Chapter 3. FINITE MORPHISMS

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exp $(t\,\xi)$ must leave x invariant, and therefore ξ must vanish at x (this was the case in the preceding example).

The Zariski tangent space. The Zariski tangent space at a point x of an analytic space X is the dual over $\mathbf{C}$ of $\mathfrak{M}_x/\mathfrak{M}_x^2$; here $\mathfrak{M}_x$ denotes as usual the maximal ideal of $\mathcal{O}_{X,x}$. If X is defined by the ideal $\mathcal{I} \subset \mathcal{O}_U$, U an open set in $\mathbf{C}^n$, the tangent space may be identified with the linear variety defined by the linear parts of all germs $\in \mathcal{I}_x$.

The Zariski tangent space of X_{red} may be strictly contained in that of X. For instance, if X is a double point, $\mathfrak{M}_x/\mathfrak{M}_x{}^2$ has dimension 1 over $\mathbf{C}$ whereas $\mathfrak{M}_x/\mathfrak{M}_x{}^2 = \{0\}$ for X_{red}, the corresponding simple point.

The tangent cone. The tangent cone at a point x of a local model $(X, \mathcal{O}_X)$ is the algebraic variety (with nilpotents, in general) defined by the ideal generated by the first non-vanishing homogeneous parts of the elements in $\mathcal{I}_x$, $\mathcal{I}$ being the ideal defining X. Since the Zariski tangent space is defined, in the local model, by the ideal spanned by the first-degree parts of the elements of $\mathcal{I}_x$ it is clear that it contains, and in general strictly, the tangent cone. If ξ is a vector field, $\xi\,(x)$ belongs to the reduced tangent cone at x, but since the possible values of $\xi\,(x)$ form a linear space, it is in general not equal to the whole cone.

Example 3. Let, again, X be the analytic subspace of $\mathbf{C}^2$ defined by the ideal $(x^3 - y^2)$. Then, as noted before, $\xi\,(x) = 0$ for all possible vector fields; the tangent cone is the algebraic variety defined by the ideal (y^2), and the reduced tangent cone is the variety $y = 0$; finally, the Zariski tangent space is the whole space $\mathbf{C}^2$, for $x^3 - y^2$ contains no linear terms.

Chapter 3.

FINITE MORPHISMS

3. 1. *Local theory.*

As elsewhere in these notes, we denote by $\mathbf{C}\{x_1, ..., x_n\}$ the ring of convergent power series in n variables $x_1, ..., x_n$ First, we recall the so-called " Weierstrass preparation theorem ".

Theorem 3.1.1. (Späh, Rückert). Given $\Phi \in \mathbf{C}\{x_1, ..., x_n\}$, with $\Phi\,(0, ..., 0, x_n) = x_n{}^p +$ (higher order terms), any $f \in \mathbf{C}\{x_1, ..., x_n\}$ can be written

$$f = \Phi Q + \sum_{i=0}^{p-1} x_n^i a_i$$

with $Q \in \mathbf{C}\{x_1, ..., x_n\}$, $a_i \in \mathbf{C}\{x_i, ..., x_{n-1}\}$
This representation is unique.

Corollary 3.1.2. (Weierstrass). Given Φ as in the preceding theorem, there exist $u \in \mathbf{C}\{x_1, ..., x_n\}$, with $u(0) \neq 0$ and $a_i \in \mathbf{C}\{x_1, ..., x_{n-1}\}$, with $a_i(0) = 0$ such that

$$\Phi u = x_n^p + \sum_{i=0}^{p-1} a_i x_n^i$$

u and (a_i) are unique.

The corollary results easily from the theorem, when applied to $f = x_n^p$. For the proof of theorem 3.1.1., see e.g. [5] or [9]. We recall also that theorem 3.1.1. implies the facts that $\mathbf{C}\{x_1, ..., x_n\}$ is noetherian, and is a unique factorisation domain.

Definition 3.1.3. An analytic algebra (we shall say also " analytic ring ") is a quotient $\mathbf{C}\{x_1, ..., x_n\}/\mathcal{I}$, where $\mathcal{I}$ is a non trivial ideal (*i.e.* $\mathcal{I} \neq \mathbf{C}\{x_1, ..., x_n\}$). An analytic algebra A is clearly a local $\mathbf{C}$-algebra ; we denote by $\mathfrak{M}(A)$ its maximal ideal; we have $A/\mathfrak{M}(A) \simeq \mathbf{C}$.

An analytic algebra, being a quotient of a noetherian ring, is a noetherian ring, and therefore is separated in the Krull topology (see appendix).

If A and B are two analytic algebras, and $f: A \to B$ a homomorphism (with $f(1) = 1$), we recall that f is automatically local and therefore continuous in the Krull topology (see § 1.2.). If E is a B-module (unitary), then the map $A \times E \to E$ defined by $(a, e) \to f(a)\, e$ makes E an A-module ; for simplicity, we write $f(a)\, e = a\, e$.

We can now state the preparation theorem, in the general form :

Theorem 3.1.3. Let A and B be analytic algebres, f a homomorphism $A \to B$, and E a finite B-module. Then E is finite over A if and only if $E/\mathfrak{M}(A)E$ is finite over $A/\mathfrak{M}(A) \simeq \mathbf{C}$ (by " finite over A " we mean " finitely generated as an A-module ").

This theorem can be precised as follows :

Corollary 3.1.4. Given A, B, f, E as above, suppose that the images of $e_1, ..., e_p$ in $E/\mathfrak{M}(A)E$ generate that module over $\mathbf{C}$; then $e_1, ..., e_p$ generate E over A.

Proof of the corollary, admitting the theorem. Let F be the sub-A-module of E spanned by $e_1, ..., e_p$; then, by hypothesis, we have $E = F + \mathfrak{M}(A) E$. On the other hand, using the theorem, we know that E is finite over A; we can therefore apply Nakayama's lemme (see Appendix), which proves the corollary.

The existence part of theorem 3.1.1. is a special case of the preceding result. For, we take $A = \mathbf{C}\{x_1, ..., x_{n-1}\}$, $B = \mathbf{C}\{x_1, ..., x_n\}$ and f the natural injection (or, in a more sophisticated language, $f = \pi^*$ where π is the projection $\mathbf{C}^n \to \mathbf{C}^{n-1}$ which " forgets the last coordinate ") ; choose now Φ as in theorem 3.1.1. and $E = B/(\Phi)$. Then $E/\mathfrak{M}(A)E$ is isomorphic to $\mathbf{C}\{x_n\}/(\Phi(0, ..., 0, x_n)) = \mathbf{C}\{x_n\}/(x_n{}^p)$, which is generated over $\mathbf{C}$ by the classes of $1, x_n, ..., x_n{}^{p-1}$. Therefore, the corollory 3.1.4. shows that the classes of $1, x_n, ..., x_n{}^{p-1}$ in E generates E over A, which is the existence part of theorem 3.1.1.

A direct proof of theorem 3.1.3. in a slightly less general case ($E = B$) can be found in [6] (the general case could be easily deduced of it). We shall follow here another method, used by Mather [7] in the $\mathbf{C}^\infty$-case, and deduce theorem 3.1.3. from theorem 3.1.1. We proceed in three steps.

Step 1. $A = \mathbf{C}\{x_1, ..., x_{n-1}\}$, $B = \mathbf{C}\{x_1, ..., x_n\}$, $f = \pi^*$, the natural injection $A \to B$. As in the theorem, E is a finite B-module such that $E/\mathfrak{M}(A)E$ is finite over $\mathbf{C}$.

We first prove the existence of a finite number of elements $e_1, ..., e_p$ in E such that any $e \in E$ can be written $e = \Sigma\, b_i\, e_i$, with $b_i \in f(A) + \mathfrak{M}(A) B$. To this end, let $\varepsilon_1, ..., \varepsilon_q$ generate E over B, and let $\eta_1, ..., \eta_r$ be members of E such that their classes $\bar{\eta}_1, ..., \bar{\eta}_r$ modulo $\mathfrak{M}(A) E$ generate $E/\mathfrak{M}(A) E$ over $\mathbf{C}$. Thus, for any $e \in E$, we have, for suitables $\gamma_i \in \mathbf{C}$

$$e - \Sigma\gamma_i\, \eta_i \in \mathfrak{M}(A) E$$

and therefore

$$e - \Sigma\gamma_i\, \eta_i = \Sigma b_j\, \varepsilon_j\,, \qquad b_j \in \mathfrak{M}(A) B$$

and it suffices to take $p = q + r$, $(e_1, ..., e_p) = (\eta_1, ..., \eta_r, \varepsilon_1, ..., \varepsilon_q)$

Therefore, for $1 \leqslant i \leqslant p$, we have

$$x_n\, e_i = \sum_j v_{ij}\, e_j\,, \qquad v_{ij} \in f(A) + \mathfrak{M}(A) B$$

(in other words, $v_{ij}(0, ..., 0, x_n)$ is a constant). If we put $\Phi = \det(x_n\, \delta_{ij} - v_{ij})$, we have $\Phi\, e_i = 0$ $i = 1, ..., p$, then $\Phi E = 0$. Therefore E is a module over $B/(\Phi)$, generated e.g. by $e_1, ..., e_p$. But $\Phi(0, ..., 0, x_n)$ is a monic polynomial of degree p, and therefore is not identically zero ; by theorem

3.1.1., $B/(\Phi)$ is finite over A, and is generated by $1, x_n, \ldots, x_n^k$ for some $k \leqslant p-1$. Then E is finite over A.

Step 2. We suppose that A and B are regular analytic rings:

$$A = \mathbf{C}\{x_1, \ldots, x_n\}, \qquad B = \mathbf{C}\{y_1, \ldots, y_m\}$$

and let f be any homomorphism $A \to B$.

We factorise f in the following way

$$\begin{array}{ccc} & C = \mathbf{C}\{x_1, \ldots, x_n, y_1, \ldots, y_m\} & \\ {}^{i}\nearrow & & \searrow^{\tilde{f}} \\ A = \mathbf{C}\{x_1, \ldots, x_n\} & \xrightarrow{f} & B = \mathbf{C}\{y_1, \ldots, y_m\} \end{array}$$

where i is the natural injection, and $\tilde{f}$ is "the map into the graph" defined by

$$\tilde{f}(x_i) = f(x_i), \qquad \tilde{f}(y_j) = y_j$$

By our hypothesis, E is finite over B; then, $\tilde{f}$ being surjective, E is finite over C; the problem is now reduced to one similar to the first case, except that the number of additional variables is m instead of 1. The proof follows by repeated use of step 1.

Step 3. General case. We have now $A = \mathbf{C}\{x_1, \ldots, x_n\}/\mathscr{I}$, $B = \mathbf{C}\{y_1, \ldots, y_m\}/\mathscr{J}$; and E is a finite B-module such that $E/\mathfrak{M}(A)E$ is finite over $\mathbf{C}$.

First, we put $A' = \mathbf{C}\{x_1, \ldots, x_n\}$ and we denote by f' $A \to B$ the composition of f and the natural projection $A' \to A$; it is clear that $E/\mathfrak{M}(A')E \simeq E/\mathfrak{M}(A)E$; therefore, we can replace A by A'.

Now, putting $B' = \mathbf{C}\{y_1, \ldots, y_m\}$ and π the natural projection $B' \to B$, we claim that there exists a commutative diagram of homomorphisms

$$\begin{array}{ccc} & {}^{\tilde{f}}\nearrow & B' \\ A' & & \downarrow \pi \\ & {}_{f'}\searrow & B \end{array}$$

For, let $\varphi_1, \ldots, \varphi_n \in B'$ be liftings of $f'(x_1), \ldots, f'(x_n)$; there exists a unique homomorphism $\tilde{f}: A' \to B'$ such that $\tilde{f}(x_i) = \varphi_i$; for any polynomial $a \in A$, we have $\pi \circ \tilde{f}(a) = f'(a)$; therefore, for any $a \in A$ we have $\pi \circ \tilde{f}(a) - f'(a) \in \bigcap_k \mathfrak{M}^k(B)$. But B is noetherian, hence separated in the Krull topology; therefore, we have $\bigcap_k \mathfrak{M}^k(B) = \{0\}$, and $\pi \circ \tilde{f} = f'$.

Now, we may consider E as a finite B'-module, and we are reduced to consider the situation $(A', B', \tilde{f}, E)$ instead of the given one; but that case was treated in step 2; this ends the proof of the theorem.

Remarks. *1.* The same proof applies to the real case, and, more generally, to analytic algebras over a complete valuated field.

2. In the C^{∞} case (over $\mathbf{R}$), it is known that the existence part of theorem 3.1.1. is true. Therefore steps 1 and 2 of the preceeding proof are applicable, but not step 3 (the lifting $\tilde{f}$ cannot be constructed a priori, so one has to *suppose* that such a lifting exists).

3.2. *Germs of analytic spaces.*

This concept will be introduced in terms of categories. As objects, we take triples $(X, \mathcal{O}_X, x)$ where $(X, \mathcal{O}_X)$ is an analytic space, and x a point of X ; as morphisms of $(X, \mathcal{O}_X, x)$ into $(Y, \mathcal{O}_Y, y)$ we take the germs at x of morphisms of $(X, \mathcal{O}_X)$ into $(Y, \mathcal{O}_Y)$, which map x into y. To simplify the notations, we write (X, x) for $(X, \mathcal{O}_X, x)$.

We shall prove some results on the correspondence between analytic rings and germs of analytic spaces.

Proposition 3.2.1. To any germ (X, x) of an analytic space is associated an analytic ring $\mathcal{O}_{X,x}$. Every analytic ring is obtained in this way. Every morphism $(X, x) \to (Y, y)$ of germs of analytic spaces induces a homomorphism $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ of analytic rings. Conversely every homomorphism $B \to A$ of analytic rings is obtained from a morphism of corresponding germs of analytic spaces; the latter is unique.

Proof. If (X, x) is a germ of analytic spaces, $\mathcal{O}_{X,x}$ is an analytic ring by definition. Now let $A = \mathbf{C}\{x_1, ..., x_n\}/\mathrm{I}$ be an analytic ring. We choose generators $f_1, ..., f_p$ for I and take an open neighborhood U of 0 such that representatives of $f_1, ..., f_p$ which are analytic in U can be found. These generators then define a coherent sheaf $\mathcal{I}$ of ideals on U which defines an analytic subspace X of U with $\mathcal{O}_{X,0} = A$.

If $f : B \to A$ is a homomorphism of analytic rings, we shall construct a morphism $(X, 0) \to (Y, 0)$ of corresponding germs which induces F. We may suppose

$$A = \mathbf{C}\{x_1, ..., x_n\}/(f_1, ..., f_p), \qquad B = \mathbf{C}\{y_1, ..., y_m\}/(g_1, ..., g_q);$$

as we have seen in § 1, F can be lifted into a homomorphism $F^1 : \mathbf{C}\{y_1, ..., y_m\} \to \mathbf{C}\{x_1, ..., x_n\}$; we can choose 1) open sets $U \subset \mathbf{C}^n$, $V \subset \mathbf{C}^m$ with $0 \in U, 0 \in V$ 2) holomorphic functions $\bar{f}_1, ..., \bar{f}_p$ in U and $\bar{g}_1, ..., \bar{g}_q$ in V such that their germs at 0 are precisely the f_i's and the g_j's, and 3) an holomorphic mapping $\Phi : U \to V$, with $\Phi(0) = 0$ such that Φ^* induces F' at the origin.

Denote now by $\mathscr{I}$ (resp $\mathscr{J}$) the coherent sheaf of ideals generated in U (resp. V) by the $\bar{f}_i's$ | resp. the $\bar{g}'_j s$). We have $\Phi^*(\mathscr{J})_0 \subset \mathscr{I}_0$, hence, since $\mathscr{J}$ is finitely generated by restricting U and V if necessary, we have $\Phi^*(\mathscr{J}) \subset \mathscr{I}$. Finally we take $X = \operatorname{supp} \mathcal{O}_U/\mathscr{I}$, $\mathcal{O}_X = \mathcal{O}_U/\mathscr{I} \mid_X$ and the same for Y ; it is clear that Φ induces the required morphism $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$.

Finally, if two morphisms $\varphi, \psi : (X, 0) \to (Y, 0)$ induce the same homomorphism $\mathcal{O}_{Y,0} \to \mathcal{O}_{X,0}$, we have to prove that φ and ψ are equals. We may assume that Y is given by a local model $(Y, \mathcal{O}_V | \mathscr{J} | Y)$ for some coherent sheaf $\mathscr{J}$ of ideals on an open set $V \subset \mathbf{C}^m$; by composition with the injection $Y \to V$, we may restrict ourselves to the case where $Y = \mathbf{C}^m$; the morphisms φ and ψ are now given by sections $f, g \in \Gamma(X, \mathcal{O}_X{}^m)$, and the hypothesis means that the germs of f and g at 0 coincide ; hence f and g coincide in a neighborhood of 0 in X, which proves the assertion.

3.3 *Finite morphisms*

Let $f : (X, 0) \to (Y, 0)$ be a morphism of germs of analytic spaces. Then f is called " finite " if the corresponding homomorphism $f^* : \mathcal{O}_{Y,0} \to \mathcal{O}_{X,0}$ makes $\mathcal{O}_{X,0}$ finite over $\mathcal{O}_{Y,0}$. According to the preparation theorem 3.1.3. in order that f be finite, it is necessary and sufficient that $\mathcal{O}_{X,0}/\mathfrak{M}(\mathcal{O}_{Y,0})\, \mathcal{O}_{X,0}$ be finite over $\mathbf{C}$; in geometrical terms, this means that the germ of space $f^{-1}(0)$ is finite over the point 0 (see § 1.3, example 4).

In the global case (complex or real), we give the following definition:

Definition 3.3.1. A morphism of *separated* analytic spaces $f = (f_0, f^1)$: $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is finite if the following properties hold:

1) f is proper (i.e. f_0 is proper).

2) For any point $x \in X$, the induced morphism of germs $f_x : (X, \mathcal{O}_X, x) \to (Y, \mathcal{O}_Y, f_0(x))$ is finite.

In the *complex* case, we have the following results :

Proposition 3.3.2. f is finite if and only if f is proper and, for any $b \in Y$, the *set* $f_0{}^{-1}(b)$ is finite.

This proposition is more or less equivalent to the " Nullstellensatz "; for the proof see e.g. Houzel [6] or Narasimhan [9]. In the real case, the part " if " of this proposition is not even true when Y is a point : for instance the subspace of $\mathbf{R}^2$ defined by $\mathscr{I}$ = (coherent sheaf of ideals generated by $x_1^2 + x_2^2$) has support 0 ; but $\mathbf{R}\{x_1, x_2\}/(x_1^2 + x_2^2)$ is not finite over $\boldsymbol{R}$.

Proposition 3.3.2. If $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a finite morphism, then the direct image $f_*(\mathcal{O}_X)$ is a coherent analytic sheaf of $\mathcal{O}_Y$-modules ; conversely, let $\mathscr{A}$ be a sheaf of $\mathcal{O}_Y$-algebras, which is coherent as sheaf of $\mathcal{O}_Y$-modules. Then there exists an analytic space $(X, \mathcal{O}_X)$ and a finite morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ such that $f_* (\mathcal{O}_X)$ is isomorphic with $\mathscr{A}$ as sheaf of $\mathcal{O}_Y$-algebras ; the triple $(X, \mathcal{O}_X, f)$ is unique up to an isomorphism.

We do not prove this proposition here and refer to Houzel [6] or Narasimhan [9] for this proof. We note also that a proof of the direct part can be given along the same lines as theorem 3.1.3, combined with the fact that direct images under finite morphism preserve exact sequences of sheaves of $\mathcal{O}_X$-modules (in other words, that higher direct images are zero). We note also that, for *proper* morphisms (not necessarily finite), a much deeper result has been proved by Grauert [2], [3].

Finally, we remark that, in the real case, proposition 3.3.2. is false (take, for instance, X the submanifold of $\mathbf{R}^2$ defined by $x_2 - x_1^2 = 0$, $Y = \mathbf{R}$ and $f =$ the projection on the x_2-axis ; $f_* (\mathcal{O}_X)$ has support $x_2 \geqslant 0$, which is not an analytic subset of $\mathbf{R}$, hence $f_* (\mathcal{O}_X)$ cannot be coherent!)

CHAPTER 4.

THE FINITENESS THEOREM

In this chapter, we consider only *complex* analytic spaces, separated and having a countable basis of open sets.

4.1. *Stein spaces*

Let $(X, \mathcal{O}_X)$ be an analytic space, and K a subset of X ; we denote, as usual by $\hat{K}$ the set

$$\left\{ x \in X \mid \forall f \in \Gamma(X, \mathcal{O}_x) : |f(x)| \leqslant \sup_{y \in K} |f(y)| \right\}$$

Definition 4.1.1. a) $(X, \mathcal{O}_X)$ is called holomorphically convex if, for any K compact $\subset X$, $\hat{K}$ is compact ;
b) $(X, \mathcal{O}_X)$ is called a Stein space if it is holomorphically convex, and if, for any $x \in X$, there exist sections $f_1, ..., f_p \in \Gamma(X, \mathcal{O}_X)$ with $f_i(x) = 0$, such that x is an isolated point of the counter-image of 0 in the morphism $(X, \mathcal{O}_X) \to \mathbf{C}^p$ defined by $f_1, ..., f_p$, (This last property can also by expressed as the fact that the morphism of germs : $(X, \mathcal{O}_X, x) \to (\mathbf{C}^p, 0)$ defined by $f_1, ..., f_p$ is finite).