_{-c}^{c} R(t)\, dt = 1$, then we have the inequality $R(x) \leqq C_3 n$ for all $x \in [-\frac{1}{2}c, \frac{1}{2}c]$.*

PROOF. Let $P(x) = \int_{-c}^{x} R(t)\, dt$. Then P is a polynomial of degree $\leqq 4n$ with $|P(x)| \leqq 1$ for all $x \in [-c, c]$ and the proof of the Lemma follows immediately from Bernstein's inequality (see [2], p. 62).

LEMMA 5. *For all n sufficiently large the polynomials R_n satisfy the inequality*

$$\int_{-c}^{c} t^2 R_n(t)\, dt \leqq Cn^{-2}.$$

PROOF. We shall prove the inequality only for

$$R_n(x) = c_n \left(\frac{p_{2n}(x)}{x^2 - \alpha_{2n}^2} \right)^2 .$$

The same argument shows that

$$R_n(x) = c_n \left(\frac{p_{2n+1}(x)}{x(x^2 - \alpha_{2n+1}^2)} \right)^2$$

satisfies the same inequality.

We have first by (1.1)

$$\int_{-c}^{c} t^2 R_n(t)\, dt \leqq \frac{1}{m} \int_{-c}^{c} t^2 R_n(t)\, w(t)\, dt$$

$$\leqq \frac{1}{m} \int_{-1}^{1} t^2 R_n(t)\, w(t)\, dt .$$

Since R_n is an even polynomial of degree $4n-4$, non-negative and vanishing at all zeros $x_{k,2n}$ of p_{2n} except at α_{2n} and $-\alpha_{2n}$, we have by Gauss quadrature formula based on the zeros of p_{2n}

$$\int_{-1}^{1} t^2 R_n(t)\, w(t)\, dt = \sum_{k=1}^{2n} \lambda_{k,2n} (x_{k,2n})^2 R_n(x_{k,2n})$$

$$= 2\lambda(\alpha_{2n})\, \alpha_{2n}^2 R_n(\alpha_{2n}) .$$

By Lemma 3 we can find $N_{c,\delta}$ such that for all $n \geqq N_{c,\delta}$ we have

$$0 < \alpha_{2n} \leqq C_2/2n \qquad \text{and} \qquad 0 < \alpha_{2n} < \min\left(\tfrac{1}{2}c, \tfrac{1}{2}\delta\right).$$

By Lemma 2 we have then $\lambda(\alpha_{2n}) \leqq C_1/2n$. Hence for all $n \geqq N_{c,\delta}$ we have

$$\int_{-c}^{c} t^2 R_n(t)\, dt \leqq C_1 (C_2)^2 (1/n^3)\, R_n(\alpha_{2n}) .$$

By Lemma 4, $0 \leqq R_n(x) \leqq C_3 n$ for all $x \in [-\frac{1}{2}c, \frac{1}{2}c]$. Since $\alpha_{2n} \in [0, \frac{1}{2}c]$, we have $R_n(\alpha_{2n}) \leqq C_3 n$ and the Lemma is proved.

LEMMA 6. *Let L_n be a positive linear operator defined on the set of all continuous functions on $[a, b]$, with values in the set of continuous functions on $[\alpha, \beta]$, with $a \leqq \alpha < \beta \leqq b$. Then*

$$(2.1) \quad \|L_n[f]-f\| \leqq (\|L_n[1]\| + 1)\, \omega_f(\mu_n) + \|f\|\, \|L_n[1]-1\|$$

where

$$\mu_n = \|L_n[(t-x)^2](x)\|^{1/2} .$$

Here, the operator L_n is applied to the variable $t \in [a, b]$, while the sup norm $\| \ \|$ is taken with respect to $x \in [\alpha, \beta]$.

PROOF. This is a result of O. Shisha and B. Mond (see [11], Th. 1.) Since

$$\mu_n^2 \leqq \|L_n[t^2](x) - x^2\| + 2\gamma\, \|L_n[t](x) - x\| + \gamma^2\, \|L_n[1]-1\|$$

where $\gamma = \max(|\alpha|, |\beta|)$, the well known theorem of P. P. Korovkin about convergence of sequences of positive linear operators follows immediately from the inequality (2.1) (see [12], Ch. 1).

We shall give here Shisha and Mond's proof of the inequality (2.1). We have first for any $x \in [\alpha, \beta]$

$$|L_n[f](x) - f(x)| \leqq L_n[|f(t)-f(x)|](x) + |f(x)|\, |L_n[1](x) - 1| .$$

Since for any $h>0$ we have

$$|f(x)-f(t)| \leqq \left(1+\frac{(t-x)^2}{h^2}\right)\omega_f(h),$$

it follows that

$$|L_n[f](x)-f(x)| \leqq \omega_f(h)\left(L_n[1](x)+\frac{1}{h^2}L_n[(t-x)^2](x)\right)+$$
$$+|f(x)|\,|L_n[1](x)-1|.$$

Hence

$$(2.2)\qquad ||L_n[f]-f|| \leqq \omega_f(h)\left(||L_n[1]||+\frac{1}{h^2}||L_n[(t-x)^2](x)||\right)+$$
$$+||f||\,||L_n[1]-1||.$$

If $\mu_n=||L_n[(t-x)^2](x)||^{1/2}>0$, (2.1) follows from (2.2) by choosing $h=\mu_n$. If $\mu_n=0$, the inequality (2.2) becomes

$$||L_n[f]-f|| \leqq \omega_f(h)\,||L_n[1]||+||f||\,||L_n[1]-1||$$

and (2.1) follows again since h can be chosen arbitrarily small.

3. *Proof of the Theorem.* The operator K_n defined by

$$K_n[f](x) = \int_{-c/2}^{c/2} f(t)\,R_n(x-t)\,dt$$

is clearly a positive linear operator. Hence, in view of Lemma 6, we have only to evaluate $||K_n[(t-x)^2](x)||$, $||K_n[1]-1||$ and $||K_n[1]||$, where $||g||=\sup\{|g(x)|:|x|\leqq\frac{c}{4}\}$.

We have first for $|x|\leqq\frac{c}{4}$

$$K_n[(t-x)^2](x)=\int_{-c/2}^{c/2}(x-t)^2R_n(x-t)\,dt=\int_{x-\frac{c}{2}}^{x+\frac{c}{2}}t^2R_n(t)\,dt\leqq$$
$$\leqq\int_{-\frac{3}{4}c}^{\frac{3}{4}c}t^2R_n(t)\,dt.$$

Hence

$$\mu_n^2=||K_n[(t-x)^2](x)||\leqq\int_{-c}^{c}t^2R_n(t)\,dt.$$

On the other hand, since $\int_{-c}^{c} R_n(t)\,dt = 1$, we have

$$1 - K_n[1](x) = \int_{-c}^{c} R_n(t)\,dt - \int_{x-\frac{c}{2}}^{x+\frac{c}{2}} R_n(t)\,dt$$

$$= \int_{x+\frac{c}{2}}^{c} R_n(t)\,dt + \int_{-c}^{x-\frac{c}{2}} R_n(t)\,dt\,.$$

Consequently, for $|x| \leq \frac{c}{4}$ we have

$$|1 - K_n[1](x)| \leq \left(\int_{\frac{c}{4}}^{c} + \int_{-c}^{-\frac{c}{4}} \right) R_n(t)\,dt$$

$$\leq \frac{16}{c^2} \left(\int_{\frac{c}{4}}^{c} + \int_{-c}^{-\frac{c}{4}} \right) t^2 R_n(t)\,dt$$

and so

$$||K_n[1] - 1|| \leq \frac{16}{c^2} \int_{-c}^{c} t^2 R_n(t)\,dt\,.$$

Finally, for $|x| \leq \frac{c}{4}$ we have

$$0 \leq K_n[1](x) = \int_{x-\frac{c}{2}}^{x+\frac{c}{2}} R_n(t)\,dt \leq \int_{-c}^{c} R_n(t)\,dt = 1$$

and so $||K_n[1]|| \leq 1$.

Applying Lemma 6 we find that

$$||K_n[f] - f|| \leq 2\omega_f\left(\left(\int_{-c}^{c} t^2 R_n(t)\,dt\right)^{1/2}\right) + \frac{16}{c^2}\,||f|| \int_{-c}^{c} t^2 R_n(t)\,dt\,.$$

By Lemma 5 we have for all n sufficiently large the inequality

$$||K_n[f] - f|| \leq 2(1 + \sqrt{C})\,\omega_f\left(\frac{1}{n}\right) + (16\,C\,||f||\,c^{-2})\,\frac{1}{n^2}$$

and the rest of the proof follows from elementary properties of the modulus of continuity ω_f.

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