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Proposition 10. Let $f : X \underset{m}{\to} Y$, $f'_1 : X \underset{m}{\to} Y_1$, $g : Y \underset{m}{\to} Z$ be meromorphic mappings, assume that $g \triangle f$ exists. Then we have:

1) If f is proper, $[f, f'_1]$ is proper,

2) If f and g are proper, $g \triangle f$ is proper,

3) If $g \triangle f$ is proper, f is proper,

4) If $g \triangle f$ is proper and f surjective, g is proper.

4. Extension of meromorphic mappings

We start with some classical results. Let D be a domain in $\mathbf{C}^n$ and $A \neq D$ an irreducible analytic set in D. Let $\varphi : D - A \to \mathbf{C}$ be a holomorphic mapping and $f : D - A \underset{m}{\to} \mathbf{P}_1$ a meromorphic mapping. Then we have (see [2], [8], [14] and the references given there):

1) If codim $A > 1$, then φ and f have extensions over A.

2) Assume codim $A = 1$. Then

a) φ has an extension over A if for some $z_0 \in A$ there is a neighborhood U of z_0 such that φ is bounded in $U - (A \cap U)$,

b) f has an extension over A if for some $z_0 \in A$ f has an extension into a neighborhood of z_0.[1]

We shall see that these statements can be generalized in some respects.[2]

Throughout this section, X and Y are irreducible complex spaces, $A \neq X$ is an irreducible analytic set in X, $f : X - A \underset{m}{\to} Y$ a meromorphic mapping. We shall study conditions under which f has an extension over A, which means that there exists a meromorphic mapping $g : X \underset{m}{\to} Y$ such that $g \mid X - A = f$.

The meromorphic mapping f can always be extended topologically to a correspondence $\bar{f} : X \underset{k}{\to} Y$ by setting $G_{\bar{f}} = \overline{G_f}$ where the closure is with respect to $X \times Y$. On the other hand, if $\tilde{f} : X \underset{m}{\to} Y$ is an extension of f, then

1) The generalization 2a) of Riemann's classical theorem on removable singularities is due to Kistler and Hartogs. 2b) is due to Hartogs and E. E. Levi. 1) follows easily from 2); the statement 1) for holomorphic functions φ is sometimes called " the second Riemann theorem on removable singularities " (2. Riemannscher Hebbarkeitssatz).

2) The extension problem for holomorphic maps is also treated in [1] and [6].

$\tilde{f} = \bar{f}$. We are thus led to study the properties of $\bar{f}$. Of essential use is the following extension theorem for analytic sets.

Theorem 1. Let Z be a complex space and M an irreducible analytic set in Z. Let further N be a pure dimensional (all irreducible components have the same dimension) analytic set in $Z-M$ such that dim $N =$ dim M. Then the closure $\bar{N}$ of N with respect to Z is an analytic set in Z if it is analytic in at least one point of M.

This theorem was proved by Thullen [21] in the case where Z is a domain in $\mathbf{C}^n$ and where dim $M =$ dim $N = n-1$. In [13] the theorem is stated without restriction on the dimension of M but likewise for a domain Z in $\mathbf{C}^n$ (the special case treated by Thullen is used here in the proof). From this one can obtain the theorem in the form above by using imbeddings of open sets of Z into domains of number space.

Corollary 1. If dim $N>$dim M, then $\bar{N}$ is analytic in Z.

This can be deduced from Theorem 1 by imbedding arguments in an obvious manner. A direct proof is contained in [8].

Corollary 2. Let Z and M be as in the theorem and $\{N_i\}$ a set of mutually different irreducible analytic sets in $Z-M$ for which dim $N_i \geqslant$ dim M, and $\cup N_i$ is analytic in $Z-M$. If every neighborhood of a point $z_0 \in M$ intersects an infinite number of sets N_i, then every point of M has this property.

This is a simple consequence of Theorem 1 and Corollary 1.

Proposition 11. Let D be a domain in $\mathbf{C}^n$, M an irreducible analytic set in D, N a pure dimensional analytic set in $D-M$ such that dim $N =$ dim M. Suppose there exists an analytic plane E_0 through a point $z_0 \in M$ such that the following conditions hold:

1) E_0 is in general position with respect to M, i.e., dim $(E_0 \cap M) =$ dim E_0+dim $M-$dim D,

2) There exists a neighborhood U of z_0 such that for every analytic plane E with dim $E =$ dim E_0 which is parallel to E_0 and which intersects U, $\bar{N} \cap E$ is analytic in D ($\bar{N}$ is the closure of N with respect to D).

Then $\bar{N}$ is analytic in z_0 and hence in D by Theorem 1.

As to the proof we refer to [13], p. 301.[1]

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1) The statement actually proved in [13] is a little more special than Proposition 11, but by suitable supplementary arguments one can obtain the proposition in the form above.

We turn now to the study of two problems:

1) When is $\bar{f}$ weakly holomorphic?

2) When is $\bar{f}$ continuous?

If $\bar{f}$ is weakly holomorphic, then $\bar{f}$ is irreducible, because the irreducibility of G_f implies that of $G_{\bar{f}}$. Hence $\bar{f}$ is a meromorphic mapping if it is weakly holomorphic and continuous.

Moreover, if $\bar{f}$ is weakly holomorphic, then the closure $\overline{f^{-1}(y)}$ of $f^{-1}(y)$ with respect to X is analytic in X for every $y \in Y$: $f^{-1}(y)$ is analytic in $X - A$ and $\bar{f}^{-1}(y)$ is analytic in X; since $\overline{f^{-1}(y)} \subset \bar{f}^{-1}(y)$ and $\overline{f^{-1}(y)} \cap (X-A)$ $= \bar{f}^{-1}(y) \cap (X-A) = f^{-1}(y)$, it follows that $\overline{f^{-1}(y)}$ is analytic in X.

We assume now, in the rest of this section, that dim $X-$dim $Y \geqslant$ dim A. We set $Z = X \times Y$, $M = A \times Y$, $N = G_f$. Then dim $M =$ dim $A+$dim Y, dim $N =$ dim $G_f =$ dim X and, by our assumption, dim N $\geqslant$ dim M. If dim $X-$dim $Y>$dim A, i.e., if dim $N>$dim M, Corollary 1 of Theorem 1 implies that $\bar{f}$ is weakly holomorphic. Furthermore, we have

Proposition 12. Assume dim $X-$dim $Y =$ dim A. Then the correspondence $\bar{f}$ is weakly holomorphic if there exists a non-empty open set $V \subset Y$ such that the closure $\overline{f^{-1}(v)}$ of $f^{-1}(v)$ with respect to X is analytic in X for all $v \in V$.

Proof. The condition dim $X-$dim $Y =$ dim A implies that dim $N =$ dim M. Hence, by Theorem 1, $\bar{N} = G_{\bar{f}}$ is analytic in $Z = X \times Y$, i.e., $\bar{f}$ is weakly holomorphic, if there is a point of $M = A \times Y$ in which $\bar{N}$ is analytic. We show that this is the case for points of $A \times V$. Choose a point $(a_0, v_0) \in A \times V$ such that A is irreducible in a_0 and such that v_0 is an ordinary point of Y. There are open neighborhoods $U_1 \subset X$ of a_0 and $U_2 \subset V$ of v_0 with the following properties: $A' = A \cap U_1$ is an irreducible analytic set in U_1; U_1 can be mapped biholomorphically onto an analytic set X' in a domain D_1 of a number space $\mathbf{C}^{n_1}$; U_2 can be mapped biholomorphically onto a domain D_2 of a number space $\mathbf{C}^{n_2}$ ($n_2 =$ dim Y). It is enough to show that the closure $\overline{N'}$ of $N' = G_f \cap (U_1 \times U_2)$ with respect to $U_1 \times U_2$ is analytic in $U_1 \times U_2$. Set $D = D_1 \times D_2$, $M' = A' \times D_2$ and, for $w \in D_2$, $E_w = \mathbf{C}^{n_1} \times \{ w \}$. Then we have dim $(E_w \cap M') =$ dim $(A' \times \{ w \}) =$ dim $A' =$ dim A, on the other hand dim $E_w +$ dim M' $-$ dim $D = n_1 +$ (dim $A' + n_2)-(n_1 + n_2) =$ dim A. The hypothesis on the analyticity of $\overline{f^{-1}(v)}$ for all $v \in V$ implies that $\overline{N'} \cap E_w$ is analytic in D for every $w \in D_2$. Hence, by Proposition 11, $\overline{N'}$ is analytic in D; then $\overline{N'}$ is, in particular, analytic in $X' \times D_2 = U_1 \times U_2$.

Concerning the continuity of $\bar{f}$ we have

Proposition 13. The correspondence $\bar{f}$ is continuous if it is continuous at one point $a_0 \in A$.

Proof. We assume first that the topology of Y has a countable base. Then $\bar{f}$ is continuous at $a \in A$ if and only if the following condition holds: If (x_ν) and (y_ν), $\nu = 1, 2, \ldots$, are sequences of points such that $x_\nu \in X-A$, $x_\nu \to a$, $y_\nu \in f(x_\nu)$, then the sequence (y_ν) has a point of accumulation in Y. Suppose that $\bar{f}$ is continuous at a point $a_0 \in A$ and let (x_ν), (y_ν) be sequences as above. Then the fibres $f^{-1}(y_\nu)$ are non-empty analytic sets in $X-A$, and the condition dim $X-$dim $Y \geqslant$ dim A implies dim $F_\nu^{(\mu)} \geqslant$ dim A for every irreducible component $F_\nu^{(\mu)}$ of $f^{-1}(y_\nu)$. Suppose that $L = \cup f^{-1}(y_\nu)$ is not analytic in $X-A$. Then there exists a subsequence (y_{ν_i}) such that one can find points $x_i' \in f^{-1}(y_{\nu_i})$ which converge to a point $x_0' \in X-A$. By continuity at x_0' it follows that (y_{ν_i}) has a point of accumulation on $f(x_0')$. Let now L be analytic in $X-A$. Assume first:

(α) There are infinitely many fibres $f^{-1}(y_{\nu_i})$ which have a common irreducible component N.

In this case we take a point of N and use similarly the continuity of f at this point. Suppose now that (α) is not satisfied. Then we apply Corollary 2 of Theorem 1 to the set of irreducible components $F_\nu^{(\mu)}$ of the fibres $f^{-1}(y_\nu)$. Since every neighborhood of a intersects infinitely many components $F_\nu^{(\mu)}$ (this implies, in particular, that the closure $\bar{L}$ of L with respect to X is not analytic in a), the same holds with respect to a_0. The y_ν have then a point of accumulation on $\bar{f}(a_0)$ because $\bar{f}$ is continuous at a_0.

Now we drop the assumption that Y has countable topology. We remark first: To show that $\bar{f}$ is continuous at $a \in A$ we may replace X by any irreducible open subspace which contains the points a and a_0. Therefore we may assume that X has countable topology. Secondly: All points of Y used in the proof above belong to the topological subspace $f(X-A) \cup \bar{f}(a_0) \subset Y$ which has countable topology since X has. If we now restrict Y to an irreducible open subspace with countable topology containing $f(X-A) \cup \bar{f}(a_0)$, the proof given above applies.

Corollary. If dim $X-$dim $Y >$ dim A, then $\bar{f}$ is always continuous.

In this case the hypothesis on the continuity of $\bar{f}$ at a point $a_0 \in A$ is not needed in the proof of Proposition 13: We have now dim $F_\nu^{(\mu)} >$ dim A. If L is analytic in $X-A$, Corollary 1 of Theorem 1 implies that $\bar{L}$ is analytic in every point of A, and the condition (α) is necessarily satisfied.

Combining the preceding statements we have the following result.

Theorem 2. Let $f: X-A \underset{m}{\rightarrow} Y$ be a meromorphic mapping and $\dim X-\dim Y \geqslant \dim A$. Then $\bar{f}$ is a meromorphic mapping if and only if

1) there exists a non-empty open set $V \subset Y$ such that $\overline{f^{-1}(v)}$ is analytic in X for all $v \in V$, and

2) $\bar{f}$ is continuous at a point $a_0 \in A$.

If $\dim X-\dim Y > \dim A$, then $\bar{f}$ is always a meromorphic mapping.

Corollary. Assume there is an open subset $U \subset X$ and a compact set $K \subset Y$ different from Y such that $U \cap A \neq \varnothing$ and $f(U-(U \cap A)) \subset K$. Then $\bar{f}$ is a meromorphic mapping.

To conclude this from Theorem 2 we remark first that the set $V = Y-K$ satisfies the above condition 1): If $v \in V$, then $f^{-1}(v)$ does not intersect U, hence $\overline{f^{-1}(v)}$ is analytic in every point of $U \cap A$ and therefore, by Theorem 1, analytic in X. On the other hand, $\bar{f}$ is continuous at every point $a_0 \in U \cap A$. For $\bar{f}(a_0)$ is compact since it is a closed subset of K. Moreover, let V_0 be a neighborhood of $\bar{f}(a_0)$; we assert that there is a neighborhood U_0 of a_0 such that $\bar{f}(U_0) \subset V_0$. If this were false, then there would exist points x in $U-(U \cap A)$ arbitrarily near a_0 such that $f(x) \cap (K-(K \cap V)) \neq \varnothing$. But then it follows that $f(a_0) \cap (K-(K \cap V_0)) \neq \varnothing$, which is a contradiction.

As to the extension of holomorphic maps we state:

Theorem 3. Let X be, in addition to the earlier assumptions, a complex manifold and $f: X-A \rightarrow Y$ a holomorphic map. Then

1) If $\dim X-\dim Y > \dim A+1$, $\bar{f}$ is a holomorphic map,

2) If $\dim X-\dim Y = \dim A+1$, then $\bar{f}$ is either a holomorphic map or $\bar{f}$ is a meromorphic mapping and $\bar{f}(a) = Y$ for all $a \in A$.

Proof. Assume $\dim X-\dim Y \geqslant \dim A+1$. Then, by Theorem 2, $\bar{f}$ is a meromorphic mapping; if $S = S(f) = \varnothing$, $\bar{f}$ is even a holomorphic map. Suppose $S \neq \varnothing$, set $T = \check{f}^{-1}(S)$ and let T_0 be an irreducible component of T. Set $S_0 = \check{f}(T_0)$. By Remmert's mapping theorem S_0 is an irreducible analytic set in X. We have

$$\dim T_0 = \dim S_0 + \inf_{z \in D_0} \dim_z (g^{-1}(g(z)) \text{ where } g = \check{f} \mid T_0,$$

furthermore $\dim S_0 \leqslant \dim S \leqslant \dim A$ because $S \subset S_0 \subset A$. Every fibre

$g^{-1}(g(z)), z \in T_0$, is mapped injectively into Y by $\hat{f}$, hence $\dim (g^{-1}(g(z)) \leqslant \dim Y$. Thus we obtain the inequalities

$$(*) \quad \dim T_0 \leqslant \dim A + \dim Y \leqslant \dim X - 1 .$$

Now we shall see that $\dim T_0 = \dim X - 1$. Therefore we have equality in (*), hence $\dim X - \dim Y = \dim A + 1$. We obtain also $\dim S_0 = \dim S = \dim A$, hence $S_0 = S = A$, since A is irreducible; moreover, $\dim (g^{-1}(a)) = \dim Y$ for every $a \in A$, consequently $\bar{f}(a) = \hat{f}(g^{-1}(a)) = Y$.

In order to show that $\dim T_0 = \dim X - 1$, we use the following theorem due to Grauert and Remmert [5] (a proof was also given by Kerner [7]):

Let X be a complex manifold, Z a normal complex space, K an analytic set in Z with codim $K \geqslant 2$, $\tau : Z \to X$ a holomorphic map such that $\tau \mid Z - K$ is locally biholomorphic. Then τ is locally biholomorphic.

Now assume first that $G_{\bar{f}}$ is a normal complex subspace of $X \times Y$. The holomorphic map $\check{f}: G_{\bar{f}} \to X$ is locally biholomorphic in a point $\zeta \in G_{\bar{f}}$ if and only if $\zeta \in T = \check{f}^{-1}(S)$. Hence, by the theorem of Grauert and Remmert, T is puredimensional and $\dim T = \dim X - 1$. If $G_{\bar{f}}$ is not normal, we take a normalization $(\tilde{G}, \nu)$ of $G_{\bar{f}}$ and look at $\check{f} \circ \nu : \tilde{G} \to X$ and $\tilde{T} = (\check{f} \circ \nu)^{-1}(S)$ instead of $\bar{f}$ and T. We see then that $\tilde{T}$ is puredimensional with $\dim \tilde{T} = \dim X - 1$, but then it follows that $\nu(\tilde{T}) = T$ has the same properties.

Remark. If Y is not compact, then $\bar{f}$ is always a holomorphic map under the hypothesis of Theorem 3 since $\bar{f}(a)$ is compact for $a \in A$. If the assumption that X be a complex manifold is dropped, then both assertions of Theorem 3 become false as can be shown by examples.

5. Maximal meromorphic mappings

All complex spaces in this section are irreducible. Before we state the problem we give the necessary definitions.

Let $f : X \underset{k}{\to} Y$ be weakly holomorphic and not empty. The *rank* rk f of f is by definition the global rank of the holomorphic mapping $\hat{f} : G_f \to Y$, i.e.,

$$\operatorname{rk} f = \sup_{z \in G_f} \operatorname{codim}_z \hat{f}^{-1}(\hat{f}(z)).$$