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# A characterization of the tangent function

Real-valued functions  $T$  satisfying the identity  $T(u + v) = \frac{T(u) + T(v)}{1 - T(u)T(v)}$  are studied. It is proved that the tangent function is the only such function having domain  $\{x: x \text{ real}, x \neq \frac{\pi}{2} + m\pi, m \text{ an integer}\}$  and satisfying  $T'(0) = 1$ .

During the past decade, several of our articles ([1], [2], [3], [4], [5]) have suggested a more fundamental role for the tangent function,  $\tan$ , in the curriculum. Many of these suggestions depend on the fact that  $\tan(u + v) = \frac{\tan u + \tan v}{1 - \tan u \tan v}$  whenever the right-hand side is

defined. It seems natural to ask whether this functional identity characterizes  $\tan$ . Accordingly, this note considers the class of *tangential functions*, by which we mean real-valued

functions  $T$  of a real variable such that  $T(u + v) = \frac{T(u) + T(v)}{1 - T(u)T(v)}$  whenever the right-

hand side is defined. In proposition 1(c), we produce infinitely many discontinuous tangential functions, thus answering the above question in the negative. On the other hand, Theorem 3 establishes that  $\tan$  is the only tangential function  $T$  which is defined at all real numbers other than  $\frac{\pi}{2} + m\pi$  (for  $m$  an integer) and satisfies  $T'(0) = 1$ . We hope that the material in this note will find use as enrichment material in introductory courses on calculus.

We begin by collecting some examples of tangential functions.

**Proposition 1.** Each of the following functions  $T$  is tangential:

- (a)  $T(x) = 1$  for each real number  $x$ ;
- (b)  $T(x) = -1$  for each real number  $x$ ;
- (c) Let  $p$  be a fixed prime number. If  $x$  is a real number, put

$$T(x) = \begin{cases} 0 & \text{if } x = \frac{m}{p^n} \text{ for some integers } m \text{ and } n \\ 1 & \text{otherwise.} \end{cases}$$

*Proof.* (a) and (b): The functional identity holds by default since  $1 - T(u)T(v) \equiv 0$  means that the identity's right-hand side is never defined.

(c): We shall verify the functional identity. Without loss of generality,  $T(u)T(v) \neq 1$ . Thus at least one of  $u, v$  is of the form  $\frac{m}{p^n}$ . If both  $u$  and  $v$  have this form, so does  $u + v$ , in which

case the functional identity reduces to the truism  $0 = \frac{0 + 0}{1 - 0}$ . On the other hand, if only

one of  $u, v$  has the form  $\frac{m}{p^n}$ , then  $u + v$  does not have this form, in which case the identity

reduces to either  $1 = \frac{1 + 0}{1 - 0}$  or  $1 = \frac{0 + 1}{1 - 0}$ . ■

A tangential function need not be continuous (and, hence, need not be differentiable).

Indeed, if  $T$  is as in Proposition 1 (c), then for each real number  $c$ ,  $\lim_{x \rightarrow c} T(x)$  does not exist. In Proposition 2(b), (c), we examine what can be said about a tangential function which is continuous (or differentiable).

**Proposition 2.** Let  $T$  be a tangential function. Then:

- (a) If  $T(0)$  is either 1 or  $-1$ , then  $T(x) = T(0)$  for each  $x$  in the domain of  $T$ .
- (b) Suppose that the domain of  $T$  contains a neighbourhood of 0. If  $T$  is continuous at 0, then  $T$  is a continuous function.
- (c) Suppose that the domain of  $T$  contains a neighbourhood of 0 and that  $T'(0) = 1$ . Then  $T$  is a differentiable function. In fact,  $T'(x) = 1 + T(x)^2$  for each  $x$  in the domain of  $T$ . Moreover,  $T(0) = 0$  and  $T$  is increasing on each subinterval of its domain.

*Proof.* (a) Suppose  $T(0) = \pm 1$  and  $T(x) \neq T(0)$ . Then  $1 - T(x)T(0) \neq 0$ , and so the functional identity of tangential functions leads to

$$T(x) = T(x + 0) = \frac{T(x) + T(0)}{1 - T(x)T(0)} = \frac{T(x) \pm 1}{1 \mp T(x)},$$

whence  $T(x)[1 \mp T(x)] = T(x) \pm 1$  and  $T(x)^2 = -1$ . This contradicts the fact that  $T$  is real-valued

(b) Since constant functions are continuous, (a) allows us to suppose that  $T(0) \neq \pm 1$ . Hence  $1 - T(0)^2 \neq 0$ , and so the functional identity gives

$$T(0) = T(0 + 0) = \frac{T(0) + T(0)}{1 - T(0)^2},$$

whence  $T(0)[1 - T(0)^2] = 2T(0)$  and  $0 = T(0)^3 + T(0) = T(0)[T(0)^2 + 1]$ . As  $T(0)^2 + 1 \neq 0$ , we have  $T(0) = 0$ . By hypothesis,  $\lim_{h \rightarrow 0} T(h) = T(0) = 0$ . For each  $x$  in the domain of  $T$ ,

$$\lim_{h \rightarrow 0} T(x + h) - T(x) = \lim_{h \rightarrow 0} \frac{T(x) + T(h)}{1 - T(x)T(h)} - T(x) = \lim_{h \rightarrow 0} \frac{[1 + T(x)^2]T(h)}{1 - T(x)T(h)},$$

which, by limit theorems, is just  $\frac{[1 + T(x)^2]0}{1 - (T(x))0} = 0$ . Thus  $\lim_{h \rightarrow 0} T(x + h) = T(x)$ , and so  $T$  is continuous at  $x$ , proving (b).

(c) Since  $T'(0) \neq 0$ , it follows from (a) that  $T(0) \neq \pm 1$ . Hence, by the proof of (b), we have

$T(0) = 0$ . It follows that  $\lim_{h \rightarrow 0} \frac{T(h)}{h} = \lim_{h \rightarrow 0} \frac{T(h) - T(0)}{h} = T'(0) = 1$ . Now, for each  $x$  in the

domain of  $T$ , we see, as in the proof of (b), that

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{T(x + h) - T(x)}{h} &= \lim_{h \rightarrow 0} [1 + T(x)^2] \frac{T(h)}{h} \left[ \frac{1}{1 - T(x)T(h)} \right] \\ &= [1 + T(x)^2] \cdot 1 \cdot \left[ \frac{1}{1 - T(x) \cdot 0} \right]; \end{aligned}$$

that is,  $T'(x) = 1 + T(x)^2$ . In particular,  $T'(x) > 0$ . The final assertion is a standard consequence of the Mean Value Theorem. ■

We next obtain the desired characterization of  $\tan$ . For motivation, note that  $\tan'(0) = \sec^2(0) = 1^2 = 1$ .

**Theorem 3.** *Let  $T$  be a tangential function such that  $T'(0) = 1$  and each real number  $x \neq \frac{\pi}{2} + m\pi$  (for  $m$  an integer) is in the domain of  $T$ . Then  $T = \tan$ .*

*Proof.* First, we restrict attention to  $x$  in the open interval  $(-\frac{\pi}{2}, \frac{\pi}{2})$ . By Proposition 2(c),  $T$  satisfies the variables-separable differential equation  $y' = 1 + y^2$ . This leads to

$$\begin{aligned} x - 0 &= \int_0^x dt = \int_{T(0)}^{T(x)} \frac{ds}{1+s^2} = \int_0^{T(x)} \frac{ds}{1+s^2} \\ &= \tan^{-1}(T(x)) - \tan^{-1}(0) = \tan^{-1}(T(x)) - 0 = \tan^{-1}(T(x)). \end{aligned}$$

Hence,  $T(x) = \tan(\tan^{-1}(T(x))) = \tan(x)$  for all  $x$  in  $(-\frac{\pi}{2}, \frac{\pi}{2})$ . [Remark: A short classroom discussion might well end here, as we have just used/reinforced the fundamental theorem of calculus and the chain rule, in the guise of definite integration by change of variable.]

Next, we focus on  $\frac{\pi}{2} < x (\neq \frac{\pi}{2} + m\pi)$ . First, suppose  $\frac{\pi}{2} < x < \pi$ . Then  $x = 2u = u + v$ , where  $u = v$  is in  $(\frac{\pi}{4}, \frac{\pi}{2})$ . As  $T$  and  $\tan$  are both tangential, we may argue as follows, using the result of the preceding paragraph:

$$T(x) = T(u + v) = \frac{T(u) + T(v)}{1 - T(u)T(v)} = \frac{\tan(u) + \tan(v)}{1 - \tan(u)\tan(v)} = \tan(u + v) = \tan(x).$$

Moreover, Proposition 2(b) yields that  $T$  is continuous, and so, since  $\tan$  is also continuous, we have  $T(\pi) = \lim_{x \rightarrow \pi^-} T(x) = \lim_{x \rightarrow \pi^-} \tan(x) = \tan(\pi) = 0$ . Hence, by mathematical

induction (and the fact that  $T$  is tangential), we have  $T(n\pi) = 0$  for each positive integer  $n$ . It now follows, by reasoning as above, that  $T$  and  $\tan$  agree on  $(n\pi - \frac{\pi}{2}, n\pi + \frac{\pi}{2})$ . Indeed, if  $x$  is in this interval, then  $x - n\pi = \tan^{-1}(T(x)) - \tan^{-1}(T(n\pi))$ , whence  $T(x) = \tan(x - n\pi) = \tan(x)$ . Hence,  $T$  and  $\tan$  agree on  $(-\frac{\pi}{2}, \infty) \setminus \{\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots\}$ .

Next, we focus on  $-\frac{\pi}{2} > x (\neq \frac{\pi}{2} + m\pi)$ . First, suppose  $-\pi < x < -\frac{\pi}{2}$ . Then  $x = 2u = u + v$ , where  $u = v$  is in  $(-\frac{\pi}{2}, -\frac{\pi}{4})$ . In particular,  $T(u) = \tan(u)$ . It follows via tangentiality as above that  $T(x) = T(u + v) = \tan(u + v) = \tan(x)$ . Then, by considering the limit as  $x$  approaches  $-\pi$  from the right and invoking continuity, we see that  $T(-\pi) = 0$ . By mathematical induction and tangentiality,  $T(-n\pi) = 0$  for each positive integer  $n$ . It now follows, by reasoning as above, that  $T$  and  $\tan$  agree on  $(-n\pi - \frac{\pi}{2}, -n\pi + \frac{\pi}{2})$ . Hence,  $T$  and  $\tan$  agree on the domain of  $\tan$ .

We have seen that  $T(x) = \tan(x)$  if  $x \neq \frac{\pi}{2} + m\pi$  (for  $m$  an integer). To complete the proof, it suffices to show that  $T(\frac{\pi}{2} + m\pi)$  is not defined. This, however, is a consequence of the continuity of  $T$ , since  $\lim_{x \rightarrow \frac{\pi}{2} + m\pi} T(x) = \lim_{x \rightarrow \frac{\pi}{2} + m\pi} \tan(x)$  does not exist. ■

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## Kleine Mitteilungen

## An explicit formula about the convex hull of random points

Denote by  $V_n^{(d)}$  the expected volume of the convex hull of  $n$  points chosen independently according to a given probability measure  $\mu$  in Euclidean  $d$ -space  $E^d$ . For  $d = 2, 3$  and  $\mu$  the uniform distribution on a convex body in  $E^d$ , Affentranger [1], [2] has shown that

$$V_{d+2m}^{(d)} = \sum_{k=1}^m \gamma_k \binom{d+2m}{2k-1} V_{d+2m-2k+1}^{(d)} \quad (m = 1, 2, \dots), \quad (1)$$

where the  $\gamma_k$  can be obtained recursively from  $\gamma_1 = \frac{1}{2}$ ,  $2\gamma_k = 1 - \sum_{i=1}^{k-1} \binom{2k-1}{2i-1} \gamma_i$  ( $k \geq 2$ ).

Recently, Buchta [3] has extended this result to arbitrary dimensions  $d$  and to arbitrary probability measures  $\mu$  on  $E^d$ . The key point in [3] is the existence of a moment functional  $\mathcal{M}$  such that

$$V_{d+1+n}^{(d)} = \binom{d+1+n}{d+1} \mathcal{M}(x^n + (1-x)^n). \quad (2)$$

(See [4] for the definition of moment functionals.)

In this note we show that in formula (1) the  $\gamma_k$  can be expressed explicitly by

$$\gamma_k = (2^{2k} - 1) \frac{B_{2k}}{k} \quad (k = 1, 2, \dots). \quad (3)$$

Here the  $B_n$  are the *Bernoulli numbers* (see e.g. [5], section 1.13), defined by the generating series  $z/(e^z - 1) = \sum_{n=0}^{\infty} B_n z^n/n!$ . In our proof of formula (1) we can avoid the elimination process used in [1].