2. Heron's solution.

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In the geometric algebra of Euclid, addition and subtraction of simple numbers are, of course, performed by increasing and decreasing the lengths of lines. Multiplication is effected by construction of a rectangle using factors equivalent to the adjacent sides.

2. *Heron's solution.*

Heron proved many of the propositions of Book II by the algebraic method with the use of one line as a figure. The following excerpt is from a later Arabic commentary [1].

> " Then if we wish to demonstrate Heron's proof of this proposition, and the reasoning, we must show that the area outlined by the two parts AD and DB together with the square on line GD is equal to the square on line GB. We take two lines; one of them AD, is divided by point G, and the other line, DB, is not divided. In the proof of proposition 1 of (Book) II, the area that is outlined by the two lines AD and DB is equal to the sum of the two areas, each outlined by line BD with the two divisions AG and GD respectively. Since AG equals GB, then the sum of the two areas, bounded respectively by the two lines GB and BD, and the two lines GD and DB, are equal to the area outlined by the two lines AD and DB. Thus, there remains to us the square on GD. We distribute it as to partners (add it to both sides equally). Then the sum of the two areas bounded by the lines GB and BD, and the lines GD and DB respectively, together with the square on GD is equal to the area outlined by the two lines AD and DB plus the square on GD. But the area that it outlined by the two lines GD and DB plus the square on GD is equal to the area outlined by the two lines BG and GD, from proposition 3 of (Book) II [8]. The sum of the two areas, one outlined by lines BG and GD, and the other by the two lines GB and BD is equal to the area outlined by the two lines AD and DB plus the square on GD. But the demonstration of proposition 2 of (Book) II, the sum of the two areas, outlined respectively by the two lines GB and BD, and the two lines BG and GD, is equal to the square on line GB. The square

on line GB thus is equal to the area that is outlined by the two divisions AD and DB plus the square on GD. This is what we wished to demonstrate."

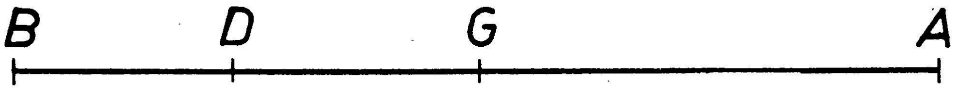

Fig. 1a

In modern symbols, the demonstration of Heron would proceed as follows:

To prove $\overline{AD} \cdot \overline{DB} + \overline{GD}^2 = \overline{GB}^2$.

Given $\overline{AD} = \overline{AG} + \overline{GD}$ and D another point on the line $\overline{AB}$.

By II, 1, $\overline{AD} \cdot \overline{DB} = \overline{BD} \cdot \overline{AG} + \overline{BD} \cdot \overline{GD}$.

But $\overline{AB} = \overline{GB}$ is given.

Then $\overline{GB} \cdot \overline{BD} + \overline{GD} \cdot \overline{DB} = \overline{AD} \cdot \overline{DB}$.

Add GD^2 to both sides of the equation:

$\overline{GB} \cdot \overline{BD} + \overline{GD} \cdot \overline{DB} + \overline{GD}^2 = \overline{AD} \cdot \overline{DB} + \overline{GD}^2$.

But by II, 3, $\overline{GD} \cdot \overline{DB} + \overline{GD}^2 = \overline{BG} \cdot \overline{GD}$.

Hence $\overline{BG} \cdot \overline{GD} + \overline{GD} \cdot \overline{BD} = \overline{AD} \cdot \overline{DB} + \overline{GD}^2$.

But, by II, 2, $\overline{GB} \cdot \overline{BD} + \overline{BG} \cdot \overline{GD} = \overline{GB}^2$.

$\therefore \overline{GB}^2 = \overline{AD} \cdot \overline{DB} + \overline{GD}^2$.

Abū Kāmil does not hesitate to utilize a variation of this procedure at a number of points. For example, he demonstrates its use in his solution of the equations $x + y = 10$, $x/y = 4$ $(x > y)$ [9]; and for $x + y = 10$, $xy = 21$ [10]. In the latter case, Abū Kāmil [11] has the following explanation:

"AG times GB equals twenty one. You divide line AB into two equal parts at point H. Then the product of AG by GB plus HG multiplied by itself equals HB multiplied by itself. The product of BH multiplied by itself is twenty five. AG times GB is twenty one. Then the remainder is HG multiplied by itself, or four; or HG is two. HB is five. Then GB remains as three and AG is seven."

Fig. 2

3. *Al-Khwārizmī's solution.*

Although al-Khwārizmī does not make use of the more abstract one line proof of Heron, nevertheless it is evident that he leans on the concrete concept of root [12] already known in ancient Babylonian times. In his discussion of the equation $x^2 + 21 = 10x$, al-Khwārizmī [13] makes it evident that he is utilizing a concept extremely practical in geometric terms.

" When a square plus twenty one dirhems are equal to ten roots, we depict the square as a square surface AD of unknown sides. Then we join it to a parallelogram, HB, whose width, HN, is equal to one of the sides of AD. The length of the two surfaces together is equal to the side HC. We know its length to be ten numbers since every square has equal sides and angles; and if one of its sides is multiplied by one, this gives the root of the surface, and if by two, two of its roots. When it is declared that the square plus twenty one equals ten of its roots, we know that the length of the side HC equals ten numbers because the side CD is a root of the square figure. We divide the line CH into two halves on the point G. Then you know that line HG equals line GC, and that line GT equals line CD. Then we extend line GT a distance equal to the difference between line CG and line GT to make the quadrilateral. Then line TK equals line KM, making a quadrilateral MT of equal sides and angles. We know that the line TK and the other sides equals five. Its surface is twenty five obtained by the