

## §4. Algebraic study of flatness

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The set of “ nice ” fibers is dense in  $X$ , so we cannot remove the  $z$ -axis and still get a closed subspace of  $\mathbf{C}_3$ .

#### § 4. Algebraic study of flatness

In the following all rings are commutative, with 1, and all modules are unitary.

*Definition 1:* An  $A$ -module  $E$  is *flat*, if for every exact sequence of  $A$ -modules

$$0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0,$$

the sequence  $0 \rightarrow E \otimes F' \rightarrow E \otimes F \rightarrow E \otimes F'' \rightarrow 0$  is also exact. We can also say, because  $\otimes$  is right exact, that  $E$  is flat, if for every injective homomorphism  $F' \rightarrow F$ ,  $E \otimes F' \rightarrow E \otimes F$  is also injective.

*Examples of modules which are not flat:*

- (1) if  $A = \mathbf{Z}$ ,  $E = \mathbf{Z}_2 = \mathbf{Z}/2\mathbf{Z}$ ,  $F = F' = \mathbf{Z}$ ; then the sequence  $0 \rightarrow \mathbf{Z} \xrightarrow{2I} \mathbf{Z} (2I : x' \rightarrow 2x)$  is exact. But now  $\mathbf{Z}_2 \otimes \mathbf{Z} = \mathbf{Z}_2$ , and the homomorphism  $\mathbf{Z}_2 \xrightarrow{2I} \mathbf{Z}_2$  is the zero homomorphism, which is not injective. So  $\mathbf{Z}_2$  is not a flat  $\mathbf{Z}$  module.
- (2) If  $A = \mathbf{C}\{x\}$ ,  $E = \mathbf{C} = \mathbf{C}\{x\}/(x)$ ,  $F = F' = \mathbf{C}\{x\}$ , then the sequence  $0 \rightarrow F \xrightarrow{xI} F' (xI : p(x) \rightarrow xp(x))$  is exact. But the homomorphism  $E \xrightarrow{xI} E$  is not injective.

*Proposition 1:* If  $A$  is an integral domain and  $E$  a flat  $A$ -module, then  $E$  is torsion-free.

*Proof:* Let  $a \in A$ ,  $a \neq 0$ . Because  $A$  is an integral domain, the sequence  $0 \rightarrow AA \xrightarrow{aI} (aI : x \rightarrow ax)$  is exact. Since  $E$  is flat, the sequence  $0 \rightarrow E \xrightarrow{aI} E$  is also exact. In other words  $E$  has no torsion elements.

*Proposition 2:* If  $A$  is a principal-ideal domain, then  $E$  is flat if and only if  $E$  is torsionfree.

*Proof:* See corollary of prop. 6.

*Examples of flat modules:*

- (1) The inductive limit of flat modules is flat, because the inductive limit preserves exactness, and it commutes with the tensor product.

- (2) Every free module is flat. In fact, if  $E$  is free and finite type, then  $E = A^n$  and  $E \otimes F = F^n$ . If  $F' \rightarrow F$  is injective, so is  $F'^n \rightarrow F^n$  too.

If  $E$  is an arbitrary free module, then it is an inductive limit of free modules of finite type, and the flatness of  $E$  follows from (1).

- (3) Let  $S$  be a multiplicative system in  $A$ . Then the ring of fractions  $S^{-1}A$  is a flat  $A$ -module. In fact the ring  $S^{-1}A$  can be identified with an inductive limit of free modules, so it is flat ((1) (2)). We assume for simplicity that  $S$  has only regular elements. We can define in the set  $S$  a partial order in the following way:

$$s' \geq s \Leftrightarrow \exists t \in A, \quad ts = s' \quad (\text{such a } t \text{ is then unique}).$$

Let  $E_s = A$  for every  $s \in S$ , and if  $s' \geq s$  (i.e.  $s' = ts$ ) then let  $f_s^{s'}$  be the homomorphism  $t \cdot I_A : E_s \rightarrow E_{s'}$ . The family  $(E_s)_{s \in S}$  with the homomorphisms  $(f_s^{s'})$  is an inductive system.

Let  $E = \lim_{\rightarrow} E_s$  be the inductive limit of this system, and  $\varphi_s$  the canonical homomorphism  $E_s \rightarrow E$ . We shall define an isomorphism  $\psi : E \rightarrow S^{-1}A$ .

We first define for every  $s$  a homomorphism  $\psi_s : E_s = A \rightarrow S^{-1}A$ ;  $x \mapsto x/s$ . Now if  $s' \geq s$ , then

$$(\psi_{s'} \circ f_s^{s'})(x) = \psi_{s'}(tx) = \frac{tx}{s'} = \frac{tx}{ts} = \frac{x}{s} = \psi_s(x).$$

Therefore there exists a homomorphism  $\psi : E \rightarrow S^{-1}A$ , satisfying  $\psi_s = \psi \circ \varphi_s$  for every  $s \in S$ .

Because every element of  $S^{-1}A$  has the form  $a/s$ ,  $\psi$  is surjective. On the other hand if  $\psi(\varphi_s(x)) = 0$ , then  $\psi_s(x) = x/s = 0$ . Thus  $x = 0$ , and  $\psi$  is also injective.

The above proof can be extended to the general case, not assuming that the elements of  $S$  are regular. The extended proof involves the notion of inductive limit of an inductive system indexed by a category instead of an ordered set.

From (1) and (2) above, any module which is the inductive limit of free modules, is flat. Conversely:

*Theorem 1 : (Daniel, Lazard)*

Any flat module is a inductive limit of free modules.

For the proof: See *C.R. Acad. Sci. Paris*, 258 (1964), pp. 6313-6316.

*Some elementary properties of flat modules :*

- (1) If  $E$  and  $F$  are flat  $A$ -modules, then  $E \otimes_A F$  is also flat. In fact, if  $G' \rightarrow G$  is injective, then  $F \otimes_A G' \rightarrow F \otimes_A G$  is injective, and also  $E \otimes_A (F \otimes_A G') \rightarrow E \otimes_A (F \otimes_A G)$  is injective. The result follows from the associativity of the tensor product.
- (2) Let  $\phi : A \rightarrow B$  be a ring homomorphism, and  $E$  a flat  $A$ -module. The module  $B \otimes_A E$  is a flat  $B$ -module.

If  $F$  is a  $B$ -module, then  $F \otimes_B (B \otimes_A E) = (F \otimes_B B) \otimes_A E = F \otimes_A E$  further if  $F'$  and  $F$  are  $B$ -modules, and  $F' \rightarrow F$  an injective homomorphism of  $B$ -modules, we can consider this homomorphism as an injective homomorphism of  $A$ -modules. Because  $E$  is  $A$ -flat,

$$F' \otimes_A E \rightarrow F \otimes_A E \text{ is injective.}$$

- (3) Let  $\phi : A \rightarrow B$  be a ring homomorphism, such that  $B$  is a flat  $A$ -module. If  $F$  is a flat  $B$ -module, then  $F$  is a flat  $A$ -module. In fact: if  $E' \rightarrow E$  is injective, then  $E' \otimes_A B \rightarrow E \otimes_A B$  is injective, and also  $(E' \otimes_A B) \otimes_B F' \rightarrow (E \otimes_A B) \otimes_B F$  is injective. But  $(E' \otimes_A B) \otimes_B F' = E' \otimes_A F$ ;  $(E \otimes_A B) \otimes_B F = E \otimes_A F$ .  
If an  $A$ -module  $E$  is not flat, we want to measure how far it is from being flat. For this purpose we introduce the functor  $\text{Tor}$ .

*Definition 2 :* A free resolution of  $E$  is an exact sequence:  $\dots \rightarrow L_n \rightarrow L_{n-1} \rightarrow \dots \rightarrow L_1 \rightarrow L_0 \rightarrow E \rightarrow 0$ , where all  $L_i$  are free  $A$ -modules.

The complex of the resolution is the sequence

$$(L.) \dots \rightarrow L_n \rightarrow L_{n-1} \rightarrow \dots \rightarrow L_1 \rightarrow L_0 \rightarrow 0.$$

Every module has a free resolution. Two resolutions are algebraically homotopy-equivalent. Forming the tensor products  $L_i \otimes F$ , we get

$$(L. \otimes F) \dots \rightarrow L_n \otimes F \rightarrow L_{n-1} \otimes F \rightarrow \dots \rightarrow L_1 \otimes F \rightarrow L_0 \otimes F \rightarrow 0.$$

*Definition 3 :*

$$\text{Tor}_n^A(E, F) = H_n(L. \otimes F) = \frac{\text{Ker}(L_n \otimes F \rightarrow L_{n-1} \otimes F)}{\text{Im}(L_{n+1} \otimes F \rightarrow L_n \otimes F)}$$

if  $n \geq 1$ , and  $\text{Tor}_0^A(E, F) = \text{Coker}(L_1 \otimes F \rightarrow L_0 \otimes F) = E \otimes F$ .

*Basic properties of Tor :*

- (1)  $\text{Tor}_n(E, F)$  is independent of the choice of the resolution (up to a canonical isomorphism).

- (2) If we take a free resolution of  $F$ , we get  $\text{Tor}_n(F, E) = \text{Tor}_n(E, F)$  (Symmetry of the Tor). We can also define  $\text{Tor}_n(E, F)$  by taking two free resolutions, one of  $E$  and one of  $F$ .
- (3) If  $0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$  is a short exact sequence, then we get a long exact sequence:

$$\begin{aligned} & \text{Tor}_n(E', F) \rightarrow \text{Tor}_n(E, F) \rightarrow \text{Tor}_n(E'', F) \rightarrow \\ & \rightarrow \text{Tor}_{n-1}(E', F) \rightarrow \text{Tor}_{n-1}(E, F) \rightarrow \text{Tor}_{n-1}(E'', F) \rightarrow \\ & \rightarrow \text{Tor}_1(E', F) \rightarrow \text{Tor}_1(E, F) \rightarrow \text{Tor}_1(E'', F) \rightarrow \\ & \rightarrow E' \otimes F \rightarrow E \otimes F \rightarrow E'' \otimes F \rightarrow 0. \end{aligned}$$

- (4) Tor is compatible with inductive limit, i.e. if  $E = \lim (E_i)$ , then

$$\text{Tor}_n(\lim E_i, F) = \lim (\text{Tor}_n(E_i, F)).$$

- (5) We can define  $\text{Tor}_n(E, F)$  by taking a flat resolution of  $E$ .

*Proposition 3:* Let  $E$  be an  $A$ -module. Then the following conditions are equivalent:

- (a)  $E$  is flat.
- (b) For all  $A$ -modules  $F$ , and for all  $n \geq 1$ ,  $\text{Tor}_n(E, F) = 0$ .
- (c) For all  $A$ -modules  $F$ ,  $\text{Tor}_1(E, F) = 0$ .

*Proof:* (a)  $\Rightarrow$  (b). If  $\dots \rightarrow L_n \rightarrow L_{n-1} \rightarrow \dots \rightarrow L_1 \rightarrow L_0 \rightarrow F \rightarrow 0$  is a free resolution of  $F$ , then the sequence

$$\dots \rightarrow E \otimes L_n \rightarrow E \otimes L_{n-1} \rightarrow \dots \rightarrow E \otimes L_1 \rightarrow E \otimes L_0 \rightarrow E \otimes F \rightarrow 0$$

is exact, thus  $\text{Tor}_n(E, F) = 0$  for all  $n \geq 1$ .

(b)  $\Rightarrow$  (c) clear. (c)  $\Rightarrow$  (a): If the sequence  $0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0$  is exact, so is also (by (3) above)  $\text{Tor}_1(E, F'') \rightarrow E \otimes F' \rightarrow E \otimes F \rightarrow E \otimes F'' \rightarrow 0$ . Now  $\text{Tor}_1(E, F'') = 0$ , thus  $E$  is flat.

*Proposition 4:* If  $I$  and  $J$  are two ideals in  $A$ , then  $\text{Tor}_1^A(A/I, A/J) = I \cap J / I \cdot J$ .

*Proof:* From the exact sequence  $0 \rightarrow I \rightarrow A \rightarrow A/I \rightarrow 0$ , we get the exact sequence:

$$\text{Tor}_1(A, A/J) \rightarrow \text{Tor}_1(A/I, A/J) \rightarrow I \otimes A/J \rightarrow A \otimes A/J \rightarrow A/I \otimes A/J \rightarrow 0.$$

But now  $\text{Tor}_1(A, A/J) = 0$  ( $A$  being  $A$ -free), and  $I \otimes A/J = I/I \cdot J$ ;  $A \otimes A/J = A/J$ . Therefore the sequence  $0 \rightarrow \text{Tor}_1(A/I, A/J) \rightarrow I/I \cdot J \rightarrow A/J$  is exact, and  $\text{Tor}_1(A/I, A/J) = \text{Ker}(I/I \cdot J \rightarrow A/J) = I \cap J / I \cdot J$ .

*Example :* Let  $U$  be an open set in  $\mathbb{C}^n$ , and  $x \in U$ . Further let  $X, Y \subset U$  be two hypersurfaces, defined by  $I = (f)$  and  $J = (g)$ . Supposing that  $f$  and  $g$  do not have common factors:  $I_x \cap J_x = I_x J_x$ , and

$$\text{Tor}_1(\mathcal{O}_{X,x}, \mathcal{O}_{Y,x}) = \text{Tor}_1(\mathcal{O}_{U,x}/I_x, \mathcal{O}_{U,x}/J_x) = \frac{I_x \cap J_x}{I_x \cdot J_x} = 0.$$

*Heuristic remark :* The formula  $\text{Tor}_1(\mathcal{O}_{X,x}, \mathcal{O}_{Y,x}) = 0$  expresses the fact that  $X$  and  $Y$  are “in general position”. If for example  $X$  and  $Y$  are two linear subspaces in  $\mathbb{C}^n$  of dimensions  $p$  and  $q$ , we have  $\text{Tor}_1(\mathcal{O}_{X,x}, \mathcal{O}_{Y,x}) = 0$  if  $\dim(X \cap Y) = p + q - n$ , and  $\text{Tor}_1(\mathcal{O}_{X,x}, \mathcal{O}_{Y,x}) \neq 0$  otherwise.

Next we shall prove an elementary flatness criterion.

*Proposition 5 :* Let  $E$  be an  $A$ -module. The following conditions are equivalent:

- (a)  $E$  is flat.
- (b) For all finitely generated ideals  $I$  of  $A$ ,  $\text{Tor}_1(E, A/I) = 0$ .
- (c) For all monogenous  $A$ -modules  $F$ ,  $\text{Tor}_1(E, F) = 0$ .

*Proof :* (a)  $\Rightarrow$  (b), by prop. 3.

(b)  $\Rightarrow$  (c): Because  $\text{Tor}$  is compatible with inductive limit, we can suppose, that  $\text{Tor}_1(E, A/I) = 0$  for an arbitrary ideal  $I$  of  $A$ . But every monogenous  $A$ -module  $F$  can be represented by  $A/I$ .

(c)  $\Rightarrow$  (a). By prop. 3 it is sufficient to prove that  $\text{Tor}_1(E, F) = 0$  for any  $A$ -module  $F$ .

First consider the case, where  $F$  is finitely generated. We use induction, supposing that  $\text{Tor}_1(E, F) = 0$ , when  $F$  has  $n$  generators. Let  $F$  have  $(n+1)$  generators  $x_1, \dots, x_n, x_{n+1}$ . If  $F'$  is the submodule generated by  $\{x_1, \dots, x_n\}$ , then  $F' \subset F$  and  $F'' = F/F'$  is monogenous. The exact sequence  $0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0$  gives the exact sequence  $\text{Tor}_1(E, F') \rightarrow \text{Tor}_1(E, F) \rightarrow \text{Tor}_1(E, F'')$ . Now  $\text{Tor}_1(E, F') = \text{Tor}_1(E, F'') = 0$ , thus  $\text{Tor}_1(E, F) = 0$ . In the general case,  $F$  can be considered as an inductive limit of finitely generated modules, and because  $\text{Tor}$  is compatible with inductive limits,  $\text{Tor}_1(E, F) = 0$ .

*Proposition 6 :* Let  $A$  be an integral domain, and  $E$  an  $A$ -module. Then  $E$  is torsionfree if and only if  $\text{Tor}_1(E, A/(a)) = 0$ , for any element  $a \in A$ .

*Proof :* If  $E$  is  $A$ -module,  $a \in A$ , then the exact sequence  $0 \rightarrow A \xrightarrow{aI} A \rightarrow A/(a) \rightarrow 0$  gives the exact sequence  $0 \rightarrow \text{Tor}_1(E, A/(a)) \rightarrow E \xrightarrow{aI} E$ . In other words  $\text{Tor}_1(E, A/(a)) = \{x \in E \mid ax = 0\}$ , from which the result follows.

*Corollary:* Let  $A$  be a principal ideal domain.  $E$  is flat if and only if  $E$  is torsionfree.

*Proof:* We have already proved that, if  $E$  is flat, then it is torsion free. The converse follows from prop. 6 and prop. 5.

The first flatness criterion for noetherian local rings is the following:

*Theorem 2:* Let  $A$  be a noetherian local ring with maximal ideal  $m$ ;  $k = A/m$ , and  $E$  a finitely generated  $A$ -module. The following conditions are equivalent:

- (a)  $E$  is free.
- (b)  $E$  is flat.
- (c)  $\text{Tor}_1^A(E, k) = 0$ .

*Proof:* We have already proved  $(a) \Rightarrow (b) \Rightarrow (c)$ .

$(c) \Rightarrow (a)$ : We recall first Nakayma's lemma. If  $A$  is a local ring with maximal ideal  $m$ ;  $k = A/m$ , and  $E$  is a finitely generated  $A$ -module, such that  $k \otimes_A E = E/mE = 0$ , then  $E = 0$ .

The module  $\bar{E} = k \otimes_A E = E/mE$  is a finitely generated vector space over  $k$ . Let  $\{\bar{x}_1, \dots, \bar{x}_r\}$  be a base of  $\bar{E}$  (over  $k$ ), and  $\{x_1, \dots, x_r\}$   $E$  representatives of  $\bar{x}_i$ :  $s$ . Consider the homomorphism  $\phi : A^r \rightarrow E$ ,  $\phi(a_1, \dots, a_r) = \sum a_i x_i$ . Denoting by  $R$  and  $Q$  the kernel and the cokernel of  $\phi$ , we get an exact sequence:

$$(*) \quad 0 \rightarrow R \rightarrow A^r \rightarrow E \rightarrow Q \rightarrow 0$$

and  $R, Q$  are finitely generated  $A$ -modules. From  $(*)$  we get the exact sequence

$$A^r \otimes_A k \rightarrow E \otimes_A k \rightarrow Q \otimes_A k \rightarrow 0.$$

But  $\bar{E} = E \otimes_A k \simeq k^r = A^r \otimes_A k$ , so  $Q \otimes_A k = 0$ , and by Nakayama's lemma  $Q = 0$ .

Therefore we have an exact sequence

$$0 \rightarrow R \rightarrow A^r \rightarrow E \rightarrow 0.$$

From this we get:  $\text{Tor}_1(E, k) \rightarrow k \otimes_A R \rightarrow k^r \rightarrow \bar{E} \rightarrow 0$  (exact). Now:  $\bar{E} \simeq k^r$ ,  $\text{Tor}_1(E, k) = 0$  (by assumption). Therefore  $k \otimes_A R = 0$ , and once more by Nakayama's lemma  $R = 0$ , thus  $E \simeq A^r$ , i.e.  $E$  is free.

*Proposition 7:* Let  $\phi : A \rightarrow B$  be a ring homomorphism, and let  $B$  be  $A$ -flat. If  $I$  is an ideal of  $A$ , we write  $\bar{A} = A/I$ ,  $\bar{B} = B/IB = \bar{A} \otimes_A B$ . Let  $F$  be a  $B$ -module, then:  $\text{Tor}_i^A(\bar{A}, F) = \text{Tor}_i^B(\bar{B}, F)$  ( $i \geq 0$ ).

*Proof:* We choose first a  $B$ -free resolution of  $F$

$$\rightarrow L_{n+1} \rightarrow L_n \rightarrow \dots \rightarrow L_1 \rightarrow L_0 \rightarrow F \rightarrow 0.$$

If  $L$  is the respective complex of resolution, then

$$\bar{B} \otimes_B L = B/IB \otimes_B L = \bar{A} \otimes_A (B \otimes_B L) = \bar{A} \otimes_A L.$$

Because every  $L_i$  is  $B$ -free, and  $B$  is  $A$ -flat, every  $L_i$  is  $A$ -flat (Property 3 after Th. 1). Thus  $L$  is a flat  $A$ -resolution, and

$$\text{Tor}_i^A(\bar{A}, F) = H_i(\bar{A} \otimes_A L) = H_i(\bar{B} \otimes_B L) = \text{Tor}_i^B(\bar{B}, F).$$

We shall next state the second flatness criterion for noetherian local rings.

*Theorem 3:* Let  $A$  and  $B$  be two noetherian local rings, with maximal ideals  $\underline{m}$ ,  $\underline{n}$ ;  $k = A/\underline{m}$ . If  $\phi : A \rightarrow B$  is a local homomorphism (i.e.  $\phi(\underline{m}) \subset \underline{n}$ ), and  $F$  finitely generated  $B$  module then

$$F \text{ is } A\text{-flat} \Leftrightarrow \text{Tor}_1^A(k, F) = 0.$$

The proof of this theorem is much more difficult than that of th. 20 see for example:

Bourbaki: *Algèbre commutative*, Chapter III § 5, th1, (i)  $\Leftrightarrow$  (iii), p. 98.

The conditions in Bourbaki's theorem are here fulfilled:

- 1° A finitely generated module  $F$  over a noetherian local ring  $B$  is idealwise separated for  $\underline{n}$ . (*Ibid.*, § 5. 1. Ex. 1, p. 97.)
- 2° If  $\phi : A \rightarrow B$  is a local homomorphism,  $F$  is also idealwise separated for  $\underline{m}$ . (*Ibid.*, § 5, prop. 2, p. 101.)
- 3° Also the flatness condition is fulfilled, because  $k$  is a field.

*Remark:* The main interest of the theorem lies in the fact, that it is true without any assumption of finiteness on  $B$ .

*Corollary:* If the assumptions are the same as in the theorem 3, and if moreover  $B$  is  $A$ -flat, then

$$F \text{ is } A\text{-flat} \Leftrightarrow \text{Tor}_1^B(\bar{B}, F) = 0,$$

where  $\bar{B} = B/\underline{m}B$ .



*Proof:*  $\text{Tor}_1^A(k, F) = \text{Tor}_1^B(\bar{B}, F)$ , by prop. 7.

## § 5. Geometric applications of the flatness criterions

### A) Flatness for finite morphisms

*Proposition 1:* Let  $\pi: X \rightarrow S$  be a finite morphism (i.e. proper with finite fibres) of analytic spaces. Then  $\pi_*(\mathcal{O}_X)$  is a coherent analytic sheaf over  $S$ . The following conditions are equivalent:

- (a)  $\pi$  is flat (i.e. for every  $x \in X$ ,  $\mathcal{O}_{X,x}$  is a flat  $\mathcal{O}_{S,s}$ -module,  $s = \pi(x)$ ).
- (b) For every  $s$ ,  $(\pi_* \mathcal{O}_X)_s$  is a flat  $\mathcal{O}_{S,s}$ -module.
- (c)  $\pi_* \mathcal{O}_X$  is a locally free sheaf.

*Proof:* Because  $\pi$  is finite  $\pi_*(\mathcal{O}_X)_s = \bigoplus_{x \in \pi^{-1}(s)} \mathcal{O}_{X,x}$ , thus the only point to prove is (b)  $\Rightarrow$  (c).

Now if  $\mathcal{O}_{X,x}$  is a flat  $\mathcal{O}_{S,s}$ -module, then (by theorem 2)  $\mathcal{O}_{X,x}$  is free, and a coherent sheaf whose fibers are free is a locally free sheaf.

*Proposition 2:* Let  $S$  be a reduced analytic space and  $\mathcal{E}$  a coherent  $\mathcal{O}_S$ -module. Let  $E(s)$  be the finite dimensional vector space (over  $\mathbb{C}$ )  $\mathcal{E}_s \otimes_{\mathcal{O}_{S,s}} \mathbb{C}_s$ .  $\mathcal{E}$  is a locally free  $\mathcal{O}_{S,s}$ -module if and only if  $\dim_{\mathbb{C}} E(s)$  is locally constant.

*Proof:* If  $\mathcal{E}$  is locally free, then  $\dim_{\mathbb{C}} E(s)$  is locally constant. Suppose now that  $\dim_{\mathbb{C}} E(s)$  is locally constant in an open set  $U \subset S$ , and that  $\mathcal{O}_U^p \xrightarrow{d} \mathcal{O}_U^q \rightarrow \mathcal{E}_U \rightarrow 0$  is exact.  $d$  is determined by a  $p \times q$  matrix of analytic functions on  $U$ , so it gives a morphism  $\mathbb{C}_U^p \xrightarrow{d} \mathbb{C}_U^q$  of trivial vector bundles over  $U$ .

From the exact sequence  $\mathcal{O}_S^p \xrightarrow{d_s} \mathcal{O}_S^q \rightarrow \mathcal{E}_S \rightarrow 0$ , we get (by making tensor-products with  $\mathbb{C}_s$ ) the exact sequence:

$$\mathbb{C}_s^p \xrightarrow{d(s)} \mathbb{C}_s^q \rightarrow E(s) \rightarrow 0,$$

which shows that  $d$  has constant rank in  $U$ . Thus  $\text{Ker } d$  and  $\text{Im } d$  are vector bundles, and we can write

$$\mathbb{C}_U^p = F_1 \oplus G_1, \quad \mathbb{C}_U^q = F_0 \oplus G_0,$$

$$d : \begin{cases} F_1 \rightarrow 0 \\ G_1 \simeq F_0 \end{cases}.$$