6. The Atiyah-Hodge theorem

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where f is the meromorphic function determined by s_a by the procedure described just before Th. 4.1.

Now let X be a compact submanifold of $\mathbf{P}^n$ and consider hyperplanes H_c in $\mathbf{P}^n$, given in homogeneous coordinates $z_0, \ldots, z_n$ by equations

$$\sum_0^n c_j z_j = 0 \text{ where } c = (c_0, \ldots, c_n) \neq 0 .$$

Theorem 5.2. There is an open dense set Ω in $\mathbf{C}^{n+1}$ such that if $c = (c_0, \ldots, c_n) \in \Omega$, the hyperplane section $D_c = H_c \cap X$ is a non-singular analytic subset of X.

The proof is omitted here.

Let $D = H \cap X$ be a non-singular hyperplane section of X. To D is then associated a positive line bundle F on X (see Sect. 4). By Kodaira's vanishing theorem there is a k_0 such that

$$H^q(X, \Omega^p \otimes \underline{F}^k) = 0 , \qquad (\forall q \geq 1, \forall k \geqq k_0) .$$

Using the isomorphism in Lemma 5.1, we have therefore proved.

Lemma 5.3. If D is a non-singular hyperplane section of a compact submanifold X of $\mathbf{P}^n$, then there exists k_0 such that

$$H^q(X, \Omega^p(k, D)) = 0 , \qquad (\forall q \geq 1, \forall k \geq k_0) .$$

6. The Atiyah-Hodge theorem

We first recall two well-known theorems.

Let X be a paracompact C^∞ manifold and let $\mathscr{E}^p$ be the sheaf of germs of C^∞ p-forms on X ($p=0, 1, \ldots$).

Then the sequence

$$0 \to \mathbf{C} \xrightarrow{i} \mathscr{E}^0 \xrightarrow{d} \mathscr{E}^1 \xrightarrow{d} \mathscr{E}^2 \xrightarrow{d} \cdots \tag{6.1}$$

is exact (Poincaré's lemma), and

$$H^q(X, \mathscr{E}^p) = 0 , \qquad (\forall q \geq 1, \forall p \geqq 0) , \tag{6.2}$$

because the $\mathscr{E}^p$ are fine sheaves, i.e. they have partitions of unity. From (6.1) we get the sequence

$$0 \to \Gamma(X, \mathscr{E}^0) \to \Gamma(X, \mathscr{E}^1) \to \ldots ,$$

which need not be exact. Put

$$H^p(\mathscr{E}) = \frac{\mathrm{Ker}\,(\Gamma(X, \mathscr{E}^p) \to \Gamma(X, \mathscr{E}^{p+1}))}{\mathrm{Im}\,(\Gamma(X, \mathscr{E}^{p-1}) \to \Gamma(X, \mathscr{E}^p))}, \qquad (p \geq 0), \tag{6.3}$$

with $\mathscr{E}^{-1} = 0$. Then one has the following theorem of de Rham:

Theorem 6.1. There are natural isomorphisms

$$H^p(X, \mathbf{C}) \simeq H^p(\mathscr{E}), \qquad (p \geq 0).$$

If X is a Stein manifold and Ω^p the sheaf of germs of holomorphic p-forms on X (p=0, 1, ...), then the sequence

$$0 \to \mathbf{C} \xrightarrow{i} \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \to \dots \tag{6.4}$$

is exact (Gnothendieck's lemma), and

$$H^q(X, \Omega^p) = 0, \qquad (\forall q \geq 1, \forall p \geq 0) \tag{6.5}$$

(Cartan's Theorem B). Put

$$H^p(\Omega) = \frac{\mathrm{Ker}\,(\Gamma(X, \Omega^p) \to \Gamma(X, \Omega^{p+1}))}{\mathrm{Im}\,(\Gamma(X, \Omega^{p-1}) \to \Gamma(X, \Omega^p))}, \qquad (p \geq 0)$$

with $\Omega^{-1} = 0$. Then one has the following theorem

Theorem 6.2. There are natural isomorphisms

$$H^p(X, \mathbf{C}) \simeq H^p(\Omega), \qquad (p \geq 0).$$

Theorems 6.1 and 6.2 both follow if one applies the following lemma to the exact sequences (6.1) and (6.4), respectively:

Lemma 6.3. Let X be a paracompact Hausdorff space and

$$0 \to F \xrightarrow{i} F_0 \xrightarrow{d_0} F_1 \xrightarrow{d_1} \cdots \tag{6.6}$$

an exact sequence of sheaves of abelian groups, such that

$$H^q(X, F_p) = 0, \qquad (\forall q \geq 1, \forall p \geq 0). \tag{6.7}$$

Then there are natural isomorphisms

$$H^p(X, F) \simeq \mathrm{Ker}\, d_p^* / \mathrm{Im}\, d_{p-1}^*, \qquad (p \geq 0),$$

where d_p^* is the mapping $\Gamma(X, F_p) \to \Gamma(X, F_{p+1})$ induced by (6.6) (with F_{-1}=0).

Proof. Put $Z_p = \mathrm{Ker}\, d_p \subset F_p$. Then the exactness of (6.6) gives short exact sequences

$$0 \to Z_{p-1} \to F_{p-1} \to Z_p \to 0\,, \qquad (p \geq 1)\,, \tag{6.8}$$

from which we get long exact sequences of cohomology groups, which we write in part:

$$H^q(X, F_{p-1}) \to H^q(X, Z_p) \to H^{q+1}(X, Z_{p-1}) \to H^{q+1}(X, F_{p-1})\,,$$
$$(q \geq 0,\ p \geq 1)\,. \tag{6.9}$$

When $q \geqslant 1$, we get from (6.7) and (6.9)

$$H^q(X, Z_p) \simeq H^{q+1}(X, Z_{p-1})\,, \qquad (p \geq 1)\,.$$

Since F is isomorphic to Z_0, we therefore have

$$H^p(X, F) \simeq H^{p-1}(X, Z_1) \simeq \ldots \simeq H^1(X, Z_{p-1})\,, \quad (p \geq 1)\,. \tag{6.10}$$

When $q = 0$, (6.9) gives an exact sequence

$$\Gamma(X, F_{p-1}) \xrightarrow{d_{p-1}} \Gamma(X, Z_p) \to H^1(X, Z_{p-1}) \to 0\,,$$

and thus

$$H^1(X, Z_{p-1}) \simeq \Gamma(X, F_{p-1})/\mathrm{Im}\, d^*_{p-1} = \mathrm{Ker}\, d^*_p/\mathrm{Im}\, d^*_{p-1}\,,$$

which together with (6.10) proves the lemma when $p \geqslant 1$.

To prove it for $p = 0$, we observe that the exact sequence

$$0 \to F = Z_0 \to F_0 \to Z_1 \to 0$$

gives an exact sequence

$$0 \to \Gamma(X, F) \to \Gamma(X, F_0) \xrightarrow{d^*_0} (X, Z_1)$$

and thus

$$H^0(X, F) = \Gamma(X, F) \simeq \mathrm{Ker}\, d^*_0 = \mathrm{Ker}\, d^*_0/\mathrm{Im}\, d^*_{-1}\,.$$

Now let V be a compact submanifold of $\mathbf{P}^n$ and D a non-singular hyperplane section of V. Then $X = V - D$ is imbedded as a closed submanifold of $\mathbf{C}^n$, and in particular it is a Stein manifold.

Let $\Omega^p(D) = \Omega^p(V, D)$ be the sheaf of germs of meromorphic p-forms on V with poles only on D, $p = 0, 1, \ldots$. Then we have a sequence (not necessarily exact)

$$0 \to \mathbf{C} \to \Omega^0(D) \xrightarrow{d'_0} \Omega^1(D) \xrightarrow{d'_1} \ldots\,.$$

Define

$$\tilde{H}^p = \mathrm{Ker}\, d'^*_p/\mathrm{Im}\, d'^*_{p-1}\,, \qquad (p \geq 0)\,,$$

where d'^*_p is the induced mapping $\Gamma(V, \Omega^p(D)) \to \Gamma(V, \Omega^{p+1}(D))$, (with $\Omega^{-1}(D) = 0$). We shall prove the following theorem of Atiyah and Hodge:

Theorem 6.4. There are natural isomorphisms

$$H^p(X, \mathbf{C}) \simeq \tilde{H}^p, \qquad (p \geq 0).$$

Proof. Let $\mathscr{E}^p(D) = \mathscr{E}^p(V, D)$ be the sheaf on V, which is defined by the presheaf that to every open subset U of V associates the module of C^∞ p-forms on $U - D$. Then we have a commutative diagram

$$\begin{array}{ccccc} 0 \to \Omega^0(D) & \xrightarrow{d'_0} & \Omega^1(D) & \xrightarrow{d'_1} & \cdots \\ \downarrow & & \downarrow & & \\ 0 \to \mathscr{E}^0(D) & \xrightarrow{d_0} & \mathscr{E}^1(D) & \xrightarrow{d_1} & \cdots, \end{array} \tag{6.11}$$

where the vertical mappings are the inclusions.

For every p, we can regard $\Omega^p(D)$ as the direct limit of $\Omega^p(k, D) = \Omega^p(V, k, D)$ as $k \to \infty$. Now, by Lemma 5.3, there is a k_0 such that $H^q(V, \Omega^p(k, D)) = 0$ for $q \geqslant 1$ and $k \geqslant k_0$. Hence we can conclude that

$$H^q(V, \Omega^p(D)) = 0, \qquad (\forall q \geq 1, \forall p \geq 0). \tag{6.12}$$

We also have

$$H^q(V, \mathscr{E}^p(D)) = 0, \qquad (\forall q \geq 1, \forall p \geq 0), \tag{6.13}$$

because $\mathscr{E}^p(D)$ are fine sheaves.

From (6.11) we get a diagram

$$\begin{array}{ccccc} 0 \to \Gamma(V, \Omega^0(D) & \xrightarrow{d_0'^*} & \Gamma(V, \Omega^1(D)) & \xrightarrow{d_1'^*} & \cdots \\ \downarrow & & \downarrow & & \\ 0 \to \Gamma(V, \mathscr{E}^0(D)) & \xrightarrow{d_0^*} & \Gamma(V, \mathscr{E}^1(D)) & \xrightarrow{d_1^*} & \cdots \end{array} \tag{6.14}$$

The cohomology groups of the upper row in (6.14) are $\tilde{H}^p$, $(p{=}0, 1, \ldots)$, and those of the lower row are the groups $H^p(\mathscr{E})$ in (6.3), because one can obviously identify $\Gamma(V, \mathscr{E}^p(D))$ with $\Gamma(X, \mathscr{E}^p)$. In view of de Rham's theorem, it is therefore sufficient to prove that the vertical mappings in (6.14) induce isomorphisms between the cohomology groups of the rows.

To do this, we will use the following theorem:

Theorem 6.5. Let X be a paracompact Hausdorff space and suppose that two complexes $\mathscr{E}$ and $\mathscr{E}'$ of sheaves over X are given, together with mappings h such that the diagram

$$\begin{array}{ccccccc} 0 \xrightarrow{d'_{-1}} & \mathscr{E}'_0 & \xrightarrow{d'_0} & \mathscr{E}'_1 & \xrightarrow{d'_1} & \mathscr{E}'_2 & \to \cdots \\ & {\scriptstyle h_0}\downarrow & & {\scriptstyle h_1}\downarrow & & {\scriptstyle h_2}\downarrow & \\ 0 \xrightarrow[d_{-1}]{} & \mathscr{E}_0 & \xrightarrow[d_0]{} & \mathscr{E}_1 & \xrightarrow[d_1]{} & \mathscr{E}_2 & \to \cdots \end{array} \tag{6.15}$$

is commutative. (The rows are not supposed to be exact, but we have $dd = 0$ and $d'd' = 0$.)

Suppose further that

$$H^q(X, \mathscr{E}_k) = 0 \text{ and } H^q(X, \mathscr{E}_k') = 0\,, \quad (\forall q \geq 1\ \forall k \geq 0)\,, \tag{6.16}$$

and that for all $k \geqslant 0$, h induces *isomorphisms* of the cohomology sheaves

$$h_k : \operatorname{Ker} d'_k / Im\, d'_{k-1} \to \operatorname{Ker} d'_k / Im\, d'_{k-1}\,. \tag{6.17}$$

Then it follows that h induces *isomorphisms* for all $k \geqslant 0$:

$$h_k^* : \operatorname{Ker} d'^*_k / \operatorname{Im} d'^*_{k-1} \to \operatorname{Ker} d'^*_k / \operatorname{Im} d'^*_k\,, \tag{6.18}$$

where d^* and d'^* are the mappings induced by d and d' between the groups of global sections of the given sheaves:

$$\begin{array}{ccccccc} 0 \to \Gamma(X, \mathscr{E}'_0) & \xrightarrow{d'_0{}^*} & \Gamma(X, \mathscr{E}'_1) & \xrightarrow{d'_1{}^*} & (X, \mathscr{E}'_2) & \xrightarrow{d'_2{}^*} & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 \to \Gamma(X, \mathscr{E}_0) & \xrightarrow{d_0{}^*} & \Gamma(X, \mathscr{E}_1) & \xrightarrow{d_1{}^*} & \Gamma(X, \mathscr{E}_2) & \xrightarrow{d_2{}^*} & \cdots \end{array} \tag{6.19}$$

Proof of Theorem 6.5: Taking $F = 0$ in Lemma 6.3, we see that exactness of a sequence

$$0 \to F_0 \xrightarrow{\delta_0} F_1 \xrightarrow{\delta_1} \cdots \tag{6.20}$$

together with the conditions (6.7) implies exactness of the sequence

$$0 \to \Gamma(X, F_0) \xrightarrow{\delta_0{}^*} \Gamma(X, F_1) \xrightarrow{\delta_1{}^*} \cdots. \tag{6.21}$$

With the help of the " mapping cylinder " construction we will reduce the proof of Theorem 6.5 to an application of this fact. We define the sheaves and mappings in (6.20) as follows (where we take $\mathscr{E}_{-1} = 0$).

$$F_k = \mathscr{E}'_k \oplus \mathscr{E}_{k-1}\,; \qquad \delta_k(a', a) = \left(d'_k a', d_{k-1} a + (-1)^k h_k a'\right).$$

Since (6.7) follows from (6.16), it is enough to prove that the fact that (6.17) are isomorphisms for all $k \geqslant 0$ implies that (6.20) is exact, and that the exactness of (6.21) implies that (6.18) are isomorphisms. But we see that (6.21) is obtained from (6.19) by the same construction which lead from (6.15) to (6.20). Thus the proof of Theorem 6.5 will be complete if we apply the following lemma in one direction to (6.15) and (6.20) and in the other direction to (6.19) and (6.21).

Lemma 6.6. Let (6.15) be any diagram of the type considered above (with no condition (6.16) supposed) and such that (6.17) are isomorphisms, and let (6.20) be the corresponding sequence given by the above construction. Then (6.18) are isomorphisms if and only if (6.20) is exact.

Proof of Lemma 6.6. By straightforward calculation we see that $\delta\delta = 0$. Clearly h_k^* is injective if and only if

(i) For every $a' \in \mathscr{E}_k'$ and $a \in \mathscr{E}_{k-1}$ with $d'a' = 0$ and $ha' = da$ there exists $b' \in \mathscr{E}_{k-1}$ with $a' = d'b'$.

Similarly, h_{k-1}^* is surjective if and only if

(ii) For every $b \in \mathscr{E}_{k-1}$ with $db = 0$ there exist $f' \in \mathscr{E}'_{k-1}$ and $c \in \mathscr{E}_{k-2}$ with $d'f' = 0$ and $dc = b - hf'$.

Finally we want to express in a similar way the condition that (6.20) is exact at F_k. If $\alpha \in F_k$ and $\delta\alpha = 0$, the condition is that $\alpha = \delta\gamma$ for some $\gamma \in F_{k-1}$. To get rid of the signs we write $\alpha = ((-1)^{k-1} a', a)$ and $\gamma = ((-1)^{k-1} c', c)$. Then the condition may be written:

(iii) For every $a' \in \mathscr{E}_k'$ and $a \in \mathscr{E}_{k-1}$ with $d'a' = 0$ and $ha' = da$, there exist $c' \in \mathscr{E}_{k-1}'$ and $c \in \mathscr{E}_{k-2}$ such that $d'c' = a'$ and $dc = a - hc'$.

Trivially, (iii) $\Rightarrow$ (i). Taking $a' = 0$ and $a =$ b, we see that (iii) $\Rightarrow$ (ii). To complete the proof we will then assume that (i) and (ii) holds and prove (iii).

Let a' and a be as in (iii). From (i) we get b'. Then $d(a-hb') = da - ha' = 0$ by hypothesis. Apply (ii) with $b = a - hb'$ and define $c' = b' - f'$. Then $d'c' = a'$ and $a\text{-}hc' = a - hb' - hf' = dc$, which completes the proof of Theorem 6.5.

Continuation of the proof of Theorem 6.4. It only remains to prove that we may take $\mathscr{E}' = \Omega(V, D)$ and $\mathscr{E} = \mathscr{E}(V, D)$ in Theorem 6.5. In view of (6.12) and (6.13), it suffices to check that the mappings (6.17) are isomorphisms for all $k \geqslant 0$.

At any point of $V - D$, both cohomologies are trivial, and there is nothing to prove. Thus it only remains to consider points in D. Let us choose a neighbourhood U of such a point a and local coordinates $(z_1, ..., z_n)$ in U in such a way that U is the polycylinder given by $|z_i| < 1$, $(i=1, 2, ..., n)$, $U \cap D$ is the part of U where $z_1 = 0$, and a is the point where all $z_i = 0$. Now $U - D = (E - \{0\}) \times E^{n-1}$, where E is the open unit disk in $\mathbf{C}$. Since the second factor is contractible, the mapping $(E - \{0\}) \times E^{n-1} \to E - \{0\}$ induces isomorphisms of $\mathscr{H}'^k = \operatorname{Ker} d'_k / \operatorname{Im} d'_{k-1}$. Thus, by de Rham's theorem, $\mathscr{H}'^k = 0$ if $k \geqslant 2$, and $\dfrac{dz_1}{z_1}$ forms a basis for $\mathscr{H}'^1$. We claim that the same is true for $\mathscr{H}^k = \mathrm{Ker} d_k / \mathrm{Im} d'_{k-1}$. Since h is the natural inclusion, this would complete the proof.

All forms considered in the sequel are meromorphic in U and have poles at most on D. If $\gamma = \Sigma a_{i_1 \ldots i_k} dz_{i_1} \wedge \ldots \wedge dz_{i_k}$ is a k-form, we set $\frac{\partial \gamma}{\partial z_\nu} = \sum \frac{\partial a_{i_1 \cdots i_k}}{\partial z_\nu} dz_{i_1} \wedge \ldots \wedge dz_{i_k}$. Then $d\gamma = \sum_{\nu=1}^{n} dz_\nu \wedge \frac{\partial \gamma}{\partial z_\nu}$.

We also introduce the norm $|\gamma| = \sup |a_{i_1 \cdots i_k}|$. If γ does not involve dz_1, we define $\delta\gamma = \sum_{\nu=2}^{n} dz_\nu \wedge \frac{\partial \gamma}{\partial z_\nu}$. We will need the following lemma.

Lemma 6.8. If γ is a k-form ($k \geqslant 1$) not involving dz_1, and if $\delta\gamma = 0$, then there exists a form γ' not involving dz_1 and such that $\delta\gamma' = \gamma$.

Proof of Lemma 6.8. We first suppose that γ is a holomorphic. Then we have $\gamma = \sum_{\nu \geqq 0} z_1^\nu \beta_\nu$ and $0 = \Sigma z_1^\nu \delta \beta_\nu$ with convergence for $|z_1| < 1$. Thus for any $\varrho > 1$ we have $|\beta_\nu| \leqslant C \varrho^\nu$.

By the ordinary lemma in a polydisk, there exists β_ν' such that $\beta_\nu = \delta\beta_\nu'$. The mapping $\beta_\nu \to \beta_\nu'$ is a mapping onto the Fréchet space of all closed $(k-1)$-forms. Thus, by the open mapping theorem, we see that the equation $\delta\beta_\nu' = \beta_\nu$ has a solution β_ν' with $|\beta_\nu'| \leqslant C' \varrho^\nu$ on any smaller polydisk P, (C' being a constant which may depend on Thus $\gamma' = \sum_{\nu \geqq 0} z_1{}^\nu \beta_\nu'$ is convergent in $|z_1| < \frac{1}{\varrho}$, which proves the lemma in the holomorphic case. In the general case we have $\gamma = \sum_{i=0}^{k} z_1{}^{-i} \gamma_i$ with holomorphic forms γ_i. We apply the first case to the γ_i and get $\gamma' = \sum_{i=0}^{k} z_1{}^{-i} \gamma_i$ which completes the proof of the lemma.

End of proof of Theorem 6.4. Let ω be any k-form. Then we may write $\omega = dz_1 \wedge \alpha + \beta$, where α and β do not involve dz_1. Suppose now that $d\omega = 0$. This condition takes the form

$$dz_1 \wedge \delta\alpha + dz_1 \wedge \frac{\partial \beta}{\partial z_1} + \delta\beta = 0 , \tag{6.22}$$

which implies that $\delta\beta = 0$. By Lemma 6.8, we have $\beta = \delta\beta'$ for some $(k-1)$-form β'.

Now ω takes the form

$$\omega - d\beta' = dz_1 \wedge \alpha' . \tag{6.23}$$

We distinguish the two cases $k > 1$ and $k = 1$. In the first case we get from (6.23)

$$dz_1 \wedge \delta\alpha' = 0 ,$$

which implies that $\delta\alpha' = 0$. Since α' is a form of type $q-1 \geqslant 1$, we can apply once again Lemma 6.8 and get $\alpha' = \delta\alpha''$. Thus $dz_1 \wedge \alpha' = d(dz_1 \wedge \alpha'')$, and we get $\omega = d(\beta' + dz_1 \wedge \alpha'')$. This proves that the cohomology under consideration is trivial for $k > 1$.

Finally, in the case $k = 1$, α' is a meromorphic function, independent of $z_2, ..., z_n$. Thus by (6.23), $\omega = d\gamma$ for some γ if and only if in the Laurent expansion of α' the coefficient of $z_1{}^{-1}$ is zero. Thus the cohomology in dimension 1 is generated by $z_1{}^{-1}\, dz_1$, which completes the proof of Theorem 6.4.

7. Lefschetz' theorem on hyperplane sections

The Lefschetz theorem in the slightly more general setting proved by Andreotti and Frankel [1], is the following:

Theorem 7.1. Let V be a submanifold of $\mathbf{P}^n$ of complex dimension d and let D be a hyperplane section of V (not necessarily non-singular). Then there are natural isomorphisms

$$H^q(V, \mathbf{Z}) \simeq H^q(D, \mathbf{Z}) , \qquad (\forall q < d-1) ,$$

and a natural injection

$$H^{d-1}(V, \mathbf{Z}) \to H^{d-1}(D, \mathbf{Z}) .$$

Proof. $X = V - D$ is a Stein manifold, since it is imbedded as a closed submanifold of $\mathbf{C}^n$. Now one knows that

$$H^q(V, D, \mathbf{Z}) \simeq H^q_c(X, \mathbf{Z}) , \tag{7.1}$$

where the c indicates cohomology with compact support. On the other hand, since X is a topological manifold of dimension $2\,d$, Poincaré duality gives

$$H^q_c(X, \mathbf{Z}) \simeq H_{2d-q}(X, \mathbf{Z}) . \tag{7.2}$$

Now we shall use the following theorem: