

VERY SPECIAL CASE

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **14 (1968)**

Heft 1: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **29.04.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

image sheaf of S of dimension l . Our main problem is to decide whether $\psi_{(l)}(S)$ is a coherent analytic sheaf of $\mathcal{O}_Y$ -modules if S is a coherent analytic sheaf on X .

A VERY SPECIAL CASE

We shall consider a special case where our main problem is easily solved. Let X_0 be a compact analytic manifold of pure dimension $m - n$. We put $E^n(\rho_0) = \{ (t_1 \dots t_n) \in \mathbb{C}^n ; |t_i| < \rho_i^0 \}$. Here $\rho_0 = (\rho_1^0 \dots \rho_n^0)$ is a fixed n -tuple of strictly positive numbers. Let $X = E^n(\rho_0) \times X_0$ and $X(\rho) = E^n(\rho) \times X_0$ for $\rho \leq \rho_0$. We see that X is an analytic manifold of pure dimension m . Let $\psi : X \rightarrow E^n(\rho_0)$ be the projection map. Now X is fibered by the fibers $\psi^{-1}(t) = X(t) = \{t\} \times X_0 \cong X_0$ for $t \in E^n(\rho_0)$. We take the sheaf S to be $S = (q\mathcal{O})_X$. With these notations we can state the following.

Theorem: The direct image sheaf $\psi_{(l)}((q\mathcal{O})_X)$ is a coherent sheaf of $\mathcal{O}_{E^n(\rho_0)}$ -modules for every $l \geq 0$.

Proof. Because X_0 is a compact analytic manifold we can find a finite Stein covering $\mathfrak{U} = \{U_1 \dots U_{i_*}\}$ of X_0 . Let us put $\hat{U}_i = E^n(\rho_0) \times U_i$, then we see that $\hat{\mathfrak{U}} = \{\hat{U}_1 \dots \hat{U}_{i_*}\}$ is a Stein covering of X . Let $\hat{\xi} = \{\hat{\xi}_{i_0 \dots i_l}\} \in C^l(\hat{\mathfrak{U}}, (q\mathcal{O})_X)$. Now $\hat{\xi}_{i_0 \dots i_l}$ is a q -tuple of holomorphic functions on $E^n(\rho_0) \times U_{i_0 \dots i_l}$. Hence $\hat{\xi}_{i_0 \dots i_l}$ admits a Taylor series of the form $\hat{\xi}_{i_0 \dots i_l} = \sum_{|v|=0}^{\infty} \xi_{i_0 \dots i_l}^{(v)} (t/\rho_0)^v$ where $v = (v_1, \dots, v_n)$, $|v| = v_1 + \dots + v_n$ and $(t/\rho)^v = (t_1/\rho_1)^{v_1} \dots (t_n/\rho_n)^{v_n}$. The uniqueness of a Taylor series shows that $\{\xi_{i_0 \dots i_l}^{(v)}\}$ is an alternating cochain over $\mathfrak{U}$. Putting $\xi_{(v)} = \{\xi_{i_0 \dots i_l}^{(v)}\} \in C^l(\mathfrak{U}, (q\mathcal{O})_{X_0})$ we may write $\hat{\xi} = \sum \xi_{(v)} (t/\rho)^v$. Introducing the map $(v) : \hat{\xi} \rightarrow \xi_{(v)}$ we get a commutative diagram of the form:

$$\begin{array}{ccc} C^l(\hat{\mathfrak{U}}, (q\mathcal{O})_X) & \xrightarrow{\delta} & C^{l+1}(\hat{\mathfrak{U}}, (q\mathcal{O})_X) \\ (v) \downarrow & & \downarrow (v) \\ C^l(\mathfrak{U}, (q\mathcal{O})_{X_0}) & \xrightarrow{\delta} & C^{l+1}(\mathfrak{U}, (q\mathcal{O})_{X_0}). \end{array}$$

We now need a *theorem of Cartan-Serre*: Let X_0 be a compact analytic manifold. Then, for any coherent analytic sheaf S the set $H^p(X_0, S)$ is a finite dimensional vector space for all $p \geq 0$.

Using this theorem we conclude that $H^l(X_0, (q\mathcal{O})_{X_0})$ has a finite base $\mathbf{b}_1 \dots \mathbf{b}_r$. By Leray's theorem we also have $H^l(\mathcal{U}, (q\mathcal{O})_{X_0}) \cong H^l(X_0, (q\mathcal{O})_{X_0})$. Hence we can find $\mathbf{b}_1 \dots \mathbf{b}_r \in Z^l(\mathcal{U}, (q\mathcal{O})_{X_0})$ such that $\mathbf{b}_v$ maps into $\mathbf{b}_v$ under the natural homomorphism $Z^l(\mathcal{U}, (q\mathcal{O})_{X_0}) \rightarrow H^l(X_0, (q\mathcal{O})_{X_0})$. We now introduce a pseudonorm in $C^l(\mathcal{U}, (q\mathcal{O})_{X_0})$ as follows:

Norm definition. Let $\eta \in C^l(\mathcal{U}, (q\mathcal{O})_{X_0})$. Then we put $\|\eta\| = \sup_{(i_0 \dots i_l)} \|\eta_{i_0 \dots i_l}\|$ and $\|\eta_{i_0 \dots i_l}\| = \max_{1 \leq q \leq q} \sup |\eta_q(U_{i_0 \dots i_l})|$, where, $\eta_{i_0 \dots i_l} = (\eta_1, \dots, \eta_q)$. Notice that it may happen that $\|\eta\| = +\infty$. Let $\mathfrak{B} = \{V_1 \dots V_{i^*}\}$ be an open covering of X_0 . The covering $\mathfrak{B}$ is much finer than $\mathcal{U} = \{U_1 \dots U_{i^*}\}$ if $V_i \subset \subset U_i$ holds for every i . We write $\mathfrak{B} \ll \mathcal{U}$ in that case. Let us now choose Stein coverings $\mathfrak{B}_1$ and $\mathfrak{B}$ such that $\mathfrak{B}_1 \ll \mathfrak{B} \ll \mathcal{U}$. In $C^l(\mathfrak{B}, (q\mathcal{O})_{X_0})$ and $C^l(\mathfrak{B}_1, (q\mathcal{O})_{X_0})$ we introduce a pseudonorm just as in $C^l(\mathcal{U}, (q\mathcal{O})_{X_0})$. If $\xi \in C^l(\mathcal{U}, (q\mathcal{O})_{X_0})$ we have defined $\xi|_{\mathfrak{B}} \in C^l(\mathfrak{B}, (q\mathcal{O})_{X_0})$. It follows that $\|\xi|_{\mathfrak{B}}\| < \infty$ because $V_{i_0 \dots i_l} \subset \subset U_{i_0 \dots i_l}$. Let us now choose $\xi \in Z^l(\mathfrak{B}, (q\mathcal{O})_{X_0})$. Since $\mathbf{b}_1 \dots \mathbf{b}_r \in Z^l(\mathcal{U}, (q\mathcal{O})_{X_0})$ constitute a base of $H^l(X_0, (q\mathcal{O})_{X_0})$ it follows from Leray's theorem that $\xi = \sum a_v \mathbf{b}_v|_{\mathfrak{B}} + \delta\eta$ where $a_v \in \mathbb{C}^1$ and $\eta \in C^{l-1}(\mathfrak{B}, (q\mathcal{O})_{X_0})$. Now we need the following.

Lemma: There exists a constant K such that $|a_v| \leq K \|\xi\|$ and $\|\eta|_{\mathfrak{B}_1}\| \leq K \|\xi\|$.

The proof follows because by the Banach theorem the map $(a_1, \dots, a_r, \eta) \rightarrow \xi$ of the Fréchet spaces $\mathbb{C}^r \times C^{l-1}(\mathfrak{B}, q\mathcal{O}_{X_0})$ onto $Z^l(\mathfrak{B}, (q\mathcal{O})_{X_0})$ is open.

Let $\xi \in C^l(\mathcal{U}, (q\mathcal{O})_{X_0})$. We can extend each $\xi_{i_0 \dots i_l} \in qI(U_{i_0 \dots i_l})$ constantly over $\hat{U}_{i_0 \dots i_l} = E^n(\rho_0) \times U_{i_0 \dots i_l}$. We get $\hat{\xi} \in Z^l(\hat{\mathcal{U}}, (q\mathcal{O})_X)$ obtained from ξ by a constant extension. In particular we extend $\mathbf{b}_1 \dots \mathbf{b}_r$ constantly to $\hat{\mathbf{b}}_1 \dots \hat{\mathbf{b}}_r \in Z^l(\hat{\mathcal{U}}, (q\mathcal{O})_X)$. Let $\hat{\mathbf{b}}_1 \dots \hat{\mathbf{b}}_r$ be the images of $\hat{\mathbf{b}}_1 \dots \hat{\mathbf{b}}_r$ in the direct image sheaf $\psi_{(1)}((q\mathcal{O})_X)$. Let now $\xi_0 \in \psi_{(1)}((q\mathcal{O})_X)_{(0)}$ where 0 is the origin of $E^n(\rho_0)$. By definition we can find $\xi \in H^l(X(\rho_1), q\mathcal{O})$ with $0 < \rho_1 \leq \rho_0$ which maps into ξ_0 . Now $\hat{\mathcal{U}}(\rho_1) = \{E^n(\rho_1) \times U_i\}$ is a Stein covering of $X(\rho_1)$. Hence Leray's theorem shows that we can find $\hat{\xi} \in Z^l(\hat{\mathcal{U}}(\rho_1), (q\mathcal{O})_X)$ such that $\hat{\xi}$ maps into ξ_0 . Let us write $\hat{\xi} = \sum \xi_{(v)}(t/\rho_1)^v$ where $\xi_{(v)} \in Z^l(\mathcal{U}, (q\mathcal{O})_{X_0})$. Let us also choose $0 < \rho_2 < \rho_1$ and consider $\hat{\xi}|_{\mathfrak{B}(\rho_2)} = \hat{\xi}_1 \in Z^l(\hat{\mathfrak{B}}(\rho_2), (q\mathcal{O})_X)$. Let us write $\hat{\xi}_1 = \sum \xi_{(v)}^*(t/\rho_2)^v$. Obviously we get $\xi_{(v)}^* = (\rho_2/\rho_1)^v \xi_{(v)}|_{\mathfrak{B}}$. It follows easily that $\sup_v \|\xi_{(v)}^*\| < \infty$.

The previous lemma shows that $\xi_{(v)}^* = \sum a_{v\lambda} \mathbf{b}_\lambda + \delta \eta_v$ where $\eta_v \in C^{l-1}(\mathfrak{B})$ with $\|\eta_v|_{\mathfrak{B}_1}\| \leq K \|\xi_{(v)}^*\|$ and $|a_{v\lambda}| \leq K \|\xi_{(v)}^*\|$. Let us put $a_\lambda = \sum a_{v\lambda} (t/\rho_2)^v$ and $\hat{\eta} = \sum \eta_v (t/\rho_2)^v$. We see that $\hat{\eta} \in C^{l-1}(\hat{\mathfrak{B}}_1(\rho_2))$ and $a_\lambda \in I(E^n(\rho_2))$. An easy computation gives $\hat{\xi}_1|_{\hat{\mathfrak{B}}_1(\rho_2)} = \sum a_\lambda \hat{\mathbf{b}}_\lambda|_{\hat{\mathfrak{B}}_1(\rho_2)} + \delta \hat{\eta}$. It follows by definition that $\hat{\xi}_0 = \sum a_\lambda \hat{\mathbf{b}}_\lambda$. We have now proved that $\hat{\mathbf{b}}_1 \dots \hat{\mathbf{b}}_r$ generate $\psi_{(l)}((q\mathcal{O})_X)$ at the origin. It follows in the same way that $\hat{\mathbf{b}}_1 \dots \hat{\mathbf{b}}_r$ generate $\psi_{(l)}((q\mathcal{O})_X)$ for every $t \in E^n(\rho_0)$ because it is enough to do everything in a polydisc around t . Now we also prove that the sheaf $\psi_{(l)}((q\mathcal{O})_X)$ is free, i.e. there are no relations between $\hat{\mathbf{b}}_1 \dots \hat{\mathbf{b}}_r$ at any point. Say for example that $a_1 \hat{\mathbf{b}}_1 + \dots + a_r \hat{\mathbf{b}}_r = 0$ at $\psi_{(l)}((q\mathcal{O})_X)_{(0)}$ where a_i are germs of analytic functions at the origin in $E^n(\rho_0)$. Hence $\tilde{a}_1 \hat{\mathbf{b}}_1 + \dots + \tilde{a}_r \hat{\mathbf{b}}_r = 0$ in $H^l(X(\rho), (q\mathcal{O})_X)$ for some $\rho > 0$ with $\tilde{a}_i \in I(E^n(\rho))$. It follows that $\sum \tilde{a}_v \hat{\mathbf{b}}_v = \delta \hat{\xi}$ in $X(\rho)$ for some $\hat{\xi} \in C^{l-1}(\hat{\mathcal{U}}(\rho), (q\mathcal{O})_X)$. Take a point $t \in E^n(\rho)$ where some $\tilde{a}_v \neq 0$. Now we see that on $\{t\} \times X_0$ we have $\tilde{a}_1(t) \mathbf{b}_1 + \dots + \tilde{a}_r(t) \mathbf{b}_r = \partial \hat{\xi}|_{\{t\} \times X_0} \in C^{l-1}(\mathcal{U}, (q\mathcal{O})_{X_0})$. This gives a contradiction to the fact that $\mathbf{b}_1 \dots \mathbf{b}_r$ are a base of $H^l(X_0, (q\mathcal{O})_{X_0})$.

MEASURE CHARTS

Let X be a connected complex analytic manifold of dimension m . Let F be a holomorphic vector bundle of rank q on X and $\mathbf{F}$ the sheaf of holomorphic crosssections in F . This sheaf is locally free. A regular proper holomorphic map $\psi: X \rightarrow E^n$ is given. Let us put $X_0 = \psi^{-1}(0)$. Now X_0 is a compact analytic manifold of dimension $m - n$. We now introduce special open coverings around X_0 in X .

Definition. A measure chart $\mathcal{W} = (\hat{W}, \Phi, \Theta, \rho)$ is a quadruple satisfying the conditions:

- 1) $\hat{W} \subset X$ is open and $W = \hat{W} \cap X_0$ is Stein.
- 2) $\Phi: \hat{W} \rightarrow E^n(\rho) \times W$ is a biholomorphic map such that the following diagram is commutative: