

§2. Privileged polycylinders

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$$g|_W : \begin{cases} F_2 \rightarrow 0 \\ G_2 W \simeq F_3 \end{cases}.$$

If $p: E_{2W} \rightarrow F_2$ is the projection with kernel G_{2W} , the map, $p \circ f: E_{1W} \rightarrow F_2$ is a split epimorphism in x_0 . Again by prop. 2 we have over an open neighbourhood $U \subset W$ of x_0 a decomposition $E_{1U} = F_1 \oplus G_{1U}$ (with $F_1 = \text{Ker } p \circ f$)

$$(p \circ f)|_U : \begin{cases} F_1 \rightarrow 0 \\ G_{1U} \xrightarrow{\sim} F_{2U} \end{cases}.$$

The image $f|_U(F_1)$ is contained in G_{2U} . But $g|_U \circ f|_U = 0$ and $g|_{G_{2U}}$ is a monomorphism hence $f|_U: F_1 \rightarrow 0$. We get finally (restricting all our morphisms to U)

$$f|_U : \begin{cases} F_{1U} \rightarrow 0 \\ G_{1U} \simeq F_{2U} \end{cases} \quad g|_U : \begin{cases} F_{2U} \rightarrow 0 \\ G_{2U} \xrightarrow{\sim} F_{3U} \end{cases}.$$

§ 2. Privileged polycylinders

Definition 1: A polycylinder in $\mathbb{C}^n$ is a compact set K of the form $K = K_1 \times \dots \times K_n$ where each K_i is a compact, convex subset of $\mathbb{C}$, with nonempty interior. If each K_i is a disc, then K is a polydisc. We first recall the following theorem of Cartan.

Theorem 1: Let K be a polycylinder contained in an open subset U of $\mathbb{C}^n$. Let $\mathcal{F}$ be a coherent analytic sheaf on U .

- (A) There exists an open neighbourhood of K over which $\mathcal{F}$ admits a finite free resolution

$$0 \rightarrow \mathcal{L}_n \rightarrow \dots \rightarrow \mathcal{L}_1 \rightarrow \mathcal{L}_0 \rightarrow \mathcal{F} \rightarrow 0.$$

- (B) $H^q(K, \mathcal{F}) = 0$ for $q > 0$.

(Reference: For instance Gunning and Rossi.)

We have the following consequences of this theorem:

- 1) Given a finite free resolution

$$0 \rightarrow \mathcal{L}_n \rightarrow \dots \rightarrow \mathcal{L}_1 \rightarrow \mathcal{L}_0 \rightarrow \mathcal{F} \rightarrow 0$$

of a coherent sheaf $\mathcal{F}$, the sequence

$$0 \rightarrow \mathcal{L}_n(K) \rightarrow \dots \rightarrow \mathcal{L}_0(K) \rightarrow \mathcal{F}(K) \rightarrow 0$$

is an $\mathcal{O}_U(K)$ - free resolution of $\mathcal{F}(K)$.

2) Given a short exact sequence of coherent sheaves

$$0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0,$$

then the sequence

$$0 \rightarrow \mathcal{F}'(K) \rightarrow \mathcal{F}(K) \rightarrow \mathcal{F}''(K) \rightarrow 0 \quad \text{is exact.}$$

Let $\mathcal{F}$ be a coherent analytic sheaf on U , and let $K \subset U$ be a polycylinder. If V is an open neighbourhood of K , then $\mathcal{F}(V)$ can be equipped with a Fréchet-space structure (see: Malgrange).

Hence we can give $\mathcal{F}(K)$ the structure of inductive limit of Fréchet-spaces. It is however essential for certain purposes to have Banach-spaces. This can be obtained by choosing a space slightly different from $\mathcal{F}(K)$ and by choosing K in a "privileged" way.

Let $B(K) = \{f: K \rightarrow \mathbb{C} \mid f \text{ continuous on } K \text{ and analytic on } \overset{\circ}{K}\}$, then $B(K)$ is Banach algebra and $B(K) \subset C(K)$. The sections of $\mathcal{O}_U$ over K are elements of $B(K)$, and $B(K)$ is in fact the uniform closure of $\mathcal{O}_U(K)$ in $C(K)$.

If $\mathcal{L} = \mathcal{O}_U^r$, we define $B(K, \mathcal{L}) = B(K)^r$. Then $B(K; \mathcal{L})$ is a free $B(K)$ -module, and since $\mathcal{L}(K) = \mathcal{O}_U(K)^r$, we have $B(K; \mathcal{L}) = B(K) \otimes_{\mathcal{O}_U(K)} \mathcal{L}(K)$.

We now assume that $\mathcal{F}$ is a coherent sheaf on U , where $U \subset \mathbb{C}^n$ is open. Consider a free resolution

$$(R) \quad 0 \rightarrow \mathcal{L}_n \rightarrow \dots \rightarrow \mathcal{L}_1 \rightarrow \mathcal{L}_0 \rightarrow \mathcal{F} \rightarrow 0 \quad \text{of } \mathcal{F}.$$

From (R) we get an $\mathcal{O}_U(K)$ -free resolution of $\mathcal{F}(K)$

$$(R') \quad 0 \rightarrow \mathcal{L}_n(K) \rightarrow \dots \rightarrow \mathcal{L}_1(K) \rightarrow \mathcal{L}_0(K) \rightarrow \mathcal{F}(K) \rightarrow 0.$$

Taking the tensorproduct $B(K) \otimes_{\mathcal{O}_U(K)}$ we get the complex

$$B(K; \mathcal{L}.): 0 \rightarrow B(K; \mathcal{L}_n) \rightarrow \dots \rightarrow B(K; \mathcal{L}_1) \rightarrow B(K; \mathcal{L}_0).$$

Definition 2: The polycylinder K is called $\mathcal{F}$ -privileged if the complex $B(K; \mathcal{L}.)$ is split-exact in every degree > 0 .

Remark: The property of being $\mathcal{F}$ -privileged is independent of the resolution (R).

The exactness of $B(K; \mathcal{L}.)$ can be expressed by $\text{Tor}_i^{\mathcal{O}_U(K)}(B(K), \mathcal{F}(K)) = 0$, for every $i > 0$, and Tor is independent of the resolution (R). It is a little

more complicated to show, that the splitting property is independent of (R) , and this is omitted.

Since $B(K; \mathcal{L}_i)$ is a Banach space, the image and its complement are thus Banach spaces if K is $\mathcal{F}$ -privileged. In this case we define $B(K; \mathcal{F}) = \text{Coker } (B(K, \mathcal{L}_1) \rightarrow B(K; \mathcal{L}_0)) = B(K) \otimes_{\mathcal{U}} \mathcal{F}(K)$ and we get a $B(K)$ -module, which is a Banach-space.

Warning: In the definition of split-exactnes, the subspaces are splitting vector spaces, but they are not splitting $B(K)$ -modules in general.

We have the following important theorem about the existence of privileged polycylinders:

Theorem 2: Let U be an open subset of $\mathbb{C}^n$, and let $\mathcal{F}$ be a coherent analytic sheaf on U . For any $x \in U$ there exists a fundamental system of neighbourhoods of x in U , which are $\mathcal{F}$ -privileged polycylinders.

For the proof, see Douady: § 7, 4, th 1.

Example: (Curves in $\mathbb{C}^2$) Let $U \subset \mathbb{C}^2$ be an open connected neighbourhood of the origin, and let $h: U \rightarrow \mathbb{C}$ be analytic and $h \neq 0$.

Let X be the curve given by h , that is $X = h^{-1}(0)$, $\mathcal{O}_X = \mathcal{O}_U/(h)$. We have an exact sequence $0 \rightarrow \mathcal{O}_U \xrightarrow{h, I} \mathcal{O}_U \rightarrow \mathcal{O}_X \rightarrow 0$. Consider a polycylinder $K = K_1 \times K_2 \subset U$. By definition K is $\mathcal{O}_X$ -privileged if and only if $h: B(K) \rightarrow B(K)$ is a split monomorphism.

Let $\dot{K}_j$ denote the boundary of K_j , and define $\ddot{K} = \dot{K}_1 \times \dot{K}_2$ ($\ddot{K}$ is called the Šilov Boundary of K).

Proposition 1: (a) The following conditions are equivalent:

- (i) $h: B(K) \rightarrow B(K)$ is a monomorphism.
- (i') $\exists a > 0$ such that $\|hf\| \geq a\|f\|$, $\forall f \in B(K)$.
- (ii) $X \cap \ddot{K} = \emptyset$.

(b) If $(K_1 \times K_2) \cap X = \emptyset$, then h is a split monomorphism (i.e. K is $\mathcal{O}_X$ privileged).

Proof: (a) (i) $\Leftrightarrow$ (i') is a well known fact from the theory of normed vector spaces.

(ii) $\Rightarrow$ (i'). Assume $X \cap \ddot{K} = \emptyset$. If $f \in B(K)$, then it follows from the maximum principle that $\|f\| = \sup_K |f(x)| = \sup_{\ddot{K}} |f(x)|$. Since $h(x) \neq 0$

whenever $x \in \ddot{K}$, we get $a = \inf_K |h(x)| > 0$. Hence $\|hf\| = \sup_K |hf(x)| \geq a \sup_K |f(x)| = a \|f\|$.

(i') $\Rightarrow$ (ii). Suppose that $X \cap \ddot{K} \neq \emptyset$ and $x = (x_1, x_2) \in X \cap \ddot{K}$. We choose an analytic function $f_1 : U_1 \rightarrow \mathbb{C}$, where $U_1 \supset K_1$, and U_1 is open, such that $f_1(x_1) = 1$, $|f_1(z)| < 1$ if $z \in K_1$, $z \neq x_1$. Similarly we choose an analytic function $f_2 : U_2 \rightarrow \mathbb{C}$, with the same properties. Consider the function $f \in B(K) : (z_1, z_2) \rightarrow f_1(z_1)f_2(z_2)$. Since $h(x) = 0$ it follows that the sequence $\{hf^n\}$ converges pointwise to 0 in K .

Applying Dini's theorem we get $\|hf^n\| \rightarrow 0$. From the inequality $a\|f^n\| \leq \|hf^n\|$ we get $\|f^n\| \rightarrow 0$, which is a contradiction, because for every $n : f^n(x) = 1$.

(b) Use the Weierstrass preparation theorem (extended form).

Question. Does the condition (ii) imply that $h : B(K) \rightarrow B(K)$ is a split monomorphism?

IV. FLATNESS AND PRIVILEGE

§ 1. Morphisms from an analytic space into $B(K)$

Let S be an analytic space and K a polycylinder in an open set $U \subset \mathbb{C}^n$. We want to construct an $\mathcal{O}_S$ -algebra homomorphism $\phi : \mathcal{O}_{S \times U}(S \times U) \rightarrow \mathcal{H}(S; B(K))$.

(a) Consider first $S = U' \subset \mathbb{C}^m$, U' -open. If $h \in \mathcal{O}_{U' \times U}(U' \times U)$ and $s \in U'$, $x \in K$, define $(\phi(h)(s))(x) = h(s, x)$. Using the Cauchy integral, one can show that $\phi(h)$ is analytic. On the other hand it's obvious that ϕ is an $\mathcal{O}_{U'}$ -algebra homomorphism.

(b) Let S have a special model in the polydisc Δ in $\mathbb{C}^m$, defined by a sheaf $\mathcal{J}$ of ideals of $\mathcal{O}_\Delta$, and let $\mathcal{J}$ be generated by $f_1, \dots, f_p$, V -a polycylinder neighbourhood of K in U . By Cartan's theorem B for a polycylinder,

the sequence $0 \rightarrow \mathcal{J}(\Delta \times V) \rightarrow \mathcal{O}(\Delta \times V) \xrightarrow{\pi} \mathcal{O}(S \times V) \rightarrow 0$ is exact. If we denote by $\tilde{\pi}$ the projection $\mathcal{H}(\Delta, B(K)) \rightarrow \mathcal{H}(S, B(K))$, $(f_1, \dots, f_p) \cdot \mathcal{H}(\Delta, B(K)) \subset \text{Ker } \tilde{\pi}$. Therefore, because π is surjection, there exists a unique

$\phi : \mathcal{O}(S \times V) \rightarrow \mathcal{H}(S, B(K))$, such that the diagram