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## Complexity of degenerations of modules

R. Aehle, Ch. Riedtmann and G. Zwara

**Abstract.** A module  $M$  over an associative algebra  $A$  over an algebraically closed field  $k$  is said to degenerate to a module  $N$  if  $N$  belongs to the closure of the isomorphism class of  $M$  in the algebraic variety of  $d$ -dimensional  $A$ -modules,  $d \in \mathbb{N}$ . We associate a non-negative integer to a degeneration  $M \leq_{\text{deg}} N$ , its complexity, and study its properties.

**Mathematics Subject Classification (2000).** 14L30, 16G10.

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### 1. Introduction

Let  $k$  be an algebraically closed field,  $A$  a finite dimensional associative  $k$ -algebra with a unit and  $\text{mod } A$  the category of finite dimensional left  $A$ -modules. Let  $\mathbb{M}_d(k)$  denote the  $k$ -algebra of the  $d \times d$ -matrices with coefficients in  $k$ . We view  $A$  as a quotient of a free associative algebra  $k\langle X_1, \dots, X_r \rangle$  by a two-sided ideal  $I$ . We define the affine variety  $\text{mod}_A^d(k)$  as the set of  $r$ -tuples  $(m_1, \dots, m_r)$  such that  $m_i \in \mathbb{M}_d(k)$  and  $\rho(m_1, \dots, m_r)$  is the zero matrix for any  $\rho \in I$ . The general linear group  $GL_d(k)$  acts on  $\text{mod}_A^d(k)$  by conjugation.

As an ordinary set,  $\text{mod}_A^d(k)$  is just the set  $\text{Hom}_{k\text{-alg}}(A, \mathbb{M}_d(k))$  and hence  $GL_d(k)$ -orbits in  $\text{mod}_A^d(k)$  correspond bijectively to isomorphism classes of  $d$ -dimensional left  $A$ -modules.

Let  $M$  and  $N$  be two  $d$ -dimensional  $A$ -modules. By definition,  $M$  degenerates to  $N$ , noted  $M \leq_{\text{deg}} N$ , if  $N$  lies in the closure of the  $GL_d(k)$ -orbit of  $M$  in  $\text{mod}_A^d(k)$ , with respect to the Zariski topology. This defines a partial order on the set of isomorphism classes of  $d$ -dimensional  $A$ -modules.

Denote by  $Q$  the quiver

$$Q = 1 \begin{array}{c} \xrightarrow{a_1} \\ \xleftarrow{b_1} \end{array} 2 \begin{array}{c} \xrightarrow{a_2} \\ \xleftarrow{b_2} \end{array} 3 \dots$$

with vertex set  $Q_0 = \mathbb{N} \setminus \{0\}$  and arrows  $a_i : i \rightarrow i+1$ ,  $b_i : i+1 \rightarrow i$  for every  $i \in Q_0$ .

We call a representation

$$T = N_1 \begin{smallmatrix} \xrightarrow{\alpha_1} \\ \xleftarrow{\beta_1} \end{smallmatrix} N_2 \cdots N_i \begin{smallmatrix} \xrightarrow{\alpha_i} \\ \xleftarrow{\beta_i} \end{smallmatrix} N_{i+1} \cdots$$

of  $Q$  in  $\text{mod } A$ , the category of finite dimensional  $A$ -modules, an *exact tube* if the sequence

$$0 \rightarrow N_i \xrightarrow{\begin{pmatrix} \beta_{i-1} \\ \alpha_i \end{pmatrix}} N_{i-1} \oplus N_{i+1} \xrightarrow{(-\alpha_{i-1}, \beta_i)} N_i \rightarrow 0$$

or equivalently the square

$$\begin{array}{ccc} N_i & \xrightarrow{\alpha_i} & N_{i+1} \\ \downarrow \beta_{i-1} & & \downarrow \beta_i \\ N_{i-1} & \xrightarrow{\alpha_{i-1}} & N_i \end{array}$$

is exact for all  $i \geq 1$ . Here we set  $N_0 = 0$ . Note that  $N_i$  is an  $A$ -module, that  $\alpha_i, \beta_i$  are  $A$ -linear and that  $\alpha_i$  is injective,  $\beta_i$  is surjective, for all  $i \geq 1$ . We say that  $T$  is an  $(M, N)$ -tube if there is a natural number  $h$  such that

- (i)  $N_1 \xrightarrow[A]{\sim} N$ ,
- (ii)  $N_{h+j+1} \xrightarrow[A]{\sim} N_{h+j} \oplus M$ , for all  $j \in \mathbb{N}$ .

We call the smallest such number  $h$  the complexity  $\text{cpl}(T)$  of the tube.

Let  $T$  be an  $(M, N)$ -tube. Note that the sequence

$$0 \rightarrow N_k \xrightarrow{\alpha_k} N_{k+1} \xrightarrow{\beta_1 \cdots \beta_k} N_1 \rightarrow 0$$

is exact for any  $k$ . As  $N_{k+1}$  is isomorphic to  $N_k \oplus M$  for  $k \geq \text{cpl}(T)$ , there is an exact sequence

$$0 \rightarrow N_k \rightarrow N_k \oplus M \rightarrow N \rightarrow 0,$$

and therefore  $M$  degenerates to  $N$  [5].

Conversely, whenever  $M$  degenerates to  $N$ , there exists an  $(M, N)$ -tube: Indeed, the third author showed in [7] that there is a short exact sequence

$$0 \rightarrow Z \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix}} Z \oplus M \rightarrow N \rightarrow 0, \quad (1.1)$$

and in [6] he associated an exact tube  $T_{f,g}$  with such a sequence (see also Section 4). In fact,  $T_{f,g}$  is the cokernel of the injection  $\varphi : X \rightarrow X'$  between the following representations of  $Q$ :

$$\begin{array}{ccccccc} X : & Z & \xrightleftharpoons[1]{f} & Z & \xrightleftharpoons[1]{f} & \cdots & \xrightleftharpoons[1]{f} & Z & \xrightleftharpoons[1]{f} & Z & \cdots \\ \varphi \downarrow & \varphi_1 \downarrow & & \varphi_2 \downarrow & & & \varphi_i \downarrow & & \varphi_{i+1} \downarrow & & \\ X' : & Z \oplus M & \xrightleftharpoons[\begin{pmatrix} f & 0 \\ g & 0 \\ 0 & 1 \end{pmatrix}]{(10)} & Z \oplus M^2 & \xrightleftharpoons[\begin{pmatrix} f & 0 \\ g & 0 \\ 0 & 1 \end{pmatrix}]{(10)} & \cdots & \xrightleftharpoons[\begin{pmatrix} f & 0 \\ g & 0 \\ 0 & 1 \end{pmatrix}]{(10)} & Z \oplus M^i & \xrightleftharpoons[\begin{pmatrix} f & 0 \\ g & 0 \\ 0 & 1 \end{pmatrix}]{(10)} & Z \oplus M^{i+1} & \cdots \end{array}$$

with  $\varphi_i = (f^i, gf^{i-1}, \dots, g)^t : Z \rightarrow Z \oplus M^i$ . Both  $X$  and  $X'$  are almost exact tubes: they satisfy all requirements except those related to  $a_1$  and  $b_1$ . The only condition left to be checked for  $T_{f,g}$  is the exactness of the bottom row in the commutative diagram (Figure 1) with exact columns. This is done by diagram chasing.

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & Z & \xrightarrow{1} & Z & \xrightarrow{f} & Z & \\
 & \downarrow \begin{pmatrix} f \\ g \end{pmatrix} & & \downarrow \begin{pmatrix} f^2 \\ gf \\ g \end{pmatrix} & & \downarrow \begin{pmatrix} f \\ g \end{pmatrix} & \\
 & Z \oplus M & \xrightarrow{\begin{pmatrix} f & 0 \\ g & 0 \\ 0 & 1 \end{pmatrix}} & Z \oplus M^2 & \xrightarrow{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}} & Z \oplus M & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 \longrightarrow & N_1 & \xrightarrow{\alpha_1} & N_2 & \xrightarrow{\beta_1} & N_1 & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 & 
 \end{array}$$

Figure 1

By construction,  $N_1 = \operatorname{coker} \varphi_1$  is isomorphic to  $N$ . Using Fitting's lemma in order to replace  $Z$  by a direct summand if necessary in the exact sequence (1.1), we may assume that  $f$  is nilpotent, say  $f^h = 0$ . Then  $\varphi_{h+j}$  has the form  $\varphi_{h+j} = (0, \dots, 0, gf^{h-1}, \dots, g)^t : Z \rightarrow Z \oplus M^{h+j}$ , and its cokernel  $N_{h+j}$  is isomorphic to  $M^j \oplus N_h$  for  $j \geq 0$ . We conclude that  $T_{f,g}$  is an  $(M, N)$ -tube of complexity at most  $h$ . In fact,  $T_{f,g}$  is an  $(M, N)$ -tube even if  $f$  is not nilpotent (compare with Proposition 4.2).

We define the *complexity* of a degeneration  $M \leq_{\deg} N$  to be

$$\operatorname{cpl}(M, N) = \min \operatorname{cpl}(T),$$

where  $T$  ranges over all  $(M, N)$ -tubes. This seems to be a good way to measure how “complicated” a degeneration is.

Indeed, we will prove in Sections 3 and 4 that a degeneration  $M \leq_{\deg} N$  is of complexity 1 if and only if there exists a non-split exact sequence

$$0 \rightarrow N' \rightarrow M \rightarrow N'' \rightarrow 0$$

with  $N \xrightarrow{\sim} N' \oplus N''$ . So these are the “simplest” degenerations. In particular, any degeneration to an indecomposable  $N$  must have complexity at least 2.

It is quite difficult to compute the complexity of a degeneration. The construc-

tion described before gives an estimate from above: if

$$0 \rightarrow Z \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix}} Z \oplus M \rightarrow N \rightarrow 0$$

is an exact sequence and  $f^h = 0$ , then  $\text{cpl}(M, N) \leq h$ . Conversely, it is easy to show that

$$\text{cpl}(M, N) \geq \frac{\ell\ell(M)}{\ell\ell(N)} - 1,$$

where  $\ell\ell(X)$  is the Loewy length of  $X$ ; i.e., the smallest number  $r$  for which  $(\text{rad } A)^r \cdot X = 0$  (see Proposition 3.5). Both bounds are sharp, but in general the complexity differs from both.

The complexity of a degeneration  $M \leq_{\text{deg}} N$  obtained from two degenerations  $M \leq_{\text{deg}} P \leq_{\text{deg}} N$  seems to be quite unrelated to the sum of the complexities of  $M \leq_{\text{deg}} P$  and  $P \leq_{\text{deg}} N$ . For instance, if we take non-split exact sequences

$$0 \rightarrow A_i \rightarrow B_i \rightarrow C_i \rightarrow 0, \quad i = 1, \dots, r,$$

then there is a sequence of degenerations

$$\begin{aligned} \bigoplus_{i=1}^r B_i &\leq_{\text{deg}} \left( \bigoplus_{i=1}^{r-1} B_i \right) \oplus A_r \oplus C_r \leq_{\text{deg}} \dots \leq_{\text{deg}} \left( \bigoplus_{i=1}^s B_i \right) \oplus \bigoplus_{i=s+1}^r (A_i \oplus C_i) \\ &\leq_{\text{deg}} \dots \leq_{\text{deg}} \bigoplus_{i=1}^r (A_i \oplus C_i), \end{aligned}$$

but the complexity of

$$\bigoplus_{i=1}^r B_i \leq_{\text{deg}} \bigoplus_{i=1}^r (A_i \oplus C_i)$$

is 1. On the other hand, we give an example of a chain of degenerations  $M \leq_{\text{deg}} P \leq_{\text{deg}} N$  in Section 5.1 for which  $\text{cpl}(M, P) + \text{cpl}(P, N) < \text{cpl}(M, N)$ . By Proposition 5.1, a minimal degeneration can have arbitrarily high complexity. A degeneration  $M \leq_{\text{deg}} N$  is called minimal if  $M$  is not isomorphic to  $N$  and moreover  $M \leq_{\text{deg}} P \leq_{\text{deg}} N$  implies that  $P$  is isomorphic to either  $M$  or  $N$ .

## 2. Degenerations, bimodules and exact tubes

The following construction is explained in detail in [7] (compare also [2] and [3], pp. 176–177): If  $M \leq_{\text{deg}} N$  is a degeneration, there exists a discrete valuation  $k$ -algebra  $R$  with maximal ideal  $\mathfrak{m}$  and residue class field  $k$  and an  $A$ - $R$ -bimodule  $\mathcal{Y}$ , which is free of rank  $d$  over  $R$ , such that

- i)  $\mathcal{Y}/\mathfrak{m} \cdot \mathcal{Y} \xrightarrow[A]{\sim} N$
- ii)  $\mathcal{Y}$  contains  $R \otimes_k M$  as an  $A$ - $R$ -submodule.

These data are related to mapping a curve  $c$  to  $\text{mod}_A^d(k)$  in such a way that its image lies generically in the orbit of  $M$  and intersects the orbit of  $N$ . Assuming  $c$  to be non-singular and passing to the completion, we may assume that  $R = k[[t]]$ . The representation  $T = (N_i, \alpha_i, \beta_i)$  defined by the setting

$$N_i = \mathcal{Y}/(t^i) \cdot \mathcal{Y}$$

and letting  $\alpha_i : N_i \rightarrow N_{i+1}$  and  $\beta_i : N_{i+1} \rightarrow N_i$  be induced by multiplication by  $t$  and the identity, respectively, is easily seen to be an exact tube, and by [7] it is moreover an  $(M, N)$ -tube.

This construction associating an exact tube with a bimodule is an equivalence:

**Proposition 2.1.** *The category  $\mathcal{T}$  of exact tubes is equivalent to the category  $\text{mod } A\text{-}k[[t]]$  of  $A\text{-}k[[t]]$ -bimodules which are free of finite rank over  $k[[t]]$ .*

*Proof.* We just describe a quasi-inverse functor. For an exact tube  $T = (N_i, \alpha_i, \beta_i)$  we set

$$\mathcal{Y} = \varprojlim (N_i, \beta_i),$$

and we put

$$t \cdot (n_1, n_2, \dots) = (0, \alpha_1(n_1), \alpha_2(n_2), \dots)$$

for any infinite sequence  $(n_1, n_2, \dots)$  with  $n_i \in N_i$  and  $\beta_i(n_i) = n_{i-1}$  representing an element of  $\mathcal{Y}$ . As  $T$  is an exact tube, this defines an  $A\text{-}k[[t]]$ -bimodule structure on  $\mathcal{Y}$ . As  $t$  acts without torsion,  $\mathcal{Y}$  is free as a  $k[[t]]$ -module, and its rank equals  $\dim_k N_1$ , since clearly  $\mathcal{Y}/(t) \cdot \mathcal{Y}$  is isomorphic to  $N_1$ .  $\square$

We give a direct construction of the bimodule corresponding to  $T_{f,g}$  for an exact sequence

$$0 \rightarrow Z \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix}} Z \oplus M \rightarrow N \rightarrow 0 \quad (2.1)$$

with a nilpotent map  $f$ . Set

$$\mathcal{Y}_{f,g} = k[[t]] \otimes_k M \oplus Z$$

as an  $A$ -module and define the action of  $t$  on  $Z$  by

$$t \cdot (0, z) = (1 \otimes g(z), f(z)).$$

Clearly, this action of  $t$  is torsion free, and  $\mathcal{Y}_{f,g}/(t)\mathcal{Y}_{f,g}$  is isomorphic to  $N$ , so that  $\mathcal{Y}_{f,g}$  actually belongs to  $\text{mod } A\text{-}k[[t]]$ . It is easy to see that the exact tube associated with  $\mathcal{Y}_{f,g}$  is  $T_{f,g}$ .

We will need the following truncated version of an exact tube:

**Definition 2.2.** For  $m \geq 1$ , an exact tube of height  $m$  is a representation in  $\text{mod } A$

$$N_1 \begin{array}{c} \xrightarrow{\alpha_1} \\ \xleftarrow{\beta_1} \end{array} N_2 \dots N_{m-1} \begin{array}{c} \xrightarrow{\alpha_{m-1}} \\ \xleftarrow{\beta_{m-1}} \end{array} N_m$$

of the full subquiver  $Q_m$  of  $Q$  whose vertices are  $1, 2, \dots, m$ , such that the square

$$\begin{array}{ccc} N_i & \xrightarrow{\alpha_i} & N_{i+1} \\ \downarrow \beta_{i-1} & & \downarrow \beta_i \\ N_{i-1} & \xrightarrow{\alpha_{i-1}} & N_i \end{array}$$

is exact for  $i = 1, \dots, m-1$ . Again we set  $N_0 = 0$ .

The category of exact tubes of height  $m$  is equivalent to the category of  $A$ - $k[t]/(t^m)$ -bimodules which are free of finite rank over  $k[t]/(t^m)$ .

Obviously, an exact tube  $T$  restricts to an exact tube  $T_{\leq m}$  of height  $m$  for all  $m$ . We will see in Section 4 that an  $M$ -extendible tube  $T = (N_i, \alpha_i, \beta_i)$  of height  $h \geq 1$  (see next definition) is always the restriction of an  $(M, N_1)$ -tube.

**Definition 2.3.** A tube  $T = (N_i, \alpha_i, \beta_i)$  of height  $h$  is called  $M$ -extendible if there is a decomposition  $N_h = Z \oplus Z'$  and an exact sequence

$$0 \rightarrow Z \xrightarrow{\begin{pmatrix} a \\ b \end{pmatrix}} N_{h-1} \oplus M \xrightarrow{(c \ d)} Z' \rightarrow 0$$

such that  $a = \beta_{h-1}|_Z$  and  $c = pr_{Z'} \circ \alpha_{h-1}$ , where  $pr_{Z'} : Z \oplus Z' \rightarrow Z'$  is the natural projection.

We end this section with some questions. We do not know how to describe the full subcategory of  $\text{mod } A\text{-}k[[t]]$  corresponding to  $(M, N)$ -tubes. Conceivably, its objects are just those bimodules  $\mathcal{Y}$  which contain  $k[[t]] \otimes_k M$  as a subbimodule. This would follow if we knew that any  $(M, N)$ -tube is of the form  $\mathcal{Y}_{f,g}$  for some exact sequence (2.1).

### 3. Complexity

**Definition 3.1.** We call a map

$$\begin{pmatrix} f \\ g \end{pmatrix} : Z \rightarrow Z \oplus M$$

an  $(M, N)$ -monomorphism provided  $N$  is isomorphic to  $\text{coker} \begin{pmatrix} f \\ g \end{pmatrix}$ .

Recall that, for a degeneration  $M \leq_{\text{deg}} N$ , we defined the complexity as

$$\text{cpl}(M, N) = \min \text{cpl}(T),$$

where  $T$  ranges over all  $(M, N)$ -tubes. There always are  $(M, N)$ -tubes with different complexities. For instance, if  $(f, g)^t : Z \rightarrow Z \oplus M$  is an  $(M, N)$ -monomorphism and we set

$$f' = \begin{pmatrix} 0 & 1 \\ f & 0 \end{pmatrix} : Z^2 \longrightarrow Z^2, \quad g' = (g \ 0) : Z^2 \longrightarrow M,$$

the map  $(f', g')^t$  will be an  $(M, N)$ -monomorphism, too, and it is easy to see that

$$\text{cpl}(T_{f', g'}) = 2 \text{cpl}(T_{f, g}).$$

**Theorem 3.2.** *Let  $h \geq 1$  be a natural number and  $M \leq_{\text{deg}} N$  a degeneration. The following conditions are equivalent:*

- (i)  $\text{cpl}(M, N) \leq h$
- (ii) *There is an exact sequence*

$$0 \longrightarrow Z \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix}} Z \oplus M \longrightarrow N \longrightarrow 0$$

*such that  $\text{cpl}(T_{f, g}) \leq h$ .*

- (iii) *There exists an exact tube  $T = (N_i, \alpha_i, \beta_i)$  of height  $2h + 1$  with  $N \xrightarrow[A]{} N_1$  and such that*

$$N_{h+j+1} \xrightarrow[A]{} N_{h+j} \oplus M$$

*for  $j = 0, \dots, h$ .*

- (iv) *There exists an  $M$ -extendible exact tube  $T = (N_i, \alpha_i, \beta_i)$  of height  $h$  with  $N \xrightarrow[A]{} N_1$ .*

*Proof.* Most ingredients for the proof will be given in Section 4. Here we indicate how they fit together: The implications (ii)  $\Rightarrow$  (i)  $\Rightarrow$  (iii) are obvious. The results of Section 4 up to Proposition 4.6 give that (ii) implies (iv), and Proposition 4.8 shows (iv)  $\Rightarrow$  (ii). Finally, the implication (iii)  $\Rightarrow$  (ii) follows from Proposition 4.9 and the next lemma.  $\square$

**Lemma 3.3.** *Let  $T = (N_i, \alpha_i, \beta_i)$  be an  $(M, N)$ -tube, and assume that  $N_{h+1} \xrightarrow[A]{} N_h \oplus M$  for some  $h \geq 1$ . Then  $\text{cpl}(T) \leq h$ .*

*Proof.* As  $T$  is an  $(M, N)$ -tube, there exists a natural number  $j \geq h$  such that  $N_{i+1} \xrightarrow[A]{} N_i \oplus M$  for all  $i \geq j$ . Take an integer  $i$  with  $h < i < j$ , and consider the



two exact squares

$$\begin{array}{ccccccc}
 N_h \oplus M & \xrightarrow{\sim} & N_{h+1} & \xrightarrow{\alpha_i \dots \alpha_{h+1}} & N_{i+1} & \xrightarrow{\alpha_j \dots \alpha_{i+1}} & N_{j+1} \xrightarrow{\sim} N_j \oplus M \\
 & & \downarrow \beta_h & & \downarrow \beta_i & & \downarrow \beta_j \\
 & & N_h & \xrightarrow{\alpha_{i-1} \dots \alpha_h} & N_i & \xrightarrow{\alpha_{j-1} \dots \alpha_i} & N_j.
 \end{array}$$

The big square splits, and therefore the two small squares split as well. We conclude that  $N_{i+1}$  is isomorphic to  $N_i \oplus M$ .  $\square$

As  $N_0 = 0$ , our theorem takes the following simpler form for  $h = 1, 2$ :

**Corollary 3.4.** *Let  $M \leq_{\text{deg}} N$  be a degeneration. Then*

i)  $\text{cpl}(M, N) \leq 1$  *if and only if*  $N = Z \oplus Z'$  *and there exists an exact sequence*

$$0 \rightarrow Z \rightarrow M \rightarrow Z' \rightarrow 0.$$

ii)  $\text{cpl}(M, N) \leq 2$  *if and only if there exist two exact squares*

$$\begin{array}{ccc}
 Z & \xrightarrow{a} & N \\
 \downarrow & & \downarrow c \\
 M & \longrightarrow & Z'
 \end{array}
 \qquad
 \begin{array}{ccc}
 N & \longrightarrow & Z \\
 \downarrow c & & \downarrow a \\
 Z' & \longrightarrow & N
 \end{array}$$

**Proposition 3.5.** *For any degeneration  $M \leq_{\text{deg}} N$  we have*

$$\text{cpl}(M, N) \geq \frac{\ell\ell(M)}{\ell\ell(N)} - 1,$$

where  $\ell\ell(X)$  denotes the Loewy length of  $X$ ; i.e., the smallest integer  $r$  such that  $(\text{rad } A)^r X = 0$ .

*Proof.* Choose an  $(M, N)$ -tube  $T = (N_i, \alpha_i, \beta_i)$  of complexity  $h = \text{cpl}(M, N)$ . Then  $M$  is a direct summand of  $N_{h+1}$ , and hence  $\ell\ell(M) \leq \ell\ell(N_{h+1})$ . We claim that, for all  $i \geq 1$ ,

$$\ell\ell(N_i) \leq i \ell\ell(N_1).$$

In fact, for any exact sequence

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

the relation

$$\ell\ell(B) \leq \ell\ell(A) + \ell\ell(C)$$

holds true. Our claim follows by induction, considering the exact sequences

$$0 \rightarrow N_{i-1} \rightarrow N_i \rightarrow N_1 \rightarrow 0. \quad \square$$

#### 4. Exact tubes from monomorphisms

Throughout this section,  $(f, g)^t : Z \rightarrow Z \oplus M$  denotes an  $(M, N)$ -monomorphism.

**Definition 4.1.** We call two exact tubes  $T = (N_i, \alpha_i, \beta_i)$  and  $T' = (N'_i, \alpha'_i, \beta'_i)$  *similar* if  $N_i$  is isomorphic to  $N'_i$  for all  $i \geq 1$ .

So we do not ask for any compatibility with the maps in the tubes. Note that the property of being an  $(M, N)$ -tube is preserved under similarity, and so is complexity.

**Proposition 4.2.** *There is a direct summand  $Z'$  of  $Z$  and an exact sequence*

$$0 \rightarrow Z' \xrightarrow{\begin{pmatrix} f|_{Z'} \\ g|_{Z'} \end{pmatrix}} Z' \oplus M \rightarrow N \rightarrow 0$$

*such that  $f|_{Z'}$  is nilpotent and  $T_{f,g}$  is similar to  $T_{f|_{Z'}, g|_{Z'}}$ . As a consequence,  $T_{f,g}$  is an  $(M, N)$ -tube.*

*Proof.* By Fitting's lemma, there is a decomposition  $Z = Z' \oplus Z''$  of  $Z$  as a direct sum which is preserved under  $f$  and such that  $f' = f|_{Z'}$  is nilpotent and  $f'' = f|_{Z''}$  is an automorphism of  $Z''$ . Set  $g' = g|_{Z'}$  and  $g'' = g|_{Z''}$ . Obviously the maps

$$\begin{pmatrix} f'^i & 0 & g' f'^{i-1} & \cdots & g' \\ 0 & f''^i & g'' f''^{i-1} & \cdots & g'' \end{pmatrix}^t : Z' \oplus Z'' \longrightarrow Z' \oplus Z'' \oplus M^i$$

and

$$(f'^i \quad g' f'^{i-1} \cdots g')^t : Z' \longrightarrow Z' \oplus M^i$$

have isomorphic cokernels as  $(f'')^i$  is an isomorphism for  $i \geq 1$ . Since  $f'$  is nilpotent,  $T_{f',g'}$  is an  $(M, N)$ -tube.  $\square$

**Remark 4.3.** Suppose that  $f^h = 0$ . As

$$\varphi_{h+j} = (0, \dots, 0, g f^{h-1}, \dots, g)^t : Z \longrightarrow Z \oplus M^{h+j},$$

for  $j \in \mathbb{N}$ , the exact tube  $T_{f,g}$  has the following particularly simple form:

$$\begin{aligned} N_{h+j} &= Z \oplus M^j \oplus Z', \quad N_{h+j+1} = Z \oplus M^{j+1} \oplus Z', \\ \alpha_{h+j} &= \begin{pmatrix} f & 0 \\ g & 0 \\ 0 & 1 \end{pmatrix} : Z \oplus (M^j \oplus Z') \rightarrow Z \oplus M \oplus (M^j \oplus Z'), \\ \beta_{h+j} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & k & l \end{pmatrix} : (Z \oplus M^j) \oplus M \oplus Z' \rightarrow (Z \oplus M^j) \oplus Z', \end{aligned}$$

for  $j \in \mathbb{N}$ , where  $Z'$  is a cokernel of

$$\psi = (g \circ (f^{h-1}, \dots, f, 1))^t : Z \longrightarrow M^h$$

and

$$(k, \quad l) : M \oplus Z' \longrightarrow Z'$$

is obtained from the commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & Z & \xrightarrow{\varphi_{h+1}} & Z \oplus M^{h+1} & \longrightarrow & N_{h+1} = Z \oplus M \oplus Z' \longrightarrow 0 \\
 & & \downarrow f & & \downarrow (1 \ 0) & & \downarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & k & l \end{pmatrix} \\
 0 & \longrightarrow & Z & \xrightarrow{\varphi_h} & Z \oplus M^h & \longrightarrow & N_h = Z \oplus Z' \longrightarrow 0
 \end{array}$$

with exact rows.

Our next goal is to show that, up to similarity, we may choose  $g \in \text{rad}(Z, M)$ . We start with an auxiliary result:

**Lemma 4.4.** *The tube  $T_{f,g}$  is similar to  $T_{f',g}$  with  $f' = f - hg$ , where  $h : M \rightarrow Z$  is any homomorphism.*

*Proof.* It suffices to check the identity  $\psi_i \circ \varphi'_i = \varphi_i$ , for  $i \geq 1$ , where

$$\begin{aligned}
 \varphi_i &= (f^i, g f^{i-1}, \dots, g)^t : Z \longrightarrow Z \oplus M^i, \\
 \varphi'_i &= (f'^i, g f'^{i-1}, \dots, g)^t : Z \longrightarrow Z \oplus M^i
 \end{aligned}$$

and

$$\psi_i := \begin{pmatrix} 1 & h & fh & f^2h & \dots & f^{i-1}h \\ 0 & 1 & gh & gfh & \dots & g f^{i-2}h \\ 0 & 0 & 1 & gh & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & gh \\ \vdots & & \ddots & \ddots & \ddots & gh \\ 0 & \dots & \dots & \dots & 0 & 1 \end{pmatrix} : Z \oplus M^i \rightarrow Z \oplus M^i.$$

The key is the equation

$$f^r = f'^r + \sum_{s=0}^{r-1} f^s (hg) f'^{r-1-s}, \quad r \geq 1,$$

which is proved by induction.  $\square$

**Proposition 4.5.** *There exists a direct summand  $Z'$  of  $Z$  and an exact sequence*

$$0 \rightarrow Z' \xrightarrow{\begin{pmatrix} f' \\ g' \end{pmatrix}} Z' \oplus M \rightarrow N \rightarrow 0 \quad (4.1)$$

*with  $g' \in \text{rad}(Z', M)$  and such that  $T_{f,g}$  is similar to  $T_{f',g'}$ .*

*Proof.* If  $g \in \text{rad}(Z, M)$ , there is nothing to be proved. Otherwise, we prove that a sequence (4.1) exists such that  $T_{f,g}$  is similar to  $T_{f',g'}$  and  $\dim Z' < \dim Z$  and then proceed by induction on  $\dim Z$ . We choose a non-zero direct summand  $Z_2$  of  $Z$  for which  $g|_{Z_2}$  is a section. Replacing  $Z$  by an isomorphic module if

necessary, which leads to an isomorphic tube, we may assume that  $Z = Z_1 \oplus Z_2$ ,  $M = M_1 \oplus Z_2$ ,

$$g = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

Applying the preceding lemma for

$$h = \begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix},$$

we obtain a monomorphism  $\begin{pmatrix} f'' \\ g \end{pmatrix}$  of the form

$$\begin{pmatrix} f'' \\ g \end{pmatrix} = \begin{pmatrix} a & 0 \\ b & 0 \\ q & 0 \\ 0 & 1 \end{pmatrix} : Z_1 \oplus Z_2 \rightarrow Z_1 \oplus Z_2 \oplus M_1 \oplus Z_2.$$

Now we may take  $Z' = Z_1$ ,  $M = Z_2 \oplus M_1$ ,  $f' = a$  and  $g' = \begin{pmatrix} b \\ q \end{pmatrix}$ .  $\square$

**Proposition 4.6.** *Set  $h = \text{cpl}(T_{f,g})$ , and suppose that  $g \in \text{rad}(Z, M)$  and that  $f$  is nilpotent. Then  $(T_{f,g})_{\leq h}$  is  $M$ -extendible.*

*Proof.* Our assumptions on  $f$  and  $g$  imply that, for some  $i$ , the restriction  $\psi|_Z$  of the composition

$$\psi = \begin{pmatrix} \varphi_i & 0 \\ 0 & 1_{M^h} \end{pmatrix} : Z \oplus M^h \rightarrow Z \oplus M^i \oplus M^h$$

of the maps

$$Z \oplus M^h \xrightarrow{\begin{pmatrix} f & 0 \\ g & 0 \\ 0 & 1 \end{pmatrix}} Z \oplus M^{1+h} \rightarrow \dots \xrightarrow{\begin{pmatrix} f & 0 \\ g & 0 \\ 0 & 1 \end{pmatrix}} Z \oplus M^{i+h}$$

belongs to  $\text{rad}(Z, Z \oplus M^{i+h})$ . By construction of  $T_{f,g}$ , the square

$$\begin{array}{ccc} Z \oplus M^h & \xrightarrow{\psi} & Z \oplus M^i \oplus M^h \\ \downarrow \pi_h & & \downarrow \pi_{h+i} \\ N_h & \xrightarrow{\alpha_{i+h-1} \cdots \alpha_h} & N_{i+h} \end{array}$$

is exact, where  $\pi_j : Z \oplus M^j \rightarrow N_j$  is the projection to the cokernel of  $\varphi_j : Z \rightarrow Z \oplus M^j$ , and it splits, since  $h = \text{cpl}(T_{f,g})$ . Therefore,  $\pi_h|_Z$  is a section, and replacing  $N_h$  by an isomorphic module, we may assume that

$$N_h = Z \oplus Z', \quad \pi_h = \begin{pmatrix} 1 & * & * \\ 0 & d & * \end{pmatrix} : Z \oplus M \oplus M^{h-1} \longrightarrow Z \oplus Z',$$

where  $*$  is an arbitrary map.

Now consider the exact squares

$$\begin{array}{ccc} Z \oplus M^{h-1} & \xrightleftharpoons[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}]{\begin{pmatrix} f & 0 \\ g & 0 \\ 0 & 1 \end{pmatrix}} & Z \oplus M \oplus M^{h-1} \\ \downarrow \pi_{h-1} & & \downarrow \pi_h = \begin{pmatrix} 1 & * & * \\ 0 & d & * \end{pmatrix} \\ N_{h-1} & \xrightleftharpoons[\beta_{h-1}]{\alpha_{h-1}} & Z \oplus Z' \end{array}$$

It is easy to see that the square

$$\begin{array}{ccc} Z & \xrightarrow{g} & M \\ \downarrow \pi_{h-1}|_Z & & \downarrow d \\ N_{h-1} & \xrightarrow{\text{pr}_{z'} \circ \alpha_{h-1}} & Z' \end{array}$$

is exact as well. Moreover, we have

$$\pi_{h-1}|_Z = \beta_{h-1}|_Z.$$

□

Next we recall a different construction for  $T_{f,g}$ , which has been presented for the most part in [6]. From  $(f, g)^t$  we obtain the commutative diagram (Figure 2) with exact rows and  $(k_i, l_i) = \beta_i(k_{i+1}, l_{i+1})$  for  $i \leq m-1$ .

$$\begin{array}{ccccccc} 0 & \longrightarrow & Z & \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix}} & Z \oplus M & \xrightarrow{(k_1, l_1)} & N_1 = N \longrightarrow 0 \\ & & \downarrow k_{m-1} & & \downarrow (k_m, l_m) & & \parallel \\ 0 & \longrightarrow & N_{m-1} & \xrightarrow{\alpha_{m-1}} & N_m & \xrightarrow{\beta_1 \dots \beta_{m-1}} & N_1 \longrightarrow 0 \\ & & \downarrow \beta_{m-2} & & \downarrow \beta_{m-1} & & \parallel \\ 0 & \longrightarrow & N_{m-2} & \xrightarrow{\alpha_{m-2}} & N_{m-1} & \xrightarrow{\beta_1 \dots \beta_{m-2}} & N_1 \longrightarrow 0 \\ & & \vdots & & \vdots & & \vdots \\ 0 & \longrightarrow & N_2 & \xrightarrow{\alpha_2} & N_3 & \xrightarrow{\beta_1 \beta_2} & N_1 \longrightarrow 0 \\ & & \downarrow \beta_1 & & \downarrow \beta_2 & & \parallel \\ 0 & \longrightarrow & N_1 & \xrightarrow{\alpha_1} & N_2 & \xrightarrow{\beta_1} & N_1 \longrightarrow 0 \end{array}$$

Figure 2

The next step is always obtained by squeezing the push-out of the top sequence by  $k_m$  between the two top rows.

We claim that the exact tube  $(N_i, \alpha_i, \beta_i)$  of height  $m$  thus obtained is isomorphic to the restriction  $(T_{f,g})_{\leq m}$  of  $T_{f,g}$ .

By induction, we obtain the following series of exact squares:

$$\begin{array}{ccccccc}
 Z & \xrightarrow{\begin{pmatrix} f & 0 \\ g & 0 \end{pmatrix}} & Z \oplus M & \xrightarrow{\begin{pmatrix} f & 0 \\ g & 0 \\ 0 & 1 \end{pmatrix}} & Z \oplus M^2 & \cdots & Z \oplus M^{m-1} \xrightarrow{\begin{pmatrix} f & 0 \\ g & 0 \\ 0 & 1 \end{pmatrix}} & Z \oplus M^m \\
 \downarrow & & \downarrow \psi_1 & & \downarrow \psi_2 & & \downarrow \psi_{m-1} & & \downarrow \psi_m \\
 0 & \longrightarrow & N_1 & \xrightarrow{\alpha_1} & N_2 & \cdots & N_{m-1} & \xrightarrow{\alpha_{m-1}} & N_m
 \end{array}$$

with  $\psi_i = (k_i, l_i, \alpha_{i-1}l_{i-1}, \dots, \alpha_{i-1} \dots \alpha_1 l_1) : Z \oplus M^i \rightarrow N_i$ .

Note that the composition of the first  $i$  maps of the top row is just  $\varphi_i : Z \rightarrow Z \oplus M^i$  and that the sequence

$$0 \rightarrow Z \xrightarrow{\varphi_i} Z \oplus M^i \xrightarrow{\psi_i} N_i \rightarrow 0$$

is exact for  $i = 1, \dots, m$ . So  $N_i \xrightarrow{\sim} \text{coker } \varphi_i$ , and the maps  $\alpha_i$  are the ones we claim. As for  $\beta_i$ , it suffices to show that

$$\psi_i \circ (1 \ 0) = \beta_i \circ \psi_{i+1}.$$

This follows easily from the explicit formulas for  $\psi_i$ ,  $\psi_{i+1}$ , the equation

$$(k_i, l_i) = \beta_i(k_{i+1}, l_{i+1})$$

and the fact that  $(N_i, \alpha_i, \beta_i)$  is an exact tube of height  $m$ . As a consequence we have:

**Remark 4.7.** Let  $(f, g)^t$  be an  $(M, N)$ -monomorphism and  $T' = (N'_i, \alpha'_i, \beta'_i)$  an exact tube of height  $m$ . Then  $T'$  is isomorphic to  $(T_{f,g})_{\leq m}$  if and only if there exists an exact square

$$\begin{array}{ccc}
 Z & \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix}} & Z \oplus M \\
 \downarrow \beta'_{m-1} \circ k & & \downarrow (k, l) \\
 N'_{m-1} & \xrightarrow{\alpha'_{m-1}} & N'_m
 \end{array}$$

**Proposition 4.8.** Any  $M$ -extendible exact tube  $T = (N_i, \alpha_i, \beta_i)$  of height  $m$  with  $N_1 \xrightarrow[A]{\sim} N$  is the restriction of the exact tube  $T_{f,g}$  to  $Q_m$  for some  $(M, N)$ -monomorphism  $(f, g)^t$ .

*Proof.* Let

$$0 \rightarrow Z \xrightarrow{\begin{pmatrix} a \\ b \end{pmatrix}} N_{m-1} \oplus M \xrightarrow{(c \ d)} Z' \rightarrow 0$$

be an exact sequence with  $N_m = Z \oplus Z'$ ,  $a = \beta_{m-1}|_Z$  and  $c = \text{pr}_{Z'} \circ \alpha_{m-1}$ . The square

$$\begin{array}{ccc} N_m = Z \oplus Z' & \xrightarrow{\begin{pmatrix} c'a & c'a' \\ b & 0 \\ 0 & 1 \end{pmatrix}} & Z \oplus M \oplus Z' \\ \downarrow (a,a')=\beta_{m-1} & & \downarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & -d & ca' \end{pmatrix} \\ N_{m-1} & \xrightarrow{(c')=\alpha_{m-1}} & N_m = Z \oplus Z' \end{array}$$

is exact. Setting

$$N_{m+1} = Z \oplus M \oplus Z', \quad \alpha_m = \begin{pmatrix} c'a & c'a' \\ b & 0 \\ 0 & 1 \end{pmatrix}, \quad \beta_m = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -d & ca' \end{pmatrix}$$

we may extend  $T$  to an exact tube of height  $m+1$ . By construction, the map

$$\begin{pmatrix} c'a \\ b \end{pmatrix} : Z \longrightarrow Z \oplus M$$

is an  $(M, N)$ -monomorphism, and the square

$$\begin{array}{ccc} Z & \xrightarrow{\begin{pmatrix} c'a \\ b \end{pmatrix}} & Z \oplus M \\ \downarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} & & \downarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \\ N_m = Z \oplus Z' & \xrightarrow{\alpha_m} & N_{m+1} = Z \oplus M \oplus Z' \end{array}$$

is exact with

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \beta_m \circ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The result now follows from Remark 4.7.  $\square$

**Proposition 4.9.** *Let  $T = (N_i, \alpha_i, \beta_i)$  be an exact tube of height  $h+m$ , for some  $h \geq 1$  and  $m \geq 1$ . Suppose that*

$$N_{h+j+1} \xrightarrow[A]{\sim} N_{h+j} \oplus M$$

*for  $j \in \{0, \dots, m-1\}$ . Then there is an  $(M, N)$ -monomorphism*

$$\begin{pmatrix} f \\ g \end{pmatrix} : N_{h+m-1} \longrightarrow N_{h+m-1} \oplus M$$

*such that the restrictions  $T_{\leq m}$  and  $(T_{f,g})_{\leq m}$  are isomorphic.*

*Proof.* We wish to choose

$$\begin{pmatrix} f \\ g \end{pmatrix} = \chi \circ \alpha_{h+m-1}$$

for a suitable isomorphism  $\chi : N_{h+m} \rightarrow N_{h+m-1} \oplus M$  and to apply Remark 4.7 to the diagram

$$\begin{array}{ccccc} N_{h+m-1} & \xrightarrow{\alpha_{h+m-1}} & N_{h+m} & \xrightarrow[\chi]{\sim} & N_{h+m-1} \oplus M \\ \beta = \beta_{m-1} \cdots \beta_{h+m-2} \downarrow & & \beta_m \cdots \beta_{h+m-1} \downarrow & & \swarrow \\ N_{m-1} & \xrightarrow{\alpha_{m-1}} & N_m & & \end{array}$$

In order to do this, we only need to construct a section

$$s : N_{h+m-1} \longrightarrow N_{h+m}$$

satisfying

$$\beta_{m-1}\beta_m \cdots \beta_{h+m-1}s = \beta\beta_{h+m-1}s = \beta.$$

By our hypothesis, the square

$$\begin{array}{ccc} N_{h+1} & \xrightarrow{\alpha} & N_{h+m} \\ \downarrow \beta_h & & \downarrow \beta_{h+m-1} \\ N_h & \xrightarrow{\alpha'} & N_{h+m-1} \end{array}$$

splits, where  $\alpha = \alpha_{h+m-1} \cdots \alpha_{h+1}$  and  $\alpha' = \alpha_{h+m-2} \cdots \alpha_h$ . Choose a maximal direct summand  $A$  of  $N_{h+m}$  for which  $\beta_{h+m-1}|_A$  is a section. Replacing  $T$  by an isomorphic exact tube, we may assume that we have

$$\beta_{h+m-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \gamma & \delta \end{pmatrix} : A \oplus B \oplus M \longrightarrow A \oplus B,$$

$$\alpha' = \begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix} : C \oplus B \longrightarrow A \oplus B$$

for some maps  $\gamma, \delta, \varepsilon$ . Setting

$$s = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} : A \oplus B \longrightarrow A \oplus B \oplus M,$$

we obtain

$$1_{A \oplus B} - \beta_{h+m-1}s = \begin{pmatrix} 0 & 0 \\ 0 & 1 - \gamma \end{pmatrix} : A \oplus B \longrightarrow A \oplus B,$$

which factors through  $\alpha'$ . But the sequence

$$0 \longrightarrow N_h \xrightarrow{\alpha'} N_{h+m-1} \xrightarrow{\beta} N_{m-1} \longrightarrow 0$$

is exact, which implies  $\beta = \beta\beta_{h+m-1}s$  as required.  $\square$



## 5. Examples

All our examples are representations of quivers with relations. Let  $Q$  be a quiver with vertex set  $Q_0 = \{1, \dots, n\}$ ,  $I$  an admissible two-sided ideal in the quiver algebra  $kQ$ ,  $\underline{d} = (d_1, \dots, d_n)$  a vector in  $\mathbb{N}^n$ , and denote by

$$\text{Rep}(Q, I, \underline{d})$$

the affine algebraic variety of representations  $X$  of  $(Q, I)$  with  $X(i) = k^{d_i}$ ,  $i \in Q_0$ . The dimension vector of  $X$  in  $\text{Rep}(Q, I, \underline{d})$  is  $\underline{d}$ . The group  $G(\underline{d}) = \prod_{i=1}^n GL(d_i)$  acts on  $\text{Rep}(Q, I, \underline{d})$  by

$$(g \cdot X)(\alpha) = g_j \circ X(\alpha) \circ g_i^{-1}$$

for an arrow  $\alpha : i \rightarrow j$  and  $g = (g_1, \dots, g_n) \in G(\underline{d})$ .

If we view  $M, N$  in  $\text{Rep}(Q, I, \underline{d})$  as modules over  $kQ/I$  of dimension  $d = \sum_{i=1}^n d_i$ , then  $M$  degenerates to  $N$  if and only if the representation  $N$  belongs to the closure of the orbit  $G(\underline{d}) \cdot M$  of  $M$  in  $\text{Rep}(Q, I, \underline{d})$  [1]. This allows us to work with the smaller group  $G(\underline{d})$ .

**5.1.** We begin with an example of a degeneration whose complexity is easy to compute: Choose a natural number  $n \geq 2$  and let  $\vec{A}_n$  be the equioriented quiver with underlying graph  $A_n$ :

$$\vec{A}_n = 1 \xleftarrow{\gamma_1} 2 \xleftarrow{\gamma_2} \dots \xleftarrow{\gamma_{n-1}} n.$$

Denote by  $X_i$  the indecomposable representation of  $\vec{A}_n$  given by

$$X_i(j) = \begin{cases} k & j \leq i, \\ 0 & j > i, \end{cases}$$

$$X_i(\gamma_j) = \begin{cases} 1 & j < i, \\ 0 & j \geq i. \end{cases}$$

Then  $M = X_n$  has a filtration

$$M = X_n \supset X_{n-1} \supset \dots \supset X_2 \supset X_1,$$

and it is well-known that  $M$  degenerates to the associated graded module

$$N = \bigoplus_{i=1}^n X_i / X_{i-1},$$

where we set  $X_0 = 0$ . We wish to compute the complexity  $\text{cpl}(M, N)$ , thereby

showing again that  $M$  actually degenerates to  $N$ . Set

$$Z = \bigoplus_{i=1}^{n-1} X_i,$$

$$f = \begin{pmatrix} 0 & & & 0 \\ \iota_2 & \ddots & & \\ & \ddots & \ddots & \\ & & \iota_{n-1} & 0 \end{pmatrix} : Z \longrightarrow Z \quad \text{and}$$

$$g = (0 \cdots 0 \ \iota_n) : Z \longrightarrow M = X_n,$$

where  $\iota_i : X_{i-1} \rightarrow X_i$  is the inclusion. It is easy to check that  $(f, g)^t$  is an  $(M, N)$ -monomorphism. Moreover,  $f^{n-1} = 0$ , and thus

$$\text{cpl}(M, N) \leq n - 1.$$

On the other hand, the Loewy lengths of  $M$  and  $N$  are  $n$  and 1, respectively, which implies

$$\text{cpl}(M, N) \geq \frac{\ell\ell(M)}{\ell\ell(N)} - 1 = n - 1$$

by Proposition 3.5. This example shows that there are degenerations of arbitrary complexity.

Note that for  $n = 4$  we obtain the following chain of degenerations:

$$M = k \xleftarrow{1} k \xleftarrow{1} k \xleftarrow{1} k \leq_{\deg} P = k \xleftarrow{1} k \xleftarrow{0} k \xleftarrow{1} k$$

$$\leq_{\deg} N = k \xleftarrow{0} k \xleftarrow{0} k \xleftarrow{0} k.$$

The complexities are

$$\text{cpl}(M, P) = 1 = \text{cpl}(P, N) \quad \text{and}$$

$$\text{cpl}(M, N) = 3 > \text{cpl}(M, P) + \text{cpl}(P, N).$$

Comparing with the example given in the introduction, we see that  $\text{cpl}(M, P) + \text{cpl}(P, N)$  can be either smaller or greater than  $\text{cpl}(M, N)$  for a chain

$$M \leq_{\deg} P \leq_{\deg} N.$$

**5.2.** Next we give an example of a minimal degeneration of arbitrary complexity: Let  $Q$  be the quiver

$$Q = 1 \xrightarrow{\alpha} 2 \begin{array}{c} \curvearrowright \\ \beta \end{array},$$

choose a natural number  $n \geq 2$ , and let  $I$  be the ideal generated by  $\beta^n$ . Define  $M$  and  $N$  to be the representations of dimension vector  $(1, n)$  given by

$$M(\alpha) = e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad N(\alpha) = e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

and

$$M(\beta) = N(\beta) = J_n, \text{ respectively,}$$

where  $e_1, \dots, e_n$  is the standard basis of  $k^n$  and  $J_m$  is the Jordan block

$$J_m = \begin{pmatrix} 0 & & & 0 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{pmatrix}$$

in  $\mathbb{M}_m(k)$ , for  $m \in \mathbb{N}$ .

**Proposition 5.1.** *There is a degeneration  $M \leq_{\deg} N$ , which is minimal, and  $\text{cpl}(M, N) = n$ .*

*Proof.* Denote by  $Z$  the indecomposable representation with dimension vector  $(0, n)$ , given by  $Z(\beta) = J_n$ , and let  $(f, g)^t : Z \rightarrow Z \oplus M$  be given by

$$f = (0, J_n), \quad g = (0, 1).$$

It is easy to see that  $(f, g)^t$  is an  $(M, N)$ -monomorphism, so  $M$  degenerates to  $N$ . Moreover, we have  $f^n = 0$ , and therefore  $\text{cpl}(M, N) \leq n$ . As

$$\dim \text{End } M = 1 \text{ and } \dim \text{End } N = 2,$$

the orbit of  $N$  has codimension 1 in the closure of the orbit of  $M$ , which implies that the degeneration is minimal.

Suppose  $\text{cpl}(M, N) \leq n - 1$ , and choose an  $(M, N)$ -tube  $T = (N_i, \alpha_i, \beta_i)$  with  $N_n \xrightarrow{\sim} N_{n-1} \oplus M$ . Let  $\psi_n : N_n \rightarrow M$  be the surjection obtained from this decomposition.

**Claim.** *For  $i = 1, \dots, n$ , there exists a surjection*

$$\psi_i : N_i \rightarrow M^{(i)},$$

where  $M^{(i)}$  has dimension vector  $(1, i)$  and is given by

$$M^{(i)}(\alpha) = (1, 0, \dots, 0)^t, \quad M^{(i)}(\beta) = J_i.$$

Using the claim for  $i = 1$ , we obtain a surjection  $\psi_1 : N_1 = N \rightarrow M^{(1)}$ , which is impossible.

We prove the claim by descending induction on  $i$ . Observe that any map from  $N$  to  $M^{(i)}$  factors through the socle  $\text{soc } M^{(i)}$  and that  $M^{(i)}/\text{soc } M^{(i)} \xrightarrow{\sim} M^{(i-1)}$ . Writing this factorization for  $\psi_i \circ \alpha_{i-1} \cdots \alpha_1$ , we obtain  $\psi_{i-1} : N_{i-1} \rightarrow M^{(i-1)}$  from the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 & \longrightarrow & N & \xrightarrow{\alpha_{i-1} \cdots \alpha_1} & N_i & \xrightarrow{\beta_{i-1}} & N_{i-1} \longrightarrow 0 \\
& & \downarrow & & \downarrow \psi_i & & \downarrow \psi_{i-1} \\
0 & \longrightarrow & \text{soc} M^{(i)} & \longrightarrow & M^{(i)} & \longrightarrow & M^{(i-1)} \longrightarrow 0.
\end{array}$$

As  $\psi_i$  is surjective,  $\psi_{i-1}$  is as well.  $\square$

A version of this argument implies the following result, which we will not use:

$$\text{cpl}(M, M_i) = \left\lfloor \frac{n-1}{i-1} \right\rfloor + 1, \quad i \geq 2.$$

The representation  $M_i$  is given by

$$M_i(\alpha) = e_i, \quad M_i(\beta) = J_n.$$

**5.3.** We now exhibit a degeneration  $M \leq_{\text{deg}} N$  of complexity 2 with the property that  $f^2 \neq 0$  for all  $(M, N)$ -monomorphisms  $(f, g)^t$ . Therefore the complexity can be strictly less than the “index of nilpotence of  $M$  and  $N$ ”; i.e., the number

$$\min\{r : f^r = 0\},$$

where the minimum is taken over all  $(M, N)$ -monomorphisms  $(f, g)^t$ . We stay with the same quiver  $Q$ , and we choose  $I$  to be generated by  $\beta^3$ ; i.e., we set  $n = 3$  in the preceding example. Note that  $kQ/I$  is representation-finite: it admits 29 indecomposables [4].

We let  $M$  and  $N$  be given by

$$M(\alpha) = e_2, \quad N(\alpha) = e_3, \quad M(\beta) = N(\beta) = J_3,$$

where  $e_1, e_2, e_3$  is the standard basis of  $k^3$ . Choose

$$\begin{array}{c}
Z' = 0 \longrightarrow k^3 \xrightarrow{\quad J_3 \quad} k^3 \\
f' = (0, J_3) : Z' \longrightarrow Z' \quad \text{and} \\
g' = (0, 1) : Z' \longrightarrow M.
\end{array}$$

Then  $(f', g')^t$  is an  $(M, N)$ -monomorphism. As  $f'^2$  factors through  $g'$ , the cokernel  $N_3$  of the map

$$\varphi_3 = (f'^3, g'f'^2, g'f', g')^t : Z' \longrightarrow Z' \oplus M^3$$

used to define the tube  $T_{f', g'} = (N_i, \alpha_i, \beta_i)$  is isomorphic to the cokernel of

$$(f'^2, 0, g'f', g')^t : Z' \longrightarrow Z' \oplus M^3$$

and thus isomorphic to  $M \oplus N_2$ . By Lemma 3.3, we know that

$$\text{cpl}(M, N) \leq 2.$$

On the other hand, as  $N$  is indecomposable, the complexity must exceed 1, so

$$\text{cpl}(M, N) = 2.$$

**Claim.** *For any  $(M, N)$ -monomorphism*

$$(f, g)^t : Z \longrightarrow Z \oplus M,$$

*we have  $f^2 \neq 0$ .*

First we show:

**Lemma 5.2.** *For any  $(M, N)$ -monomorphism*

$$(f, g)^t : Z \longrightarrow Z \oplus M,$$

*$Z'$  is a direct summand of  $Z$ .*

*Proof.* Consider the exact sequences

$$\Sigma' : 0 \longrightarrow Z' \xrightarrow{\begin{pmatrix} f' \\ g' \end{pmatrix}} Z' \oplus M \xrightarrow{(k', l')} N \longrightarrow 0$$

and

$$\Sigma : 0 \longrightarrow Z \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix}} Z \oplus M \xrightarrow{(k, l)} N \longrightarrow 0.$$

It is easy to check that

$$\begin{aligned} \dim \text{Hom}(Z', M) &= \dim \text{Hom}(Z', N) = 3, \\ \dim \text{End } M &= \dim \text{Hom}(M, N) = 2. \end{aligned}$$

Therefore the sequence of vector spaces

$$0 \longrightarrow \text{Hom}(Z' \oplus M, Z) \longrightarrow \text{Hom}(Z' \oplus M, Z \oplus M) \longrightarrow \text{Hom}(Z' \oplus M, N) \longrightarrow 0$$

obtained from mapping  $Z' \oplus M$  into  $\Sigma$  is exact. In particular,  $(k', l') : Z' \oplus M \rightarrow N$  factors through  $(k, l) : Z \oplus M \rightarrow N$ , and hence we have the following commutative diagram (Figure 3) with exact rows and columns.

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \uparrow & & \uparrow & \\
 \Sigma : & 0 \longrightarrow & Z & \longrightarrow & Z \oplus M & \xrightarrow{(k,l)} & N \longrightarrow 0 \\
 & & \parallel & & \uparrow & & \uparrow (k',l') \\
 & 0 \longrightarrow & Z & \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} & Z \oplus (Z' \oplus M) & \xrightarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}} & Z' \oplus M \longrightarrow 0 \\
 & & & & \uparrow \begin{pmatrix} s' \\ f' \\ g' \end{pmatrix} & & \uparrow \begin{pmatrix} f' \\ g' \end{pmatrix} \\
 & & & & Z' & \xlongequal{\quad} & Z' \\
 & & & & \uparrow & & \uparrow \\
 & & & & 0 & & 0
 \end{array}$$

Figure 3

So the middle column splits as well, and since by construction  $f', g'$  lie in the radical,

$$s : Z' \longrightarrow Z$$

must be a section.  $\square$

Let  $(f, g)^t : Z \rightarrow Z \oplus M$  be an  $(M, N)$ -monomorphism, suppose  $f^2 = 0$ , and consider the commutative diagram (Figure 4) with exact rows and columns.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 \longrightarrow & \ker f & \longrightarrow & M & \longrightarrow & X & \longrightarrow 0 \\
 & \downarrow & & \downarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} & & \downarrow & \\
 0 \longrightarrow & Z & \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix}} & Z \oplus M & \xrightarrow{(k,l)} & N & \longrightarrow 0 \\
 & \downarrow & & \downarrow \begin{pmatrix} 1, 0 \end{pmatrix} & & \downarrow & \\
 0 \longrightarrow & \operatorname{im} f & \longrightarrow & Z & \longrightarrow & \operatorname{coker} f & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 & 
 \end{array}$$

Figure 4

Then  $X$  must be a quotient of  $M$  and a submodule of  $N$ , which is possible in

exactly two ways:

$$(i) \quad X = k \xrightarrow{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} k^2 \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} J_2$$

$$(ii) \quad X = 0 \longrightarrow k \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} 0$$

In the first case, we have

$$\ker f = 0 \longrightarrow k \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} 0,$$

and our assumption  $f^2 = 0$  implies that  $\dim Z(2) \leq 2$ . But then  $Z$  cannot contain  $Z'$  as a direct summand.

In the second case, we see that

$$\ker f = k \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} k^2 \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} J_2.$$

Now  $f^2 = 0$  implies that  $\dim Z(2) \leq 4$ . But then necessarily  $Z(\beta\alpha) = 0$ , since  $Z'$  must be a direct summand of  $Z$ , and  $Z$  cannot contain  $\ker f$  as a submodule.

**5.4.** As our last example, we find a degeneration  $M \leq_{\text{deg}} N$  of complexity 2 for which there exists an exact sequence

$$\Sigma : 0 \longrightarrow N \xrightarrow{\alpha_1 = \begin{pmatrix} f \\ g \end{pmatrix}} N \oplus M \xrightarrow{\beta_1 = (f, -l)} N \longrightarrow 0.$$

So we have an exact tube

$$T = (N_1 = N, N_2 = N \oplus M, \alpha_1, \beta_1)$$

of height 2. If this tube were the restriction of an (infinite) exact tube, the complexity  $\text{cpl}(M, N)$  would have to equal 1. So the number  $2h + 1$  in condition (iii) of our main theorem cannot be replaced by  $2h$ .

Choose  $A = k[\alpha, \beta]/(\alpha^2, \beta^2)$ , let  $M$  and  $N$  be 4-dimensional with

$$M(\alpha) = N(\alpha) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad M(\beta) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad N(\beta) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

and set

$$f = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad g = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad l = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

It is easy to check that the sequence  $\Sigma$  obtained from these choices is exact. So  $M$  degenerates to  $N$ . As  $N$  is indecomposable and  $f^2$  factors through  $g$ , the same argument as in Section 5.3 implies that  $\text{cpl}(M, N) = 2$ .

This example has another surprising feature: For any degeneration  $M \leq_{\text{deg}} N$  we obtain

$$\text{cpl}(M^r, N^r) \leq \text{cpl}(M, N), \quad r \geq 1,$$

by taking for  $M^r \leq_{\text{deg}} N^r$  the direct sum of  $r$  copies of an  $(M, N)$ -tube of minimal complexity. In our example, we have

$$\text{cpl}(M^2, N^2) = 1 < \text{cpl}(M, N) = 2.$$

Indeed,  $M^2$  is a projective cover for  $N$ , and the kernel of an epimorphism  $M^2 \rightarrow N$  is  $N$  again. So there is an exact sequence

$$0 \longrightarrow N \longrightarrow M^2 \longrightarrow N \longrightarrow 0.$$

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