spread holomorphically, i.e. for any $x \in X$ there exists $f_1 \dots f_N \in I(X)$ such that x is an isolated common zero of $f_1 \dots f_N$.

Let X be a complex analytic manifold. A Stein covering $\mathfrak{U} = \{ U_\iota \}_{\iota \in J}$ of X is an open covering of X such that every U_ι is Stein. We shall often use the following result:

Leray's Theorem: If $\mathfrak{U}$ is a Stein covering of X then $H^l(\mathfrak{U}, S) \to H^l(X, S)$ is an isomorphism for every coherent analytic sheaf S.

The isomorphism between $H^l(\mathfrak{U}, S)$ and $H^l(X, S)$ means the following: If $\underline{\xi} \in H^l(X, S)$ there exists $\xi \in Z^l(\mathfrak{U}, S)$ such that ξ maps into $\underline{\xi}$ under the natural homomorphism $Z^l(\mathfrak{U}, S) \to H^l(X, S)$ and moreover if $\xi \in Z^l(\mathfrak{U}, S)$ is mapped into zero in $H^l(X, S)$ there exist $\eta \in C^{l-1}(\mathfrak{U}, S)$ such that $\xi = \delta\eta$ in $C^l(\mathfrak{U}, S)$.

Direct images of sheaves

Let X and Y be complex analytic manifolds. Let $\psi : X \to Y$ be a holomorphic map and let S be an analytic sheaf on X. Now X is fibered by the fibers $X(y) = \psi^{-1}(y)$ for $y \in Y$. Let U be an open neighborhood of a point $y \in Y$, then $V = \psi^{-1}(U)$ is an open set in X. Hence V is a complex analytic manifold and the restriction of S to V gives an analytic sheaf on V. We can now define $H^l(V, S)$. Let us put $H^l_y = \bigcup_U H^l(\psi^{-1}(U), S)$ where U runs over all open neighborhoods of y in Y. In H^l_y we introduce an equivalence relation as follows: $\xi_1 \in H^l(\psi^{-1}(U_1), S)$ and $\xi_2 \in H^l(\psi^{-1}(U_2), S)$ are equivalent iff there exists $U = U(y)$ in Y such that $U \subset U_1 \cap U_2$ and $\xi_1 | \psi^{-1}(U) = \xi_2 | \psi^{-1}(U)$ in $H^l(\psi^{-1}(U), S)$. We let $\psi_{(l)}(S)_{(y)}$ denote the set of equivalence classes in H^l_y. The equivalence class generated by $\xi \in H^l(\psi^{-1}(U), S)$ is denoted by ξ_y. The set $\psi_{(l)}(S)_{(y)}$ is called the set of germs of cohomology classes of dimension l along the fiber $X(y)$. Now $\psi_{(l)}(S)_{(y)}$ is an $\mathcal{O}_{y,Y}$-module. For if $g_y \in \mathcal{O}_{y,Y}$ we have a representative $g \in I(U)$ for some open neighborhood U of y. Then $g \circ \psi \in I(\psi^{-1}(U))$. If $\xi_y \in \psi_{(l)}(S)_{(y)}$ and U is sufficiently small we can find a representative $\xi \in H^l(\psi^{-1}(U), S)$ for ξ_y. Then we put $g_y \cdot \xi_y = ((g \circ \psi)\xi)_y$. Now we form $\psi_{(l)}(S) = \bigcup_{y \in Y} \psi_{(l)}(S)_{(y)}$ where we introduce a sheaf topology. A base of the open sets are $\{ \xi_y : y \in U \}$ for $\xi \in H^l(\psi^{-1}(U), S)$. If $\xi \in H^l(X, S)$ then the map $y \to \xi_y$ is a cross-section in $\psi_{(l)}(S)$. We call it the direct image of ξ and denote it by $\psi_{(l)}(\xi)$. The sheaf $\psi_{(l)}(S)$ is the direct image sheaf of S of dimension l. Our main problem is to decide whether $\psi_{(l)}(S)$ is a coherent analytic sheaf of $\mathcal{O}_Y$-modules if S is a coherent analytic sheaf on X.

A very special case

We shall consider a special case where our main problem is easily solved. Let X_0 be a compact analytic manifold of pure dimension $m - n$. We put $E^n(\rho_0) = \{ (t_1 \dots t_n) \in C^n \,;\, | t_i | < \rho_i^0 \}$. Here $\rho_0 = (\rho_1^0 \dots \rho_n^0)$ is a fixed n-tuple of strictly positive numbers. Let $X = E^n(\rho_0) \times X_0$ and $X(\rho) = E^n(\rho) \times X_0$ for $\rho \leqslant \rho_0$. We see that X is an analytic manifold of pure dimension m. Let $\psi : X \to E^n(\rho_0)$ be the projection map. Now X is fibered by the fibers $\psi^{-1}(t) = X(t) = \{ t \} \times X_0 \cong X_0$ for $t \in E^n(\rho_0)$. We take the sheaf S to be $S = (q\mathcal{O})_X$. With these notations we can state the following.

Theorem: The direct image sheaf $\psi_{(l)}((q\mathcal{O})_X)$ is a coherent sheaf of $\mathcal{O}_{E^n(\rho_0)}$-modules for every $l \geqslant 0$.

Proof. Because X_0 is a compact analytic manifold we can find a finite Stein covering $\mathfrak{U} = \{ U_1 \dots U_{\iota^*} \}$ of X_0. Let us put $\hat{U}_\iota = E^n(\rho_0) \times U_\iota$, then we see that $\hat{\mathfrak{U}} = \{ \hat{U}_1 \dots \hat{U}_{\iota^*} \}$ is a Stein covering of X. Let $\hat{\xi} = \{ \hat{\xi}_{\iota_0 \dots \iota_l} \} \in C^l(\hat{\mathfrak{U}}, (q\mathcal{O})_X)$. Now $\hat{\xi}_{\iota_0 \dots \iota_l}$ is a q-tuple of holomorphic functions on $E^n(\rho_0) \times U_{\iota_0 \dots \iota_l}$. Hence $\hat{\xi}_{\iota_0 \dots \iota_l}$ admits a Taylor series of the form $\hat{\xi}_{\iota_0 \dots \iota_l} = \sum_{|\nu|=0}^{\infty} \xi^{(\nu)}_{\iota_0 \dots \iota_l} (t/\rho_0)^\nu$ where $\nu = (\nu_1, \dots, \nu_n)$, $| \nu | = \nu_1 + \dots + \nu_n$ and $(t/\rho)^\nu = (t_1/\rho_1)^{\nu_1} \dots (t_n/\rho_n)^{\nu_n}$. The uniqueness of a Taylor series shows that $\{ \xi^{(\nu)}_{\iota_0 \dots \iota_l} \}$ is an alternating cochain over $\mathfrak{U}$. Putting $\xi_{(\nu)} = \{ \xi^{(\nu)}_{\iota_0 \dots \iota_l} \} \in C^l(\mathfrak{U}, (q\mathcal{O})_X)$ we may write $\hat{\xi} = \sum \xi_{(\nu)} (t/\rho)^\nu$. Introducing the map $(\nu) : \hat{\xi} \to \xi_{(\nu)}$ we get a commutative diagram of the form:

$$\begin{array}{ccc} C^l(\hat{\mathfrak{U}}, (q\mathcal{O})_X) & \xrightarrow{\delta} & C^{l+1}(\hat{\mathfrak{U}}, (q\mathcal{O})_X) \\ (\nu)\downarrow & & \downarrow(\nu) \\ C^l(\mathfrak{U}, (q\mathcal{O})_{X_0}) & \xrightarrow{\delta} & C^{l+1}(\mathfrak{U}, (q\mathcal{O})_{X_0}) . \end{array}$$

We now need a *theorem of Cartan-Serre:* Let X_0 be a compact analytic manifold. Then, for any coherent analytic sheaf S the set $H^p(X_0, S)$ is a finite dimensional vector space for all $p \geqslant 0$.