# 2. Notation

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# A NEW FIXED POINT THEOREM FOR CONTINUOUS MAPS OF THE CLOSED $n$-CELL

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## 1. INTRODUCTION

In this paper the authors prove two fixed point theorems for continuous maps of a closed $n$-cell $\eta^n$ into the euclidean space $R^n \supset \eta^n$. Neither theorem requires that $\eta^n$ be mapped into itself.

The main theorem is Theorem 1 in which it is proved that *a continuous mapping of a closed n-cell* $\eta^n$ *into* $R^n \supset \eta^n$ *which maps the boundary of* $\eta^n$ *into* $\eta^n$, *has a fixed point.* It is believed that this theorem is new and is stronger than Brouwer's classical fixed point theorem inasmuch as it implies the latter and has weaker hypotheses.

Although the same theorem can be proved in a much shorter way by using Tietze's extension theorem followed by the classical Brouwer's fixed point theorem, however, in the proofs given below no knowledge of these two theorems is presupposed.

In this paper the proofs of the theorems are based in part on use of homologies, and in part on the turning index (defined below), which is essentially a generalization to the $n$ dimensional case of the idea involved in [1], pp. 251-5, for the case of a circular disc.

## 2. NOTATION

In what follows, $R^n$ denotes an oriented euclidean $n$-space, fixed once and for all.

All closed solid $n$-spheres and $(n-1)$-spheres are assumed to be triangulated with solid $n$-spheres oriented to agree with

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$R^n$, and $(n-1)$-spheres oriented with orientations induced by their interiors.

Symbols $c^{n-1}$, $g^{n-1}$, ... denote oriented $(n-1)$-cycles in $R^n$; $D^{n-1}$, $V^{n-1}$, ... denote $(n-1)$-spheres in $R^n$. $E^n$ denotes a closed solid $n$-sphere in $R^n$, and the boundary of $E^n$ is denoted by $S^{n-1}$. $\eta^n$ denotes a closed $n$-cell in $R^n$ and the boundary of $\eta^n$ is denoted by $\sigma^{n-1}$.

In this paper $\eta^n$ is assumed to be the image of $E^n$ under homeomorphism $\theta$, and $\eta^n$ and $\sigma^{n-1}$ obtain their orientations from $E^n$ and $S^{n-1}$ respectively.

## 3. The Turning Index

Let $c^{n-1}$ be an $(n-1)$-cycle in $R^n$ and $g$ a continuous map of $c^{n-1}$ into $R^n$ having no fixed point. Let $D^{n-1}$ be an $(n-1)$-sphere with center 0, called a *direction sphere* [2]. Let $c^{n-1}$ be mapped on $D^{n-1}$ as follows. To a point $c \in c^{n-1}$ there corresponds a point $d \in D^{n-1}$ such that the line segment from 0 to $d$ has the same sense and direction as that from $c$ to $g(c)$. The resulting $(n-1)$-cycle $g^{n-1}$ on $D^{n-1}$ is called, in the sequel, *the $(n-1)$-cycle $g^{n-1}$ resulting from $g$ applied to $c^{n-1}$*, and the degree of the resulting map, that is, the multiple of $D^{n-1}$ which is homologous to $g^{n-1}$ (which is clearly independent of the radius of $D^{n-1}$ and the location of 0) is called the *turning index* of $c^{n-1}$ under $g$.

If $p$ is a point not on $c^{n-1}$, the *index of $p$ relative to $c^{n-1}$* is defined as the turning index of the map which maps every point of $c^{n-1}$ into $p$. (For odd $n$, this is the negative of the corresponding definition given in [3], as shown by Theorem 1.5, page 105).

## 4. Preliminary Lemmas

Lemma 1. *Let $g$ and $h$ be two continuous maps into $R^n$ of an $(n-1)$-cycle $c^{n-1}$, such that neither leaves any point of $c^{n-1}$ fixed, and, for no point $c \in c^{n-1}$ are the directions from $c$ to $g(c)$ and from $c$ to $h(c)$ exactly opposite. Then the turning indices of $c^{n-1}$ under $g$ and $h$ are equal.*

*Proof.* For each $c \in c^{n-1}$, the directions of the two vectors $\overline{c,g(c)}$ and $\overline{c,h(c)}$ are not opposite and hence, if not identical,