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Concavity of the Lagrangian for quasi-periodic orbits

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Abstract. Percival introduced a "Lagrangian" for finding quasi-periodic orbits. For suitable area preserving mappings, we show that Percival's "Lagrangian" is strictly concave with respect to an appropriate affine structure on its domain. Consequently, the "Lagrangian" admits a unique maximum in the case of irrational frequencies.

Introduction

In [3, 4], Percival sketched a method of finding quasi-periodic orbits numerically, by maximizing a function, which he called the "Lagrangian." Percival was looking for invariant tori. In the case we study in this paper (area preserving mappings), the invariant tori would be invariant circles.

It is well known that frequently invariant circles (of a given frequency) do not exist. But the author proved in [2] that, under suitable hypotheses, Percival's "Lagrangian" always has a maximum, and there is an invariant set associated to to this maximum. If there is an invariant circle of the given frequency, it contains the invariant set associated to the maximum; otherwise, the invariant set associated to the maximum is a Cantor set.

In this paper, we will show that for irrational frequencies, Percival's "Lagrangian" is strictly concave with respect to a suitable affine structure on its domain. As a consequence, we obtain that the maximum of Percival's "Lagrangian" is unique.

We impose slightly stronger hypotheses than in [2].

§2. Definitions and main results

We retain the notations and hypotheses on f from [2]. In addition, we suppose that f is C^1 and $\partial f(x, y)_1/\partial y > 0$. (Under the hypotheses of [2], this inequality need not be strict.) We also suppose that $\rho(f_0) < \omega < \rho(f_1)$.

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In [2], we proved the existence of quasi-periodic orbits of frequency ω. These were associated to a maximum of Percival's "Lagrangian" F_ω.

Let W denote the set of weakly order preserving, left continuous mappings $\phi:\mathbf{R}\to\mathbf{R}$ such that $\phi(t)\to\pm\infty$ as $t\to\pm\infty$. Define $I: W\to W$ by

$$(x, y)\in \text{graph } I(\phi) \Leftrightarrow (y, x)\in \text{graph } \phi.$$

In other words,

$$I(\phi)(t)=\sup\{s:\phi(s)<t\}.$$

When ϕ is a homeomorphism, $I(\phi)=\phi^{-1}$. Obviously, $I^2=\text{id}$.

We let $Y_\omega^-=I(Y_\omega)$. Thus, Y_ω^- is the set of weakly order-preserving, left-continuous mappings $\psi:\mathbf{R}\to\mathbf{R}$ such that $\psi(x+1)=\psi(x)+1$ and $\psi(f_0(x))\le \psi(x)+\omega\le\psi(f_1(x))$.

Obviously, Y_ω^- is a convex subset of the space of all real valued functions of a real variable.

THEOREM 1. *$F_\omega I: Y_\omega^-\to\mathbf{R}$ is a concave function.*

The statement that $F_\omega I$ is concave means that if $\psi_0, \psi_1\in Y_\omega^-$ and $0\le s\le 1$, then

$$(1-s)F_\omega I(\psi_0)+sF_\omega I(\psi_1)\le F_\omega I((1-s)\psi_0+s\psi_1). \tag{2.1}$$

Let $X_\omega^-=\{\psi\in Y_\omega:\psi(0)=0\}$. Then $I(X_\omega^-)\subset X_\omega$ and we have an identification

$$Y_\omega^-=X_\omega^-\times\mathbf{R},$$

where $\psi\in Y_\omega^-$ is identified with $(\psi-\psi(0), \psi(0))\in X_\omega^-\times\mathbf{R}$. From the translation invariance of F_ω (cf. [2, §3]), it follows that $F_\omega I(\psi)=F_\omega I(\psi-\psi(0))$, for all $\psi\in Y_\omega^-$.

We let $X_{\omega c}^-$ denote the set of continuous $\psi\in X_\omega^-$.

THEOREM 2. *If ω is irrational, then $F_\omega I: X_{\omega c}^-\to\mathbf{R}$ is strictly concave.*

In other words, if ψ_0, ψ_1 are distinct members of $X_{\omega c}^-$ and $0<s<1$, then

$$(1-s)F_\omega I(\psi_0)+sF_\omega I(\psi_1)< F_\omega I((1-s)\psi_0+s\psi_1). \tag{2.2}$$

For $\phi, \psi\in W$, we set

$$d(\phi,\psi)=\max\{\sup_\xi\inf_\eta|\xi-\eta|, \sup_\eta\inf_\xi|\xi-\eta|\},$$

where ξ ranges over graph ϕ, the element η ranges over graph ψ, and $|\ |$ denotes the Euclidean norm on $\mathbf{R}^2$. This may be infinite. However, we have

$$d(\phi_1, \phi_2) \le d(\phi_1, \phi_3) + d(\phi_3, \phi_2)$$
$$d(\phi_1, \phi_2) = d(\phi_2, \phi_1)$$
$$d(\phi_1, \phi_2) \ge 0$$
$$d(\phi_1, \phi_2) = 0 \Rightarrow \phi_1 = \phi_2,$$

for $\phi_1, \phi_2, \phi_3 \in W$. In [2], we showed that the restriction of d to Y_ω is always finite. In view of the conditions which d satisfies, this implies that d is a metric on Y_ω. Likewise the restriction of d to Y_ω^- is always finite, and d is a metric on Y_ω^-. Obviously $I: Y_\omega \to Y_\omega^-$ is an isometry.

We let $\pi: Y_\omega^- \to X_\omega^-$ denote the mapping defined by

$$\pi(\psi) = \psi - \psi(0).$$

We call π the *projection* of Y_ω^- on X_ω^-. The subspace X_ω^- of Y_ω^- is *not* closed. For this reason, the induced topology and metric on X_ω^- are not convenient; instead, we provide X_ω^- with the quotient topology associated to this projection and the quotient metric $\bar{d}$ defined by

$$\bar{d}(\psi_1, \psi_2) = \inf_{a,b \in \mathbf{R}} d(\psi_1 + a, \psi_2 + b).$$

The triangle inequality for $\bar{d}$ is an easy consequence of

$$\bar{d}(\psi_1, \psi_2) = \inf_{a \in \mathbf{R}} d(\psi_1 + a, \psi_2) = \inf_{a \in \mathbf{R}} d(\psi_1, \psi_2 + a),$$

which, in turn, follows from the obvious translation invariance of d:

$$d(\psi_1 + a, \psi_2 + a) = d(\psi_1, \psi_2).$$

It is easily verified that the quotient topology on X_ω^- (associated to the projection π) is the underlying topology of the metric $\bar{d}$.

Provided with the metric $\bar{d}$, the space X_ω^- is compact. For, $\pi I: X_\omega \to X_\omega^-$ is a continuous surjective mapping, and we proved in [2, §5] that X_ω is compact.

In [2, §6], we proved that $F_\omega: Y_\omega \to \mathbf{R}$ is continuous. Since $I: Y_\omega^- \to Y_\omega$ is an isometry, it follows that $F_\omega I: Y_\omega^- \to \mathbf{R}$ is continuous. Since $F_\omega I \pi = F_\omega I$, it follows that $F_\omega I: X_\omega^- \to \mathbf{R}$ is continuous.

To summarize, $F_\omega I: X_\omega^- \to \mathbf{R}$ is a concave, continuous function on a compact, convex set, and it is strictly concave on $X_{\omega c}^-$ when ω is irrational.

Since $F_\omega I$ takes its maximum only in $X_{\omega c}^-$, this proves the uniqueness of the maximum, when ω is irrational.

§3. Outline of the Proof of Theorem 1

We will say that $\psi \in Y_\omega^-$ is *smooth* if it is C^2 and its first derivative never vanishes. We let $Y_{\omega s}^-$ denote the set of smooth members of Y_ω^-. We will prove in §§4, 5 that $Y_{\omega s}^-$ is dense in Y_ω^-. Since $F_\omega I$ is continuous on Y_ω^-, and $Y_{\omega s}^-$ is dense in Y_ω^-, (2.1) will follow if we verify it whenever $\psi_0, \psi_1 \in Y_{\omega s}^-$.

Suppose $\psi_0, \psi_1 \in Y_{\omega s}^-$. Set $\psi_s = s\psi_0 + (1-s)\psi$, $\dot{\psi} = \psi_1 - \psi_0$, $\phi_s = \psi_s^{-1}$. We have

$$F_\omega I(\psi_s) = \int_0^1 h(\phi_s(t), \phi_s(t+\omega))\, dt. \tag{3.1}$$

As we observed in [2, §1], h is a C^1 function on B. We set

$$h_1(x, x') = \frac{\partial h}{\partial x}(x, x'), \qquad h_2(x, x') = \frac{\partial h}{\partial x'}(x, x').$$

Since ψ_0, ψ_1 are C^2, we have that $\psi_s(x)$ is a C^2 function of $x \in \mathbf{R}$ and $s \in [0, 1]$. Since the first derivatives of ψ_0 and ψ_1 never vanish and both ψ_0 and ψ_1 are weakly increasing, we have

$$\frac{d\psi_s}{dx}(x) > 0,$$

everywhere. Hence, $\phi_s(t)$ is a C^2 function of $t \in \mathbf{R}$ and $s \in [0, 1]$. Consequently,

$$\frac{d}{ds} F_\omega I(\psi_s) = \int_0^1 \left[h_1(\phi_s(t), \phi_s(t+\omega)) \frac{\partial \phi_s(t)}{\partial s} + h_2(\phi_s(t), \phi_s(t+\omega)) \frac{\partial \phi_s(t+\omega)}{\partial s} \right] dt$$

Obviously,

$$0 = \frac{\partial(\psi_s \phi_s)}{\partial s}(t) = \frac{\partial \psi_s}{\partial s}(x) + \frac{d\psi_s}{dx}(x) \frac{\partial \phi_s}{\partial s}(t),$$

where $x = \phi_s(t)$. Hence

$$\frac{\partial \phi_s}{\partial s}(t) = -\frac{\partial \psi_s}{\partial s}(x) \Big/ \frac{d\psi_s}{dx}(x) = -\dot{\psi}(x) \Big/ \frac{dt}{dx},$$

since $t=\psi_s(x)$. Changing the independent variable from t to x in the first summand of the above integral, we obtain

$$\int_0^1 h_1(\phi_s(t), \phi_s(t+\omega))\frac{\partial\phi_s(t)}{\partial s}\,dt$$
$$=-\int_0^1 h_1(x, x'(s,x))\frac{\dot\psi(x)}{dt/dx}\,dt=-\int_0^1 h_1(x, x'(s,x))\dot\psi(x)\,dx.$$

where $x'(s,x)=\phi_s(\psi_s(x)+\omega))$. Similarly, the change of variables $x=\phi_s(t+\omega)$ gives

$$\int_0^1 h_2(\phi_s(t), \phi_s(t+\omega))\frac{\partial\phi_s(t+\omega)}{\partial s}\,dt$$
$$=-\int_0^1 h_2(\bar x(s,x), x)\frac{\dot\psi(x)}{dt/dx}\,dt=-\int_0^1 h_2(\bar x(s,x), x)\dot\psi(x)\,dx,$$

where $\bar x(s,x)=\phi_s(\psi_s(x)-\omega))$. Note that when we change variables, we may take 0 and 1 as the limits of integration, since everything under the integral signs in periodic (in x and t) of period 1. From the formulas which we have just derived, we obtain

$$\frac{d}{ds}F_\omega I(\psi_s)=-\int_0^1 [h_1(x, x'(s,x))+h_2(\bar x(s,x),x)]\dot\psi(x)\,dx. \tag{3.2}$$

From the fact that f is C^1 and $\partial f(x,y)_1/\partial y>0$, we obtain that the functions g and g', defined in the introduction of [2], are C^1 on B. In view of the definition of h given in [2, §1], it follows that h is C^2 on the interior of B and the second partial derivatives of h extend continuously to the boundary of B. We set

$$h_{12}(x,x')=\frac{\partial^2 h}{\partial x\,\partial x'}(x,x').$$

Set $f(x,y)=(x',y')$. Taking x and y as independent variables, our "twist" condition on f states $\partial x'/\partial y>0$. Recall that the "twist" condition implies that for $(x,x')\in B$, there exists unique $y, y'\in[0,1]$ such that $f(x,y)=(x',y')$. Thus, we may take x and x' as independent variables, and the condition $\partial x'/\partial y>0$ becomes

$$\frac{\partial g(x,x')}{\partial x'}=\frac{\partial y}{\partial x'}>0.$$

Since $\partial h(x, x')/\partial x = g(x, x')$, by the definition of h, it follows that

$$h_{12}(x, x') > 0, \tag{3.3}$$

for all $(x, x') \in B$.

From (3.2), we obtain

$$\frac{d^2}{ds^2} F_\omega I(\psi_s) = -\int_0^1 \left[h_{12}(x, x'(s, x)) \frac{\partial x'(s, x)}{\partial s} + h_{12}(\bar{x}(s, x), x) \frac{\partial \bar{x}(s, x)}{\partial s} \right] \dot{\psi}(x)\, dx.$$

From the definition of $x'(s, x)$, we obtain

$$\frac{\partial x'}{\partial s} = \frac{\partial \phi_s}{\partial s}(\psi_s(x) + \omega) + \frac{d\phi_s}{dt}(\psi_s(x) + \omega)\dot{\psi}(x).$$

Moreover,

$$0 = \frac{\partial(\phi_s \psi_s)}{\partial s}(x) = \frac{\partial \phi_s}{\partial s}(\psi_s(x)) + \frac{d\phi_s}{dt}(\psi_s(x))\dot{\psi}(x),$$

so

$$\frac{\partial \phi_s}{\partial s}(\psi_s(x) + \omega) = -\frac{d\phi_s}{dt}(\psi_s(x) + \omega) \cdot \dot{\psi}(x'(s, x)).$$

Hence

$$\begin{aligned} &\int_0^1 h_{12}(x, x'(s, x)) \frac{\partial x'(s, x)}{\partial s} \dot{\psi}(x)\, dx \\ &\quad = \int_0^1 h_{12}(x, x'(s, x)) \frac{d\phi_s(\psi_s(x) + \omega)}{dt} [\dot{\psi}(x) - \dot{\psi}(x'(s, x))]\dot{\psi}(x)\, dx. \end{aligned}$$

From the definition of $\bar{x}(s, x)$, we obtain

$$\begin{aligned} \frac{\partial \bar{x}}{\partial s} &= \frac{\partial \phi_s}{\partial s}(\psi_s(x) - \omega) + \frac{d\phi_s}{dt}(\psi_s(x) - \omega) \cdot \dot{\psi}(x) \\ &= \frac{d\phi_s}{dt}(\psi_s(x) - \omega)[\dot{\psi}(x) - \dot{\psi}(\bar{x}(s, x))]. \end{aligned}$$

Hence,

$$\int_0^1 h_{12}(\bar{x}(s,x),x)\frac{\partial \bar{x}(s,x)}{\partial s}\dot{\psi}(x)\,dx$$
$$=\int_0^1 h_{12}(\bar{x}(s,x),x)\frac{d\phi_s}{dt}(\psi_s(x)-\omega)[\dot{\psi}(x)-\dot{\psi}(\bar{x}(s,x))]\dot{\psi}(x)\,dx$$
$$=\int_0^1 h_{12}(x,x'(s,x))\frac{d\phi_s}{dt}(\psi_s(x))[\dot{\psi}(x'(s,x))-\dot{\psi}(x)]\dot{\psi}(x'(s,x))\,dx'(s,x)$$
$$=\int_0^1 h_{12}(x,x'(s,x))\frac{d\phi_s}{dt}(\psi_s(x)+\omega)[\dot{\psi}(x'(s,x))-\dot{\psi}(x)]\dot{\psi}(x'(s,x))\,dx,$$

since

$$\frac{dx'(s,x)}{dx}=\frac{d\phi_s}{dt}(\psi_s(x)+\omega)\Big/\frac{d\phi_s}{dt}(\psi_s(x)).$$

Combining the above integrals, we get

$$\frac{d^2}{ds^2}F_\omega I(\psi_s)=-\int_0^1 h_{12}(x,x'(s,x))\frac{d\phi_s}{dt}(\psi_s(x)+\omega)[\dot{\psi}(x'(s,x))-\dot{\psi}(x)]^2\,dx. \tag{3.4}$$

In view of the fact that h_{12} and $d\phi_s/dt$ are everywhere positive, we get

$$\frac{d^2}{ds^2}F_\omega I(\psi_s)\le 0.$$

Since this is satisfied for $0\le s\le 1$, we obtain (2.1).

The only thing which remains to be done in order to finish the proof of Theorem 1 is to prove that $Y^-_{\omega s}$ is dense in Y^-_ω.

§4. Proof that $Y^-_{\omega s}$ is dense in $Y^-_{\omega c}$

DEFINITION. We let $Y^-_{\omega c}$ denote the set of continuous $\psi\in Y^-_\omega$.

LEMMA. *There exists a homeomorphism $\psi\in Y^-_\omega$ and $\delta>0$ such that*

$$\psi(f_0(x))+\delta\le\psi(x)+\omega\le\psi(f_1(x))-\delta. \tag{4.1}$$

Proof. When ω is irrational, let $g_0=\rho*(f_0+\varepsilon)$, $g_1=\rho*(f_1-\varepsilon)$, where ρ is a bump function, i.e., ρ is infinitely differentiable, $\rho\geq 0$ everywhere, $\int\rho=1$, and supp ρ is contained in a small interval $[-\delta,\delta]$ above the origin. Here, $u*v$ denote the convolution of u and v, i.e.,

$$(u*v)(x)=\int_{-\infty}^{\infty}u(\xi)v(x-\xi)\,d\xi=\int_{-\infty}^{\infty}u(x-\xi)v(\xi)\,d\xi.$$

We suppose $\varepsilon>0$ and then choose $\delta>0$ such that

$$|x-x'|<\delta\Rightarrow|f_i(x)-f_i(x')|<\varepsilon$$

$i=0,1$. Then g_i is infinitely differentiable, $g_i(x+1)=g_i(x)+1$, and $dg_i/dx>0$ everywhere, for $i=0,1$. Moreover, $g_0>f_0$, $g_1<f_1$, by our hypotheses on δ and the assumption that supp $\rho\subset[-\delta,\delta]$. Obviously, $g_i\rightarrow f_i$, uniformly as $\varepsilon\rightarrow 0$, so $\rho(g_i)\rightarrow\rho(f_i)$.

In view of our standing assumption that $\rho(f_0)<\omega<\rho(f_1)$, we may choose ε and the bump function ρ such that $\rho(g_0)<\omega<\rho(g_1)$.

Let $g_s=(1-s)g_0+sg_1$. Then $\rho(g_s)$ is a continuous function of s, so there exists $s(0)$, satisfying $0<s(0)<1$, such that $\rho(g)=\omega$, where $g=g_{s(0)}$. Clearly, g is a C^∞ diffeomorphism and $g(x+1)=g(x)+1$, so there exists a homeomorphism $\psi:\mathbf{R}\rightarrow\mathbf{R}$ satisfying $\psi(x+1)=\psi(x)+1$, and

$$\psi g(x)=\psi(x)+\omega,$$

for all $x\in\mathbf{R}$, by Denjoy's theorem [1].

From the construction of g, it is obvious that there exists $\delta_1>0$ such that

$$f_0(x)+\delta_1<g(x)<f_1(x)-\delta_1,$$

for all $x\in\mathbf{R}$. Since ψ is a homeomorphism and $\psi(x+1)=\psi(x)+1$, it follows that there exists $\delta>0$ such that

$$\psi(x+\delta_1)>\psi(x)+\delta,\qquad \psi(x-\delta_1)<\psi(x)-\delta,$$

for all $x\in\mathbf{R}$. Then

$$\psi(f_0(x))+\delta<\psi(f_0(x)+\delta_1)<\psi(g(x))=\psi(x)+\omega<\psi(f_1(x)-\delta_1)<\psi(f_1(x))-\delta,$$

proving our assertion.

Proof when ω is rational. Let ω_0, ω_1 be irrational numbers such that

$$\rho(f_0)<\omega_0<\omega<\omega_1<\rho(f_1).$$

By what we have just proved there exist homeomorphisms $\psi_0, \psi_1 \in Y_\omega^-$ and $\delta>0$, such that

$$\psi_i(f_0(x))+\delta \leq \psi_i(x)+\omega_i \leq \psi_i(f_1(x))-\delta, \qquad i=0, 1.$$

Let λ be such that

$$\omega=(1-\lambda)\omega_0+\lambda\omega_1.$$

Then $\psi=(1-\lambda)\psi_0+\lambda\psi_1$ has the required properties. □

End of proof that $Y_{\omega s}^-$ is dense in $Y_{\omega c}^-$. Let $\psi_1 \in Y_{\omega c}^-$ and let ψ be as in the lemma. Let $\psi_s=(1-s)\psi+s\psi_1$. Then ψ_s is a homeomorphism in $Y_{\omega c}^-$, satisfying

$$\psi_s(f_0(x))+(1-s)\delta \leq \psi_s(x)+\omega \leq \psi_s(f_1(x))-(1-s)\delta. \tag{4.2}$$

If ρ is a bump function (as above), it is clear that $\rho * \psi_s$ is C^∞ and has non-vanishing derivative everywhere. Moreover, as $\operatorname{supp}\rho \to \{0\}$, we have $\rho*\psi_s \to \psi_s$ uniformly; in particular, we have $\rho*\psi_s \in Y_{\omega s}^-$, when $\operatorname{supp}\rho$ is small enough. Since $\psi_s \to \psi$ as $s \to 1$, this finishes the proof.

§5. Proof that $Y_{\omega c}^-$ is dense in Y_ω^-

Let $\psi_1 \in Y_\omega^-$. Let $\psi \in Y_\omega^-$ be as in the lemma of §4. Let $\psi_s=(1-s)\psi+s\psi_1$. Obviously, ψ_s satisfies (4.2) and is strictly increasing. If $\psi': \mathbf{R} \to \mathbf{R}$ is left continuous, weakly order preserving, satisfies $\psi'(x+1)=\psi'(x)+1$, and $|\psi'(x)-\psi_s(x)|<(1-s)\delta/2$, then $\psi' \in Y_\omega^-$. Obviously, there exists such a ψ' such that ψ' has only finitely many discontinuities in $[0, 1)$, and is strictly order preserving. Since we may take ψ_s arbitrarily close to ψ_1 and ψ' arbitrarily close to ψ_s, it follows that any member of Y_ω^- may be arbitrarily well approximated by a member which has only finitely many discontinuities in $[0, 1)$ and which is strictly order preserving.

So, we suppose from now on that ψ_1 has only finitely many discontinuities in $[0, 1)$ and is strictly increasing.

One of the conditions for ψ_1 to be in Y_ω^- is

$$\psi_1(f_0(x)) \leq \psi_1(x)+\omega \leq \psi_1(f_1(x)).$$

This implies the two conditions:

$$\psi_1(f_0(x)) - \omega \le \psi_1(x) \le \psi_1(f_0^{-1}(x)) + \omega \tag{5.1}$$

$$\psi_1(f_1^{-1}(x)) + \omega \le \psi_1(x) \le \psi_1(f_1(x)) - \omega. \tag{5.2}$$

This leads us to introduce the following two quantities:

$$L(\psi_1, x) = \min\,(\psi_1(f_0^{-1}(x)-) + \omega,\ \psi_1(f_1(x)-) - \omega)$$

$$U(\psi_1, x) = \max\,(\psi_1(f_0(x)+) - \omega,\ \psi_1(f_1^{-1}(x)+) + \omega).$$

LEMMA 5.1. *Let x_0 be a point of discontinuity of ψ_1. Let $\frac{1}{2} > \delta > 0$, and suppose x_0 is the only point of discontinuity of ψ_1 in $[x_0-\delta, x_0+\delta]$. Then there exists $\psi' \in Y_\omega^-$ arbitrarily close to ψ_1, such that ψ' is strictly increasing, $\psi' = \psi_1$ on $[x_0+\delta, x_0+1-\delta]$, and the following holds: If $U(\psi_1, x_0) \le L(\psi_1, x_0)$, then ψ' is continuous in $[x_0-\delta, x_0+\delta]$. If $L(\psi_1, x_0) < U(\psi_1, x_0)$, then x_0 is the only point of discontinuity of ψ' in the interval $[x_0-\delta, x_0+\delta]$, and*

$$\psi'(x_0-) = L(\psi_1, x_0), \qquad \psi'(x_0+) = U(\psi_1, x_0). \tag{5.3}$$

Proof. Consider ψ' which is left-continuous, strictly order preserving, and satisfies $\psi'(x+1) = \psi'(x) + 1$ and $\psi' = \psi_1$ on $[x_0+\delta, x_0+1-\delta]$. For ψ' to be in Y_ω^-, it is enough that (5.1,2 – with ψ_1 replaced by ψ') be satisfied for $x \in [x_0-\delta, x_0+\delta]$. When $f_0(x_0) - x_0 \notin \mathbf{Z}$ and $f_1(x_0) - x_0 \notin \mathbf{Z}$, it is possible to alter ψ_1 in a small neighborhood of x_0 without changing it near $f_0(x_0)$ or $f_1(x_0)$. Conditions (5.1, 2) then become

$$\psi_1(f_0(x)) - \omega \le \psi'(x) \le \psi_1(f_0^{-1}(x)) + \omega \tag{5.4}$$

$$\psi_1(f_1^{-1}(x)) - \omega \le \psi'(x) \le \psi_1(f_1(x)) - \omega, \tag{5.5}$$

where it is enough that these conditions should be satisfied in the set of x where ψ' differs from ψ_1, which may be taken to be an arbitrarily small neighborhood of x_0. It is easy to see that there exists such a ψ' which is continuous on $[x_0-\delta, x_0+\delta]$ except possibly at x_0, is continuous at x_0 if $U(\psi_1, x_0) \le L(\psi_1, x_0)$, and satisfies (5.3), otherwise.

If $f_0(x_0) - x_0 \in \mathbf{Z}$, then $f_0(x_0) - x_0 = \rho(f_0) < \omega$. In view of the periodicity condition on ψ_1, condition (5.1), for ψ' in place of ψ_1, may be rewritten as

$$\psi'(f_0'(x)) - \omega' \le \psi'(x) \le \psi'(f_0'^{-1}(x)) + \omega', \tag{5.6}$$

where $f_0' = f_0 - \rho(f_0)$ and $\omega' = \omega - \rho(f_0)$. Then x_0 is a fixed point of f_0' and $\omega' > 0$. When $f_1(x_0) - x_0 \notin \mathbf{Z}$, then we must satisfy (5.5) and (5.6), when ψ' is an alteration of ψ_1 in a sufficiently small neighborhood of x_0 (and its translates). In this case, it is easy to see that we can make the alteration so that ψ' is continuous in $[x_0 - \delta, x_0 + \delta]$, no matter what $L(\psi_1, x_0)$ and $U(\psi_1, x_0)$ are. (Of course, we could also arrange for (5.3) to hold, if we prefer.)

There are two more cases to be considered: namely, $f_1(x_0) - x_0 \in \mathbf{Z}$, $f_0(x_0) - x_0 \notin \mathbf{Z}$ and $f_1(x_0) - x_0 \in \mathbf{Z}$, $f_0(x_0) - x_0 \in \mathbf{Z}$. But these may be treated in exactly the same way as the case which we just studied. □

Now consider the following procedure. Let $x_1, \ldots, x_n$ be the points of discontinuity of ψ_1 in the interval $[0, 1)$. Use Lemma 5.1 to change ψ_1 in a small neighborhood of x_1. This gives $\psi_2 \in Y_\omega^-$ which may be taken arbitrarily close to ψ_1. The new element has discontinuities only at $x_2, \ldots, x_n$ and at x_1. The jumps at $x_2, \ldots, x_n$ are the same as before. The jump at x_1 is no larger than before.

Next use the lemma to change ψ_2 in a small neighborhood of x_2, getting a new element ψ_3, just as before. Continue this process, making alterations successively at $x_3, \ldots, x_n$, getting new elements $\psi_4, \ldots, \psi_{n+1}$. If ψ_{n+1} still has discontinuities, start over at x_1 and then run through $x_2, \ldots, x_n$, just as before, getting $\psi_{n+2}, \ldots, \psi_{2n+1}$. If ψ_{2n+1} still has discontinuities, start over again at x_1, etc.

LEMMA 5.2. *If ω is irrational, this procedure stops after finitely many steps, and gives a continuous $\psi \in Y_\omega^-$, arbitrarily close to ψ_1.*

Proof. Let $K_i = \overline{\psi_i(\mathbf{R})}$, $U_i = \mathbf{R} \backslash K_i$, $J_{ij} = (\psi_i(x_j -), \psi_i(x_j +))$, $j = 1, \ldots, n$. For fixed i, we have that U_i is the disjoint union of the $T^k(J_{ij})$, where $j = 1, \ldots, n$, where k ranges over all integers, and T is translation by one.

Consider a positive integer l and write $l = qn + r$, where $r, q \in \mathbf{Z}$ and $1 \le r \le n$. From the construction we have given of ψ_{l+1}, it is clear that

$$J_{l+1,j} = J_{lj}, \qquad j \neq r$$
$$J_{l+1,r} \subset J_{lr}.$$

SUBLEMMA. *If either $t + \omega \notin U_l$ or $t - \omega \notin U_l$, then $t \notin J_{l+1,r}$.*

Proof. If $L(\psi_l, x_r) \ge U(\psi_l, x_r)$, then $J_{l+1,r} = \varnothing$. So, we may suppose $L(\psi_l, x_r) < U(\psi_l, x_r)$. Since $f_0(x) < f_1(x)$, we have

$$\psi_l(f_0(x)+) - \omega \le \psi_l(f_1(x)-) - \omega.$$

Since $f_1^{-1}(x_0) < f_0^{-1}(x_0)$, we have

$$\psi_l(f_1^{-1}(x)+)+\omega \leq \psi_l(f_0^{-1}(x)-)+\omega.$$

Since $L(\psi_l, x_r) < U(\psi_l, x_r)$, we must have one of the following two possibilities:

$$L(\psi_l, x_r) = \psi_l(f_0^{-1}(x_r)-)+\omega$$
$$U(\psi_l, x_r) = \psi_l(f_0(x_r)+)-\omega \tag{5.7}$$

or

$$L(\psi_l, x_r) = \psi_l(f_1(x_r)-)-\omega$$
$$U(\psi_l, x_r) = \psi_l(f_1^{-1}(x_r)+)+\omega \tag{5.8}$$

Suppose (5.7) holds and $t-\omega \notin U_l$. Since $t \in J_{lr}$, we have that $\psi_l(x_r+) > t$. By (5.1), with ψ_1 replaced by ψ_l, we have

$$\psi_l(f_0^{-1}(x_r)+) > t-\omega.$$

Since $t-\omega \notin U_l$, this implies

$$\psi_l(f_0^{-1}(x_r)-) \geq t-\omega.$$

Hence, $L(\psi_l, x_r) \geq t$, and $t \notin J_{l+1,r}$, by construction of ψ_{l+1}.

Next, suppose (5.7) holds and $t+\omega \notin U_l$. Since $t \in J_{lr}$, we have $\psi_l(x_r-) < t$. By (5.1), with ψ_1 replaced by ψ_l, we have

$$\psi_l(f_0(x_r)-) < t+\omega.$$

Since $t+\omega \notin U_l$, we have

$$\psi_l(f_0(x_r)+) \leq t+\omega.$$

Hence, $U(\psi_l, x_r) \leq t$, and $t \notin J_{l+1,r}$, by the construction of ψ_{l+1}.

We have thus proved the sublemma when (5.7) holds. The case when (5.8) holds may be treated in the same way. □

End of the proof of Lemma 5.2. K_1 has non-empty interior, because ψ_1 has only finitely many jumps and is strictly increasing. Since ω is irrational, it follows that there is a positive integer N such that for any $t\in\mathbf{R}$, there exists $\alpha\in\mathbf{Z}$, $|\alpha|\leq N$, such that $t+\alpha\omega\in K_1$. Then $t\in K_{n\alpha+1}$, by the sublemma. Hence $K_{nN+1}=\mathbf{R}$, and our procedure stops after finitely many steps, as we asserted. □

By Lemma 5.2, $Y^-_{\omega c}$ is dense in Y^-_ω, when ω is irrational. Now we will prove that $Y^-_{\omega c}$ is dense in Y^-_ω, when ω is rational, using the fact that it is dense when ω is irrational.

Suppose ω is rational, and let $\psi_1\in Y^-_\omega$. Let $\psi\in Y^-_\omega$ satisfy (4.1). Let $\psi_s=(1-s)\psi+s\psi_1$. Then

$$\psi_s\in Y^-_{\omega'},\quad \text{for}\quad |\omega'-\omega|<(1-s)\delta.$$

Choose irrational numbers ω_0 and ω_1 such that $\rho(f_0)<\omega_0<\omega<\omega_1<\rho(f_1)$ and $|\omega_i-\omega|<(1-s)\delta$, for $i=0, 1$. By what we have just finished proving, there exist ψ_0', ψ_1' arbitrarily close to ψ_s such that

$$\psi_0'\in Y^-_{\omega(0)c},\qquad \psi_1'\in Y^-_{\omega(1)c}.$$

Define λ by $\omega=(1-\lambda)\omega_0+\lambda\omega_1$. Then $0<\lambda<1$. Set $\psi'=(1-\lambda)\psi_0'+\lambda\psi_1'$. Obviously $\psi'\in Y^-_{\omega c}$.

Since we may choose ψ_s arbitrarily close to ψ_1, and ψ_0', ψ_1' arbitrarily close to ψ_s, it follows that this procedure produces ψ' arbitrarily close to ψ_1.

This finishes the proof that $Y^-_{\omega c}$ is dense in Y^-_ω. □

In §3, we showed that if $Y^-_{\omega s}$ is dense in Y^-_ω, then Theorem 1 is true. In §4, we showed that $Y^-_{\omega s}$ is dense in $Y^-_{\omega c}$. In this section, we showed that $Y^-_{\omega c}$ is dense in Y^-_ω. We have thereby completed the proof of Theorem 1.

§6. Outline of the Proof of Theorem 2

We set

$$A(\psi_0, \psi_1)=\int_0^1 [F_\omega I(\psi_s)-sF_\omega I(\psi_0)-(1-s)F_\omega I(\psi_1)]\,dx.$$

This is the area bounded by the graph of the function $s\mapsto F_\omega I(\psi_s)$ and the graph of the function $s\mapsto sF_\omega I(\psi_0)+(1-s)F_\omega I(\psi_1)$. By Theorem 1, $A(\psi_0, \psi_1)\geq 0$ and the necessary and sufficient condition for (2.2) to hold is that $A(\psi_0, \psi_1)>0$.

Suppose $\psi_0, \psi_1 \in Y^-_{\omega s}$. Integration by parts gives

$$A(\psi_0, \psi_1) = -\int_0^1 (s-\tfrac{1}{2})\left(\frac{d}{ds} F_\omega I(\psi_s) + F_\omega I(\psi_1) - F_\omega I(\psi_0)\right) ds.$$

A second integration by parts gives

$$A(\psi_0, \psi_1) = -\int_0^1 \frac{s(1-s)}{2} \frac{d^2}{ds^2} F_\omega I(\psi_s)\, ds.$$

Using (3.4), we obtain

$$A(\psi_0, \psi_1) = \int_0^1 \int_0^1 \frac{s(1-s)}{2} h_{12}(x, x'(s, x)) \frac{d\phi_s}{dt}(\psi_s(x)+\omega)[\dot\psi(x'(s, x)) - \dot\psi(x)]^2\, dx\, ds.$$

Recall that $x'(s, x) = \phi_s(\psi_s(x)+\omega)$. In other words, $x' = x'(s, x)$ is the unique solution of the equation

$$\psi_s(x') = \psi_s(x) + \omega,$$

i.e.,

$$(1-s)\psi_0(x') + s\psi_1(x') = (1-s)\psi_0(x) + s\psi_1(x) + \omega. \tag{6.1}$$

We wish to express the integral in terms of independent variables x and x'. Observe that (6.1) is equivalent to

$$s = \frac{\psi_0(x) - \psi_0(x') + \omega}{\dot\psi(x') - \dot\psi(x)}. \tag{6.2}$$

Hence

$$\int_0^1 \frac{s(1-s)}{2} h_{12}(x, x'(s, x)) \frac{d\phi_s}{dt}(\psi_s(x)+\omega)[\dot\psi(x'(s, x)) - \dot\psi(x)]^2\, ds$$

$$= \int_{f_0(x)}^{f_1(x)} U(x, x') h_{12}(x, x') \frac{d\phi_s}{dt}(\psi_s(x)+\omega)[\dot\psi(x') - \dot\psi(x)]^2 \left|\frac{ds}{dx'}\right| dx', \tag{6.3}$$

where s is given by (6.2), and

$$U(x, x') = \frac{s(1-s)}{2}, \quad \text{if } 0 \le s \le 1,$$
$$= 0, \qquad \text{otherwise.}$$

Of course, we must assume $\dot{\psi}(x') \neq \dot{\psi}(x)$ for this to make sense.

Note that for any fixed $x \in \mathbf{R}$, there are three possibilities, according to whether $x'(0, x) > x'(1, x)$, $x'(0, x) < x'(1, x)$, or $x'(0, x) = x'(1, x)$. In the first case, $U(x, x') \neq 0$ is equivalent to $x'(1, x) < x' < x'(0, x)$. For x' in this range,

$$\psi_0(x') < \psi_0(x'(0, x)) = \psi_0(x) + \omega$$
$$\psi_1(x') > \psi_1(x'(1, x)) = \psi_1(x) + \omega,$$

so we obtain $\dot{\psi}(x') > \dot{\psi}(x)$. Moreover, s is a strictly decreasing function of x', for $x'(1, x) \le x' \le x'(0, x)$, in view of (6.2). This justifies our change of variables in (6.3), in the first case.

In the second case, $U(x, x') \neq 0$ is equivalent to $x'(0, x) < x' < x'(1, x)$. For x' in this range, we have

$$\psi_0(x') > \psi_0(x'(0, x)) = \psi_0(x) + \omega$$
$$\psi_1(x') < \psi_1(x'(1, x)) = \psi_1(x) + \omega,$$

so we obtain $\dot{\psi}(x') < \dot{\psi}(x)$. Moreover, s is a strictly increasing function of x', for $x'(0, x) \le x' \le x'(1, x)$, in view of (6.2). This justifies our change of variables in (6.3) in the second case.

In the third case, $U(x, x') = 0$ except for $x' = x'(0, x) = x'(1, x)$, where it is undefined. Then

$$\psi_0(x') = \psi_0(x) + \omega$$
$$\psi_1(x') = \psi_1(x) + \omega,$$

so we obtain $\dot{\psi}(x') = \dot{\psi}(x)$. Moreover, $\psi_s(x') = \psi_s(x) + \omega$, for all $0 \le s \le 1$, so we obtain $x' = x'(s, x)$. It follows that the integral on the left side of (6.3) vanishes. The integral on the right side of (6.3) also vanishes, since the integrand vanishes everywhere except at one point, where it is undefined. Thus, (6.3) holds in this case, too, even though the change of variables argument does not apply.

From $\psi_s(x')-\psi_s(x)=\omega$, we obtain

$$(\dot\psi(x')-\dot\psi(x))\,ds+\frac{d\psi_s}{dx'}(x')\,dx'=0,$$

when x is held constant. Moreover, since $\psi_s=\phi_s^{-1}$, we have

$$\frac{d\psi_s(x')}{dx'}=\left(\frac{d\phi_s}{dt}(\psi_s(x)+\omega)\right)^{-1}.$$

Hence

$$\left|\frac{ds}{dx'}\right|=\left(\frac{d\phi_s}{dt}(\psi_s(x)+\omega)\,|\dot\psi(x')-\dot\psi(x)|\right)^{-1}.$$

Hence the right side of (6.3) equals

$$\int_{f_0(x)}^{f_1(x)} U(x,x')h_{12}(x,x')\,|\dot\psi(x')-\dot\psi(x)|\,dx'.$$

Substituting this in the equation we obtained previously for $A(\psi_0,\psi_1)$, we get

$$A(\psi_0,\psi_1)=\int_0^1\left[\int_{f_0(x)}^{f_1(x)} U(x,x')h_{12}(x,x')\,|\dot\psi(x')-\dot\psi(x)|\,dx'\right]dx. \tag{6.4}$$

The possibility that $\dot\psi(x')=\dot\psi(x)$ for some values of x and x' causes no difficulty, since in the above calculation, both sides contribute 0 on the set of (x,x') for which $\dot\psi(x')=\dot\psi(x)$.

In order to finish the proof of Theorem 2, two further steps are enough. First, we will show that (6.4) is valid for *all* $\psi_0,\psi_1\in F_\omega^-$. (So far, we have shown it only for $\psi_0,\psi_1\in F_{\omega s}^-$.) For this it is enough to prove that the right side of (6.4) is continuous on $F_\omega^-\times F_\omega^-$. For, we have proved that $Y_{\omega s}^-$ is dense in Y_ω^- (§§4, 5). Moreover, the left side of (6.4) is continuous in $(\psi_0,\psi_1)\in Y_\omega^-\times Y_\omega^-$, in view of the definition of $A(\psi_0,\psi_1)$ and the fact that $F_\omega I$ is continuous on Y_ω^- (§2). The proof of the continuity of the right side of (6.4) will be carried out in §7.

Second, we will show that (6.4) implies that if $A(\psi_0,\psi_1)=0$ and ψ_0 is continuous then $\psi_1=\psi_0+\text{constant}$ (§8). This will finish the proof of Theorem 2.

§7. Proof of (6.4) for all $\psi_0, \psi_1 \in Y_\omega^-$

As we just observed, it is enough to prove that the right side of (6.4) is continuous in ψ_0 and ψ_1.

Let $0<\delta<10^{-3}$. Let $\psi_0, \psi_1, \psi_0', \psi_1' \in Y_\omega^-$. We suppose that $d(\psi_i, \psi_i')<\delta$, for $i=0, 1$.

Let $\delta_1=\sqrt{\delta}+\delta$. If $\psi_i'(x)-\psi_i(x)\geq\delta_1$, then $\psi_i(x+\delta)\geq\psi_i(x)+\sqrt{\delta}$. At a point x where $\psi_i'(x)-\psi_i(x)\geq\delta_1$, the variation of ψ_i over the interval $[x, x+\delta]$ is $\geq\sqrt{\delta}$. Since the total variation of ψ_i over the interval $[0, 1]$ is ≤ 1, it follows that $\{x\in[0, 1]: \psi_i'(x)-\psi_i(x)\geq\delta_1\}$ can be covered by at most $[\delta^{-1/2}]+1$ intervals of length δ. Hence, we have the following estimate for its measure:

$$\mu\{x\in[0, 1]: \psi_i'(x)-\psi_i(x)\geq\delta_1\}\leq\delta([\delta^{-1/2}]+1)\leq\delta_1.$$

Here, μ denotes Lebesque measure. Likewise,

$$\mu\{x\in[0, 1]: \psi_i(x)-\psi_i'(x)\geq\delta_1\}\leq\delta_1,$$
$$\mu\{x\in[f_0(0), f_1(1)]: \psi_i'(x)-\psi_i(x)\geq\delta_1\}\leq N\delta_1,$$
$$\mu\{x\in[f_0(0), f_1(1)]: \psi_i(x)-\psi_i'(x)\geq\delta_1\}\leq N\delta_1,$$

where N is the smallest integer greater than $|f_1(1)-f_0(0)|$.

Let $\dot\psi'=\psi_1'-\psi_0'$. If

$$||\dot\psi(x')-\dot\psi(x)|-|\dot\psi'(x')-\dot\psi'(x)||\geq 4\delta_1,$$

then we must have at least one of the four inequalities

$$|\psi_i(x)-\psi_i'(x)|\geq\delta_1, \qquad i=0 \text{ or } 1,$$

or

$$|\psi_i(x')-\psi_i'(x')|\geq\delta_1, \qquad i=0 \text{ or } 1.$$

The Lebesque measure of the set Π of $(x, x')\in[0, 1]\times[f_0(0), f_1(1)]$ where one of these four inequalities holds is $\leq 8N\delta_1$.

Let U' be defined in terms of ψ_0' and ψ_1' in the same way as U was defined in terms of ψ_0 and ψ_1. In other words,

$$U'(x, x') = \frac{s'(1-s')}{2}, \quad \text{if } 0 \le s' \le 1,$$

$$= 0, \quad \text{otherwise,}$$

where

$$s' = \frac{\psi_0'(x) - \psi_0'(x') + \omega}{\dot\psi'(x') - \dot\psi'(x)}.$$

Then

$$|U - U'| \le \tfrac{1}{2}|s - s'|$$

$$\le \tfrac{1}{2}C(|\dot\psi'(x') - \dot\psi'(x)|^{-1} - |\dot\psi(x') - \dot\psi(x)|^{-1}) + \frac{|\psi_0'(x) - \psi_0(x)| + |\psi_0'(x') - \psi_0(x')|}{2\,|\dot\psi(x') - \dot\psi(x)|},$$

where C is an upper bound for $|\psi_0'(x) - \psi_0'(x') + \omega|$, for $x \in [0, 1]$ and $x' \in [f_0(0), f_1(1)]$. In view of the fact that ψ_0' is weakly increasing and satisfies $\psi_0'(x+1) = \psi_0'(x) + 1$, such a bound exists and is independent of ψ_0'. From this, we obtain

$$|U - U'| \le 8C\delta_1\,|\dot\psi(x') - \dot\psi(x)|^{-2} + \delta_1\,|\dot\psi(x') - \dot\psi(x)|^{-1}, \tag{7.1}$$

if

$$|\psi_i(x) - \psi_i'(x)| \le \delta_1 \quad \text{and} \quad |\psi_i(x') - \psi_i'(x')| \le \delta_1, \quad i = 0, 1 \tag{7.2}$$

and

$$8\delta_1 \le |\dot\psi(x') - \dot\psi(x)|. \tag{7.3}$$

Let $M = \max\{h_{12}(x, x') : 0 \le x \le 1, f_0(0) \le x' \le f_1(1)\}$. This maximum exists and is finite because h_{12} is continuous (§3) and $h_{12}(x+1, x'+1) = h_{12}(x, x')$.

Writing $A^*(\psi_0, \psi_1)$ for the right side of (6.4), we find

$$|A^*(\psi_0', \psi_1') - A^*(\psi_0, \psi_1)|$$

$$\le \int_0^1 \left[\int_{f_0(x)}^{f_1(x)} \tfrac{1}{2}M(|\dot\psi(x') - \dot\psi(x)| - |\dot\psi'(x') - \dot\psi'(x)|)\,dx'\right] dx$$

$$+ \int_0^1 \left[\int_{f_0(x)}^{f_1(x)} M\,|U'(x, x') - U(x, x')|\,|\dot\psi(x') - \dot\psi(x)|\,dx'\right] dx,$$

in view of $M \geq h_{12} > 0$ and $0 \leq U' \leq \frac{1}{2}$, everywhere. We estimate the first summand on the right side by breaking it into two parts: the integral over Π and the integral over the complement $\mathscr{C}\Pi$ of Π.

Our hypothesis that $d(\psi_i, \psi_i') < \delta < 10^{-3}$ implies that

$$|\psi_i'(x) - \psi_i(x)| < 1 + 10^{-3} < 2,$$

for all $x \in \mathbf{R}$, in view of the fact that ψ_i and ψ_i' are strictly order-preserving and satisfy the periodicity properties $\psi_i(x+1) = \psi_i(x) + 1$, $\psi_i'(x+1) = \psi_i'(x) + 1$. Consequently, the integrand J_1 is the first integral is $\leq 4M$, everywhere. Since the Lebesque measure of Π is $\leq 8N\delta_1$, we obtain

$$\iint_{\Pi} J_1 \, dx \, dx' \leq 32NM\delta_1.$$

By definition of Π, (7.2) holds on $\mathscr{C}\Pi$. Hence $J_1 \leq 2M\delta_1$ and

$$\iint_{\mathscr{C}\Pi} J_1 \, dx \, dx' \leq 2MN\delta_1.$$

We estimate the second summand on the right side by breaking it into three parts: the integral over Π, the integral over the set Π_1 of $(x, x') \in \mathscr{C}\Pi$ such that

$$8\sqrt{\delta_1} \leq |\dot{\psi}(x') - \dot{\psi}(x)|, \tag{7.4}$$

and the integral over the set Π_2 of $(x, x') \in \mathscr{C}\Pi$ such that (7.4) does not hold.

It is easily seen that

$$|\dot{\psi}(x') - \dot{\psi}(x)| \leq 2, \quad \text{everywhere,}$$

in view of $\dot{\psi}(x+1) = \dot{\psi}(x)$, and the fact that the variation of $\dot{\psi}$ over $[0, 1]$ is ≤ 2. Hence the integrand J_2 of the integral in the second summand is $\leq 2M$ everywhere. It follows that

$$\iint_{\Pi} J_2 \, dx \, dx' \leq 16NM\delta_1.$$

If $(x, x') \in \Pi_1$, then (7.3) holds by (7.4) and the fact that $\delta_1 < 1$. Moreover, (7.2) holds because $(x, x') \in \mathscr{C}\Pi$. Hence, (7.1) holds. From (7.1) and (7.4), we get

$$J_2 \leq CM\sqrt{\delta_1} + M\delta_1$$

on Π_1. Hence

$$\iint_{\Pi_1} J_2 \, dx \, dx' \leq MN(C\sqrt{\delta_1} + \delta_1).$$

On Π_2, we have $J_2 \leq 8M\sqrt{\delta_1}$. Hence

$$\iint_{\Pi_2} J_2 \, dx \, dx' \leq 8MN\sqrt{\delta_1}.$$

Combining all these estimates, we get

$$|A^*(\psi_0', \psi_1') - A^*(\psi_0, \psi_1)| \leq C_1\sqrt{\delta_1},$$

where $C_1 = (59 + C)NM$. Here we use the fact that $\delta_1 < \sqrt{\delta_1}$, since $\delta_1 < 1$.

Since this was obtained under the hypothesis that $d(\psi_i, \psi_i') < \delta$, $i = 0, 1$, and $\sqrt{\delta_1} = (\sqrt{\delta} + \delta)^{1/2} \to 0$ as $\delta \to 0$, it follows that the right side of (6.4) is continuous in ψ_0 and ψ_1.

Hence (6.4) holds for all $\psi_0, \psi_1 \in Y_\omega^-$.

§8. End of the Proof of Theorem 2

Suppose ω is irrational.

Suppose that there exists $(x, x') \in B$ such that

$$\psi_1(x') > \psi_1(x) + \omega \quad \text{and} \quad \psi_0(x') < \psi_0(x) + \omega \tag{8.1}$$

or

$$\psi_1(x') < \psi_1(x) + \omega \quad \text{and} \quad \psi_0(x') > \psi_0(x) + \omega. \tag{8.2}$$

In this case, we have $U(x, x') > 0$ and $|\dot{\psi}(x) - \dot{\psi}(x')| > 0$, and these inequalities still hold everywhere in a sufficiently small neighborhood of (x, x'). It follows that $A(\psi_0, \psi_1) > 0$.

Suppose that there is no $(x, x') \in B$ such that (8.1) or (8.2) holds. Let $\phi_i = I(\psi_i)$, $i = 0, 1$. (See §2 for the definition of I.) Since neither (8.1) nor (8.2) ever holds and ψ_0 is continuous,

$$\phi_1(t_1) = \phi_0(t_0) \Rightarrow \phi_1(t_1+\omega) = \phi_0(t_0+\omega).$$

Since $\phi_i(t+1) = \phi_i(t)+1$, and ω is irrational, we obtain that $\phi_1 = \phi_0 T_a$, for some $a \in \mathbf{R}$, where $T_a(x) = x + a$. Hence, $\psi_1 = \psi_0 + a$. If $\psi_1, \psi_0 \in X_{\omega c}^-$, we must have $\psi_1 = \psi_0$.

We have proved: if $\psi_0, \psi_1 \in X_{\omega c}^-$ and ω is irrational, then either $A(\psi_0, \psi_1) > 0$ or $\psi_0 = \psi_1$. So, Theorem 2 holds. □

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