ON THE PROJECTION OF A FOUR-POINT SYSTEM OF CONICS INTO A FAMILY OF CIRCLES

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In view of the frequent use of the process of the projection of two common points of a four point system of real conics into the circular (isotropic) points with gay abandon of any difficulties which may arise from the reality or non-reality of part or parts of the consequent figure, the authors have felt the need for a closer investigation. We shall show that in order to obtain real circles when two real points are projected into the circular points two projections are necessary in general, and that only two real points, at most, on a conic remain real. We shall also establish the various types of the families of circles which are obtained.

Let A, B, C, D be the four common points. Then if A, B and C, D are pairs of real points or conjugate imaginaries, AB and CD are real lines. We construct a projection in which A and B (distinct points) become the circular points and thus any conic of the system becomes a circle. Suppose the four points are distinct and let AC, BD meet at L; AD, BC at M; AB, CD at N; AB, LM at F; CD, LM at G. If C and D coincide at A or at B no projection is possible. If C and D coincide at a point other than A or B the conic family fixes the tangent direction at the point (C, D) and this direction replaces CD in the construction. If A and C coincide and B and D coincide, the tangent directions at A and B are known and replace AC and BD: the points N and F may then be taken as any pair of harmonic conjugates with respect to A and B, with M and G coinciding with F.

Suppose first that A and B are real distinct points. In any real plane through A, B the joins of A and B to the circular points intersect in two imaginary points V and V'. Projecting from V on to any plane parallel to VAB the projection is virtual. If a real system of coaxal circles is to be obtained, their plane and their radical axis, the projection of CD, must be real. If AB meets the plane at great distance in F', then F' is a real point and its harmonic conjugates with respect to A, B, say N', and with respect to the circular points in the plane VAB are both real. The unique real line through V is therefore VN' and N' is the only real finite point on AB which will project into a real point. Since AB and CD are to remain real their intersection N must project into a real point and therefore N' coincides with N, and it follows that F' must coincide with F. Hence in general before the virtual projection can be made a real projection must first be made in which F is projected to great distance and LM is now parallel to AB.

Call π, a plane parallel to VAB, the plane on to which the virtual projection is made and π' the plane of the quadrangle $ABCD$. The same lettering will be used for a point of the original figure and its projection where there is no possibility of confusion. Since VN is the only real line through V, the only real points of π' which remain real on projection are N and those on the line of intersection of π and π'. The only real lines which remain real are those through N and the line of intersection of π and π'. When L and M are determined by the quadrangle (thus excluding M in the case when A and C coincide and B and D) they become the point circles of the system: C and D become imaginary points in all cases and hence for a real system of circles π must pass through LM. In the projected figure LM is the line of centres and CD the radical axis. These lines are therefore at right angles and a conic of the system becomes a real circle if and only if the conic cuts LM in real points.

This discussion has tacitly assumed that A, B, L, M, N are finite points. This can be ensured by taking in the preliminary real projection a general finite line through F, not AB nor LM, as the vanishing line.

We give the detailed distributions for the various cases of real conics in the original system which lead to real circles.

The real projection of F to great distance does not affect the reality of points and lines. Following convention, we use the symbols $N \int AB$ and $N f AB$ to denote separation and non-separation of A, B by N.

(i) A, B, C, D real and distinct.

Quadrangle	Real conics becoming real circles surrounding L	Real conics becoming real circles surrounding M
$N \int AB$ and $\int CD$	Hyperbolas with A, C on one branch; B, D on the other.	Hyperbolas with A, D on one branch; B, C on the other.
$N \int AB$ and $f CD$	Hyperbolas with A, C, D on one branch; B on the other.	Hyperbolas with B, C, D on one branch; A on the other.
$N f AB$ and $\int CD$	Hyperbolas with A, B, D on one branch; C on the other.	Hyperbolas with A, B, C on one branch; D on the other.
$N f AB$ and $f CD$	Hyperbolas with A, B on one branch; C, D on the other.	Hyperbolas with A, B, C, D on one branch, ellipses, parabolas.

In this distribution we have considered the figures in which $C\,DN$ in the second and fourth cases, and $B\,AN$ in the third and fourth cases. If we take $D\,CN$ in the second, $A\,BN$ in the third, and either of these in the fourth, the corresponding columns must be interchanged.

It will be noted that coaxal circles with real intersections are never obtained when the four common points of the system are real.

(ii) C and D coincide not at A nor at B. L and M coincide with (C, D) and therefore all real conics become real circles.

$N \int AB$. The circles on one side of the common tangent arise from hyperbolas with A and (C, D) on one branch. Those on the other side arise from the hyperbolas with B and (C, D) on one branch.

$N f AB$. The circles on one side of the common tangent arise from hyperbolas with A, B and (C, D) on one branch, ellipses and parabolas. Those on the other side arise from the hyperbolas with A, B on one branch and (C, D) on the other.

(iii) C and D conjugate imaginaries.

N is external to all conics and its polar LM is real and cuts them in real points. Hence all real conics become real circles.

The points L, M are imaginary before and after projection and the circles, having imaginary limiting points, intersect in real points, and therefore C and D become real. Two conics become circles with centres on the same side of CD only if their poles of AB are both between or not between G and F.

(iv) A and C coincide, B and D coincide.

$N \int AB$. Hyperbolas with A, B on different branches become real circles.

$N f AB$. Hyperbolas with A, B on the same branch, ellipses and parabolas become real circles.

Since N can be chosen arbitrarily either of these groups of conics can be projected into real circles with the other group becoming imaginary circles.

It follows from (iii) that any pair of conjugate imaginary points on any real line through N and which harmonically separate N and the intersection of the line with LM project into real points on π, since these points and A, B form a quadrangle having N and the line LM as a vertex and opposite side of its harmonic triangle.

When two real points A, B of a single real conic are projected into the circular points to give a real circle the projection can be so arranged that any arbitrary real point, R, and a dependent point, S, of the conic remain real. Let the conjugate line to AB through R meet the conic in S, which is also real, and let any line through N, the pole of RS, meet the conic in C, D. $ABCD$ can be taken as the basic quadrangle and the projection of A, B into the circular points leads to a real circle with R and S the only real points remaining real.

Suppose secondly that A and B are a pair of conjugate imaginary points. After the projection of any general line through F into the line at great distance the initial construction leads to two real points V and V'. We may project directly from V on to any plane parallel to VAB without any further limitation. The projection is real and real points and loci remain real. Real conics become real circles having LM as line of centres. They are (i) a non-intersecting coaxal system with limiting points L, M if C, D are conjugate imaginaries, (ii) a system with common tangent if C, D coincide in a real point, (iii) an intersecting coaxal system if C, D are real distinct points, (iv) a concentric system if A, C coincide and B, D coincide.

We now give the analytic transformations equivalent to the processes of the projections. Let $pS_1 + qS_2 = 0$ be the equation of the family of conics referred to any axes where $S_1 = 0$, $S_2 = 0$ are conics with real equations defining the points A, B, C, D. There are three values of p/q which lead to degenerate conics $P_1 . Q_1 = 0$, $P_2 . Q_2 = 0$, $P_3 . Q_3 = 0$, where P_r, Q_r are linear functions, and one pair at least is real. Taking a real pair, say P_1, Q_1, as AB, CD, intersecting in general in a definite point N, we will first transform the axes so that AB is represented by $Y = 0$, CD by $X = 0$, and LM by $Y =$ a constant. This is equivalent to transforming a chosen line through F, other than AB or LM, into the line at great distance, and this incorporates the first real projection. Suppose

$$P_1 . Q_1 \equiv (a_1 x + b_1 y + c_1)(a_2 x + b_2 y + c_2) ,$$

and that the polar of N with respect to any conic of the system is $a_3 x + b_3 y + c_3 = 0$.

The conditions are satisfied by the identities

$$\begin{aligned} a_1 x + b_1 y + c_1 &\equiv pY/D , \\ a_2 x + b_2 y + c_2 &\equiv qX/D , \\ a_3 x + b_3 y + c_3 &\equiv (rY + s)/D , \end{aligned}$$

where $D = lX + mY + n$, if

$$\begin{vmatrix} a_1 & b_1 & pY - c_1 D \\ a_2 & b_2 & qX - c_2 D \\ a_3 & b_3 & rY + s - c_3 D \end{vmatrix} \equiv 0 .$$

Hence it is necessary and sufficient that the constants on the right hand side satisfy the relations

$$\begin{aligned} p \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} + r \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} &= m\Delta , \\ -q \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} &= l\Delta , \\ s \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} &= n\Delta , \end{aligned}$$

where

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

and cannot be zero unless $P_1 = 0$, $Q_1 = 0$ represent the same line (A coincides with C and B with D and N is not defined).

If in this special case the line $P_1 = 0$ is finite, we then take $a' x + b' y + c' = 0$ as representing the polar of the point, F, at great distance on $P_1 = 0$ (this is equivalent to taking N as the mid-point of AB), and the transform is then

$$a_1 x + b_1 y + c_1 \equiv pY/D \ ,$$
$$a' x + b' y + c' \equiv qX/D \ ,$$

and no further conditions are imposed. If $P_1 = 0$ is the line at great distance, N and F are taken as any pair of harmonic conjugates with respect to A and B. The basic equations of the transform are then

$$c_1 \equiv pY/D \ ,$$
$$a' x + b' y + c' \equiv qX/D \ ,$$

where $D = mY$ and $mc_1 = p$. A transform which may then be used is

$$a' x = (c_1 qX + h)/pY \ , \quad b' y = (- c' pY - h)/pY \ ,$$

where h is an arbitrary constant $\neq 0$.

Replacing X, Y by x, y the conic family is now in the form

$$\lambda xy = P.Q$$

in the general case, and

$$\lambda y^2 = P.Q$$

in the special case.

In all cases N is now the mid-point of AB.

We can take the equations of lines of $ABCD$ as

$$BD \ \ ax + by - 1 + ky = 0 \ , \qquad AC \ \ ax - by + 1 + ky = 0 \ ,$$
$$BG \ \ ax + by - 1 = 0 \ , \qquad AG \ \ ax - by + 1 = 0 \ ,$$
$$LM \ \ by - 1 = 0 \ ,$$

in the general case, and

$$BL \quad ax + by - 1 = 0 , \qquad AL \quad ax - by + 1 = 0 ,$$
$$LM \quad by - 1 = 0 ,$$

in the special case.

The product $P \, . \, Q$ can therefore always be put in the form

$$(ax + by - 1 + ky)\,(ax - by + 1 + ky) ,$$

with k zero in the special case. In these equations b is always real, a is real when A and B are real and can be replaced by a/ι (where a is real) when A and B are conjugate imaginaries: k is real when C and D are real and can be replaced by ιk or k/ι (where k is real) when C and D are conjugate imaginaries. There will be no loss in generality in taking all the constants as positive. We require a transformation in which A and B become the circular points. In the general case CD and LM are real, intersect in a finite point and after projection become lines at right angles which will be taken as $X = 0$ and $Y = 0$. In the special case LN and LM are real, intersect in a finite point and after projection become lines at right angles which will be taken as $X = 0$ and $Y = 0$. Consider the transformation

$$ax + by - 1 \equiv (X + \iota Y)/D ,$$
$$ax - by + 1 \equiv (X - \iota Y)/D ,$$

where $D = lX + mY + n$.

Hence $by = (\iota Y + D)/D$ and therefore, since AB becomes the line at infinity, D must be $-\iota Y + n$ where $n \neq 0$. This transformation is equivalent to

$$\frac{ax}{X} = \frac{by}{n} = \frac{by - 1}{\iota Y} .$$

The conic family

$$\lambda xy = (ax + by - 1 + ky)\,(ax - by + 1 + ky) ;$$

becomes the circle family

$$\frac{\lambda X n}{ab} = \left(X + \frac{kn}{b}\right)^2 + Y^2 ,$$

and in the special case the conic family

$$\lambda y^2 = (ax + by - 1)(ax - by + 1) ,$$

becomes the circle family

$$\lambda n^2/b^2 = X^2 + Y^2 .$$

Case I. A and B real.

From the equation of AG a must be real and in order that a real line through N shall project into a real line n must be real. The conics must be real if their equations are real since they pass through a real point A.

(i) C, D real and distinct.

From the equation of AC k must be real and not zero, and L, M become the real limiting points.

The circle will be real if and only if it meets $Y = 0$, the line of centres, in real points, that is if $\lambda(\lambda - 4ak) \geqslant 0$: λ must either be negative or positive and greater than $4ak$. Circles with negative λ surround the limiting point L $(\lambda = 0)$ and those with $\lambda > 4ak$ surround the limiting point M $(\lambda = 4ak)$.

(ii) C, D real and coincident.

Putting $k = 0$, we obtain from real conics the family of real circles

$$\lambda Xn/ab = X^2 + Y^2 ,$$

touching each other at the origin, and with their centres on the positive part of the X axis for positive λ and on the negative part for negative λ.

(iii) C, D conjugate imaginaries.

Replacing k by ιk, the equation of a conic is real if and only if $\lambda - 2a\iota k$ is real. The equation of the circle family is then

$$\lambda Xn/ab = (X + \iota kn/b)^2 + Y^2 ,$$

or

$$0 = X^2 + Y^2 - X(\lambda - 2a\iota k)\, n/ab - k^2 n^2/b^2 ,$$

a system of real coaxal circles intersecting in real points with centres on the positive or negative part of the X axis according as $\lambda - 2a\iota k$ is positive or negative.

(iv) A, C coincide and B, D coincide (the special case).

The conics of the family are real for all real λ and they transform into concentric circles which are real if and only if λ is positive.

Case II. A and B conjugate imaginaries.

In the basic equations we replace a by a/ι and n by $n\iota$, where the new a and n are real. The transformation is real and reduces to

$$\frac{ax}{X} = \frac{by}{n} = \frac{by - 1}{Y}.$$

In the conic families λ is replaced by $-\lambda$, when C and D are conjugate imaginaries, k by k/ι (k real and $\neq 0$). The results in the various cases are listed.

(i) C, D conjugate imaginaries.

For real conics $\lambda(4ak - \lambda) \geqslant 0$, giving the non-intersecting coaxal system of real circles

$$\lambda X n/ab = (X + kn/ab)^2 + Y^2 .$$

(ii) C, D real and coincident.

Putting $k = 0$ we obtain from real conics the family of real circles

$$\lambda X n/ab = X^2 + Y^2 .$$

(iii) C, D real and distinct.

For real conics $\lambda - 2a\iota k$ must be real, leading to the intersecting system of real coaxal circles

$$0 = X^2 + Y^2 - X(\lambda - 2a\iota k)\, n/ab - k^2 n^2/b^2 .$$

(iv) A, C coincide and B, D coincide (the special case).

For real conics λ must be positive, giving a real concentric circle system

$$\lambda n^2/b^2 = X^2 + Y^2 .$$