Chapter 2. DIFFERENTIAL CALCULUS ON ANALYTIC SPACES

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sheaf of ideals defining $\mathcal{O}_V$. The sheaf $\mathcal{I}'$ is coherent by the Oka-Cartan then, and $\mathcal{I}$ by assumption, hence $\mathcal{I}'/\mathcal{I}$ is coherent. Now define $(X_{red}, \mathcal{O}_{X_{red}})$ by taking X_{red} equal to X as a topological space, and $\mathcal{O}_{X_{red}} = \mathcal{O}_X/\mathcal{N}$.

For a systematic treatment of reduced analytic spaces we refer to Narasimhan [9]. We remark here that for non-reduced spaces, the decomposition into irreducible components has no meaning, even at a point.

Example. Consider the analytic subspace X of $\mathbf{C}^2$ defined by the ideal $\mathcal{I}$ generated by $x_1 x_2$ and x_2^2. It is clear that $\mathcal{I}_X = (x_2)$ if $x_1, \neq 0$, hence X is locally the one-dimensional manifold $x_2 = 0$ outside the origin. However, $\mathcal{I} = (x_2) \cap (x_1, x_2^2)$ which is strictly contained in (x_2) at the origin so the origin cannot be an ordinary point, in particular X is not an analytic subspace of the manifold $x_2 = 0$. To illustrate this further, let $\pi : X \to \mathbf{C}$ be the projection of X into $\mathbf{C}$ defined by $(x_1, x_2) \to x_1$. We shall calculate the fibers $\pi^{-1}(a) = X \times_{\mathbf{C}} \{a\}$ of this map for an arbitrary point $a \in \mathbf{C}$.

To do this, we use the characterisation of $\mathcal{O}_{\pi^{-1}(a),b}$ given in § 1.3, example 4: if $a(= x_1) \neq 0$, and $b = (a,0)$ we find immediately $\mathcal{O}_{\pi^{-1}(a),b} = \mathbf{C}$ hence $\pi^{-1}(a)$ is a simple point. But, if $a = 0$, $b = (0,0)$ we find $\mathcal{O}_{\pi^{-1}(a),b} = \mathbf{C}\{x_1, x_2\}/(x_1, x_2^2) \simeq \mathbf{C}\{x_2\}/(x_2^2)$; hence $\pi^{-1}(0)$ is a double point.

Chapter 2.

DIFFERENTIAL CALCULUS ON ANALYTIC SPACES

Very little is known yet about differential operators on spaces with singularities. We shall just give the main definitions here. Let us first consider differential operators in the regular case, i.e. on manifolds. One then usually introduces, for each point a on a complex manifold X, the vector space $\mathcal{O}_{X,a}/\mathfrak{m}_a^{k+1}$, the jets of order k at a. Here $\mathfrak{m}_a$ denotes, as usual, the maximal ideal in $\mathcal{O}_{X,a}$. The jets of order k form, in a natural way, an analytic bundle J^k. A differential operator is then by definition a morphism of J^k into the trivial bundle $X \times \mathbf{C}$. Differential operators from bundles to bundles are defined similarly.

This definition is not suitable for generalization to analytic spaces (the collection of vector spaces $\mathcal{O}_{X,a}/\mathfrak{m}_a^{k+1}$ would not define a bundle over X). However, as noted by Grothendieck [4], if we consider, instead of the bundle J^k, the sheaf of sections of it, we can generalize to any analytic space X the definition above in the following way:

Let Δ be the diagonal in $X \times X$ regarded as a closed analytic subspace, i.e. the sheaf $\mathcal{I}$ of ideal defining Δ is that generated by all germs of the form $\pi_1{}^*f - \pi_2{}^*f$ where f is a germ on X, $\pi_j : X \times X \to X$ being the projections. Similarly $\Delta^{(k)}$ denotes the analytic subspace of X^2 with sheaf of ideals $\mathcal{I}^{k+1}$ ($\Delta^{(k)}$ is not reduced for $k > 1$ even if X is). The structure sheaf $\mathcal{O}_{\Delta(k)}$ on $\Delta^{(k)}$ is moved down to X by π_1 ; its direct image will be denoted by $\pi_{1*}\, \mathcal{O}_{\Delta(k)}$, a sheaf on X. It is made into an $\mathcal{O}_X$-module by the map $\mathcal{O}_X \to \pi_{1*}\, \mathcal{O}_{\Delta(k)}$ defined by $\pi_1{}^* : \mathcal{O}_{X,x} \to \mathcal{O}_{X\times X;(x,\alpha)}$.

Definition. (Grothendieck). A linear differential operator of order $\leqslant k$ is a morphism $\pi_{1*}\, \mathcal{O}_{\Delta(k)} \to \mathcal{O}_X$, both sheaves being considered as $\mathcal{O}_X$-modules.

Let us see how this definition connects with the usual one in case X is a manifold.

Differential operators in $\mathbf{C}^n$. Let U be open in $\mathbf{C}^n$ (or a coordinate patch on a manifold). Then a differential operator in the usual sense in U is a map $Q : \mathcal{O}_U \to \mathcal{O}_U$ of the form

$$f \to \sum_{|j| \leq k} a_j D^j f$$

where a_j are analytic functions in U and

$$D^j f = \frac{\partial^{j1}}{\partial x_1^{j1}} \cdots \frac{\partial^{jn}}{\partial x_n^{jn}} f\,.$$

Clearly Q is $\mathbf{C}$-linear and continuous. Consider the map φ ; $\mathcal{O}_U \to \pi_{1*}\, \mathcal{O}_{\Delta(k)}$ defined as the composition of $\pi_2{}^* : \mathcal{O}_U \to \mathcal{O}_{\Delta(k)}$ and the natural map $\to \pi_{1*}\, \mathcal{O}_{\Delta(k)}$. In somewhat sloppy notation,

$$f(x) \xrightarrow{\varphi} \sum_{|j| \leq k} D^j f(x)\,(y-x)^j/j\,!\,,$$

where $j! = j_1! \ldots j_n!$. Now if $P : \pi_{1*}\, \mathcal{O}_{\Delta(k)} \to \mathcal{O}_U$ is a differential operator in the sense of the definition just made, we get a differential operator in the elementary sense by putting $Q = P \circ \varphi$. Here $a_j(x) = P((y - x)^j/j!)$ are sections of $\mathcal{O}_U$ over all of U, for P maps sections of $\Gamma(U, \pi_{1*}\mathcal{O}_{\Delta(k)})$ onto sections of $\Gamma(U, \mathcal{O}_U)$.

Conversely, if Q is given, P can be constructed from the requirement $P((y - x)^j) = j!\, a_j$, for the germs $(y - x)^j$, $|j| \leqslant k$, generate $\pi_{1*}\, \mathcal{O}_{\Delta(k)}$ as an $\mathcal{O}_U$-module. By this procedure every linear differential operator $\in \mathrm{Hom}_{\mathcal{O}_U}(\pi_{1*}\, \mathcal{O}_{\Delta(k)}, \mathcal{O}_U)$ defines an element of $\mathrm{Hom}_{\mathbf{C}}(\mathcal{O}_U, \mathcal{O}_U)$; hence every germ $\in$

$Hom_{\mathcal{O}_U}(\pi_{1*}\mathcal{O}_{\Delta(k)}, \mathcal{O}_V)$ of a differential operator determines an element of $Hom_{\mathbf{C}}(\mathcal{O}_U, \mathcal{O}_U)$.

Lifting of differential operators, in models. We shall also describe in a more concrete way the differential operators in a model $(X, \mathcal{O}_X)$ where $X = \operatorname{supp} \mathcal{O}_U/\mathcal{J}$, $\mathcal{O}_X = (\mathcal{O}_U/\mathcal{J}) \mid X$, U being a domain of holomorphy in $\mathbf{C}^n$, $\mathcal{J}$ a coherent sheaf of ideals in $\mathcal{O}_U$. We claim that the differential operators in X correspond to those differential operators in the elementary sense in U which map $\mathcal{J}$ into $\mathcal{J}$, taken modulo those which map $\mathcal{O}_U$ into $\mathcal{J}$. Consider the following diagram where all arrows except Q, P, P_1 are ring homomorphisms:

$$\begin{tikzcd}
\mathcal{O}_U \arrow[d, "\pi_2^*"] \arrow[drr, "Q"] & & \\
\mathcal{O}_{U\times U}|_{\Delta U} \arrow[r] \arrow[d] & \pi_{1*}\mathcal{O}_{\Delta(k)_U} \arrow[r, "P"] \arrow[d, "\Psi"] & \mathcal{O}_U \arrow[d, "\psi"] \\
\mathcal{O}_{X\times X}|_{\Delta X} \arrow[r] & \pi_{1*}\mathcal{O}_{\Delta(k)_X} \arrow[r, "P_1"] & \mathcal{O}_X
\end{tikzcd}$$

First, if P_1 is a given differential operator in X we may construct an operator P in U (and hence an operator Q in the elementary sense) as follows. To give P it is sufficient to give the sections a_j, $|j| \leqslant k$, onto which $(x - y)^j/j\,!$ are to be mapped (see the previous section). The image in $\mathcal{O}_X$ of the sections $(x - y)^j/j\,!$ by $P_1 \circ \psi$ are certain sections b_j. In view of Theorem B of Cartan these can be lifted to sections a_j of $\mathcal{O}_U$ over U. The ambiguity in constructing a_j corresponds exactly to an operator mapping $\pi_{1*}\mathcal{O}_{\Delta(k)_U}$ into $\mathcal{J}$. Let us also note that the corresponding operator $Q : \mathcal{O}_U \to \mathcal{O}_U$ has the claimed property that $Q(\mathcal{J}) \subset \mathcal{J}$. In fact, if $f \in \mathcal{J}$, the image of f down in $\mathcal{O}_{X\times X} \mid_{\Delta X}$ is already zero, a fortiori its image in $\mathcal{O}_X$ is zero. Since the diagram is commutative it follows that the image of f by Q is in $\mathcal{J}$.

Conversely, suppose that Q is given, $Q(\mathcal{J}) \subset \mathcal{J}$, and that P is constructed from Q as before. We shall then find P_1 to make the diagram commutative. We clearly have to define $P_1\, g$ by first lifting $g \in \pi_{1*}\mathcal{O}_{\Delta(k)_X}$ to $\pi_{1*}\mathcal{O}_{\Delta(k)_U}$, then take $\psi \circ P$ of the element thus obtained. To see that this definition is allowed we have to see that $\ker \psi \subset \ker(\psi \circ P)$. However, it is clear that $\ker \psi$ is generated by the images in $\pi_{1*}\mathcal{O}_{\Delta(k)_U}$ of $\pi_1{}^*\mathcal{J}$ and $\pi_2{}^*\mathcal{J}$. Now if $f \in \pi_1{}^*\mathcal{J}$, its image in $\mathcal{O}_U$ is $a_0(x) f(x)$ which belongs to $\mathcal{J} = \ker \psi$. On the other hand, if $f \in \pi_2{}^*\mathcal{J}$, its image in $\mathcal{O}_U$ is contained in $\mathcal{J}$ by our assumption on Q. This proves that P_1 is well-defined.

Example 1. Let us determine all differential operators on the double point $(0, \mathbf{C}\{x\}/(x^2))$. By the principle of lifting differential operators we shall therefore decide when a differential operator in **C**,

$$Q = \sum_{o}^{k} a_j(x) \frac{\partial^j}{\partial x^j}$$

maps (x^2) into (x^2). First we reduce the coefficients modulo (x^2) so that

$$Q = \sum_{o}^{k} (b_j + c_j x) \frac{\partial^j}{\partial x^j}$$

where b_j, c_j are complex numbers. It is clearly necessary and sufficient that

$$Q\left(\frac{x^k}{k!}\right) \equiv 0 \pmod{x^2}, \qquad k \geqslant 2.$$

This is equivalent to

$$b_k = c_k + b_{k-1} = 0, \qquad k \geqslant 2,$$

hence the differential operators are precisely

$$b_0 + c_0 x + (b_1 + c_1 x) \frac{\partial}{\partial x} - b_1 x \frac{\partial^2}{\partial x^2}.$$

This gives a space of dimension 4 on **C**, with the following basis

$$Q_1 = \text{identity}; \qquad Q_2 = x; \qquad Q_3 = x \frac{\partial}{\partial x}; \qquad Q_4 = \frac{\partial}{\partial x} - x \frac{\partial^2}{\partial x^2}$$

(this last being of order two!). Note that all the **C**-linear maps of the space of dual numbers into itself are given by differential operators.

We define the composition of differential operators as in the non singular case, by the composition of the corresponding elements of Hom $(\mathcal{O}_X, \mathcal{O}_X)$ (we leave the details to the reader) ; if P has order $\leqslant p$ and Q order $\leqslant q$, the $P\,Q$ has order $\leqslant p + q$. Denote by $\mathcal{D}_{X,x}$ the space of germs of differential operators of any order on X at the point x ; with this operation, $\mathcal{D}_{X,x}$ is a (non-commutative) ring.

Very little is known on these rings, except in the non-singular case. For instance:

1) Are they " finitely generated " in the sense that there would exist $D_1, \ldots,$ $D_k \in \mathcal{D}_{X,x}$ such that any $D \in \mathcal{D}_{X,x}$ could be written as $D = \sum f_{i_1 \ldots i_p}$ $D_{i_1} \ldots D_{i_p}$ $(f_{i_1, \ldots, i_p} \in \mathcal{O}_{X,x}$; $i_1, \ldots, i_p = 1, \ldots, k)$?
2) Are they left or right noetherians ? (In the non-singular case, to prove this result, it suffices to introduce the filtration defined by the order and

to note that the associated graded ring is a ring of polynomials on $\mathcal{O}_{X,x}$, and therefore is noetherian).

The differential. If f is a holomorphic germ on an analytic space $(X, \mathcal{O}_X)$ we define its differential $d\,f$ as the image by $\pi_1{}^*$ of the germ $\pi_2{}^* f - \pi_1{}^* f$ in $\mathcal{O}_{\Delta^{(1)}}$; $\pi_j : \Delta^{(1)} \to X$ being the two natural projections induced by the projections $X^2 \to X$. Obviously $\pi_2{}^* f - \pi_1{}^* f$ vanishes on the diagonal, i.e. it belongs to the sheaf $\tilde{\Omega}_X$ of ideals of germs in $\mathcal{O}_{\Delta^{(1)}}$ which have restriction zero to $\Delta = \Delta^{(0)}$. We call $\Omega_X = \pi_{1*} \tilde{\Omega}_X$ the *sheaf of (first order) differentials* on X. Clearly we have a natural isomorphism

$$\pi_{1*} \mathcal{O}_{\Delta^{(1)}} \cong \mathcal{O}_X \oplus \Omega_X .$$

In a local model $(V, \mathcal{O}_V)$, V an analytic subset of $U \subset \mathbf{C}^n$, U open, we can also introduce the sheaf of differentials as follows. Let Ω_U denote the sheaf of germs of differential 1-forms on U.

Suppose that the sheaf of ideals defining $\mathcal{O}_V$ is generated by $f_1, ..., f_p$ Then Ω_U modulo the subsheaf $(f_1, ..., f_p)\, \Omega_U + \mathcal{O}_U\, (d\,f_1, ..., d\,f_p)$ defines a sheaf with support equal to V which coincides with the sheaf of differentials on V as defined above.

Vector fields. A *germ* ξ *of a vector field* at a point x of an analytic space is the same as a first order homogeneous (i.e. $\xi\,(1) = 0$) differential operator at x. In other words, ξ is defined as an $\mathcal{O}_{X,x}$-linear map of the germs of differential 1-forms at x into $\mathcal{O}_{X,x}$. A *vector field* on X is, of course, a section of the sheaf of germs of vector fields so defined.

Example 2. Consider the analytic subspace of $\mathbf{C}^2$ defined by the ideal $(x^3 - y^2)$. Here all vector fields are linear combinations of the equivalence classes of $2\,x\,\partial/\partial\,x + 3\,y\,\partial/\partial\,y$ and $2\,y\,\partial/\partial\,x + 3\,x^2\,\partial/\partial y$. In particular, all vector fields vanish at the origin. To see this, it is only necessary to observe that a differential operator $a\,(x, y)\,\partial/\partial\,x + b\,(x, y)\,\partial/\partial\,y$ must give a multiple of $x^3 - y^2$ when applied to $x^3 - y^2$ if it shall operate on the ring $\mathbf{C}\,\{x, y\}$ $/(x^3 - y^2)$. Hence it must satisfy $3\,y\,a\,(x, y) - 2\,x\,b\,(x, y) \equiv 0 \pmod{x^3 - y^2}$. The space of these operators is spanned by the two just given, modulo $x^3 - y^2$.

If ξ is a vector field on X, one can define, as in the non-singular case, the " local group of automorphisms " $\exp\,(t\,\xi)$: it suffices to consider the case of a local model, when X is a closed subspace of U open, $\subset \mathbf{C}^n$, and ξ is the restriction of a vector field $\tilde{\xi}$ on U, and to note that $\exp\,(t\,\tilde{\xi})$ leaves X invariant. Suppose f. i. that X has an isolated singular point at x : then exp $(t\,\xi)$ must leave x invariant, and therefore ξ must vanish at x (this was the case in the preceding example).

The Zariski tangent space. The Zariski tangent space at a point x of an analytic space X is the dual over $\mathbf{C}$ of $\mathfrak{M}_x/\mathfrak{M}_x^2$; here $\mathfrak{M}_x$ denotes as usual the maximal ideal of $\mathcal{O}_{X,x}$. If X is defined by the ideal $\mathcal{I} \subset \mathcal{O}_U$, U an open set in $\mathbf{C}^n$, the tangent space may be identified with the linear variety defined by the linear parts of all germs $\in \mathcal{I}_x$.

The Zariski tangent space of X_{red} may be strictly contained in that of X. For instance, if X is a double point, $\mathfrak{M}_x/\mathfrak{M}_x{}^2$ has dimension 1 over $\mathbf{C}$ whereas $\mathfrak{M}_x/\mathfrak{M}_x{}^2 = \{0\}$ for X_{red}, the corresponding simple point.

The tangent cone. The tangent cone at a point x of a local model $(X, \mathcal{O}_X)$ is the algebraic variety (with nilpotents, in general) defined by the ideal generated by the first non-vanishing homogeneous parts of the elements in $\mathcal{I}_x$, $\mathcal{I}$ being the ideal defining X. Since the Zariski tangent space is defined, in the local model, by the ideal spanned by the first-degree parts of the elements of $\mathcal{I}_x$ it is clear that it contains, and in general strictly, the tangent cone. If ξ is a vector field, $\xi\,(x)$ belongs to the reduced tangent cone at x, but since the possible values of $\xi\,(x)$ form a linear space, it is in general not equal to the whole cone.

Example 3. Let, again, X be the analytic subspace of $\mathbf{C}^2$ defined by the ideal $(x^3 - y^2)$. Then, as noted before, $\xi\,(x) = 0$ for all possible vector fields; the tangent cone is the algebraic variety defined by the ideal (y^2), and the reduced tangent cone is the variety $y = 0$; finally, the Zariski tangent space is the whole space $\mathbf{C}^2$, for $x^3 - y^2$ contains no linear terms.

Chapter 3.

FINITE MORPHISMS

3. 1. *Local theory.*

As elsewhere in these notes, we denote by $\mathbf{C}\{x_1, ..., x_n\}$ the ring of convergent power series in n variables $x_1, ..., x_n$ First, we recall the so-called " Weierstrass preparation theorem ".

Theorem 3.1.1. (Späh, Rückert). Given $\Phi \in \mathbf{C}\{x_1, ..., x_n\}$, with $\Phi\,(0, ..., 0, x_n) = x_n{}^p +$ (higher order terms), any $f \in \mathbf{C}\{x_1, ..., x_n\}$ can