The Osculating Conics of Steiner`s Hypocycloid

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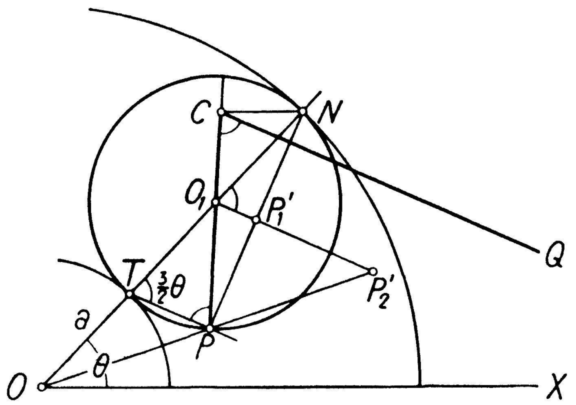
In the present note we shall give a simple construction of the osculating conic at an arbitrary point of Steiner's hypocycloid and determine the locus of the centres of these conics.

Let O be the centre of a fixed circle with radius $3\,a$ and O_1 the centre of a circle with radius a, which is rolling inside the first circle. The angle from an x-axis through O to the line OO_1 is denoted by θ and the corresponding point of Steiner's hypocycloid $\mathfrak{H}$ is called P (figure). The line OO_1 cuts the rolling circle in the points T and N such



that TP is the tangent and NP the normal to $\mathfrak{H}$ at P. Since $\sphericalangle\,(NO_1P) = 3\,\theta$ we have $\sphericalangle\,(O_1TP) = \sphericalangle\,(O_1PT) = 3\,\theta/2$. Furthermore, it is known that the radius of curvature ϱ of $\mathfrak{H}$ at P is

$$\varrho = 4\,PN = 8\,a \sin\frac{3}{2}\,\theta.$$

To determine the osculating conic of $\mathfrak{H}$ at P we use some results from the affine geometry[1]).

The *affine normal* (or axis of deviation) at P can be constructed in the following manner (l. c., p. 33): On the usual normal PN we find the centre of curvature P_1 corresponding to P and again for the evolute of $\mathfrak{H}$ the centre of curvature P_2 corresponding to P_1. On the line P_1P_2 which is parallel to the tangent TP we determine a point P_3 such that $P_3P_1 = P_3P_2/4$. The line joining P and P_3 is the required affine normal.

In consequence of a well-known property of the cycloids, the point P_2 is situated on the line OP. Consider a normal to PN through O_1 which cuts PN in a point P_1' and OP in a point P_2'. From the figure we derive that $O_1P_1' = TP/2 = O_1P_2'/4$. Hence the line PO_1 passes through P_3, i. e. PO_1 is the affine normal at P. We then have

The affine normal at a point P of Steiner's hypocycloid is the line which joins P with the centre of the rolling circle.

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[3]) W. BLASCHKE, *Vorlesungen über Differentialgeometrie*, vol. 2 (Springer, Berlin 1923).

The envelope of the affine normal is the so-called affine evolute of $\mathfrak{H}$, and the point of contact C is the centre of the osculating conic at P (l. c., p. 28). Since C can be found by drawing a perpendicular from N to the line PO_1, the locus of C is a new hypocycloid where the diameter of the rolling circle is the segment NO_1.

The locus of the centres of the osculating conics for the hypocycloid $\mathfrak{H}$ is a new hypocycloid generated by a point of a circle with radius $a/2$ which is rolling inside the fixed circle with radius $3a$[1]).

This hypocycloid consists of 6 equal arcs. Every second cusp coincides with a cusp of $\mathfrak{H}$.

All the osculating conics will be *ellipses*, because the centre C lies on the same side of the tangent TP as the arc of $\mathfrak{H}$ which contains P. Of the ellipse that osculates $\mathfrak{H}$ at P we have found the centre C and a semi-diameter CP with the length $CP = 2a \sin^2(3\theta/2)$. Let Q be the extreme point of the conjugate semi-diameter CQ parallel to TP. At P the osculating ellipse has the same radius of curvature ϱ as $\mathfrak{H}$, and since the radius of curvature of the ellipse at P can be expressed by $CQ^2/CP \sin C$, where $\sphericalangle C = 3\theta/2$, we get

$$8a \sin\frac{3}{2}\theta = \frac{CQ^2}{2a\sin^2\frac{3}{2}\theta \sin\frac{3}{2}\theta}$$

or

$$CQ = 4a \sin^2\frac{3}{2}\theta = 2\,CP.$$

The length of the diameter parallel to the tangent TP is twice the diameter through P.

We thus have obtained the required construction of the osculating ellipse at the point P of $\mathfrak{H}$.

In a similar way we may obtain the osculating ellipses of a general hypo- or epicycloid. The construction, however, will not be as simple as here, because the first theorem mentioned above can not be extended to other cycloids.

At last we notice that the reciprocal curve of $\mathfrak{H}$ with respect to the circle with centre O and radius a is a *cubic* $\mathfrak{H}'$, which consists of three open convex arcs situated in angles of 60^0 and having the same vertices as $\mathfrak{H}$. By a dualistic transformation an osculating conic will be transformed into an osculating conic. The point O being outside all the osculating ellipses of $\mathfrak{H}$, it is obvious that each osculating conic of the cubic $\mathfrak{H}'$ will be a *hyperbola*.

FR. FABRICIUS-BJERRE, Copenhagen.

La mode dans les mathématiques

La théorie élémentaire des séries insiste sur le rôle secondaire que jouent les premiers termes d'une série. Ce sont les termes de haut rang, leur allure, leur caractère, qui signifient tout.

Pourtant, la théorie des fonctions analytiques nous montre que ce point de vue, que nous inculquons à nos élèves de première année, peut induire en erreur. Le

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[1]) J. LEMAIRE, *Hypocycloïdes et Epicycloïdes* (Paris 1929), p. 55.