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Autor(en): **Munkholm, H.J. / Pedersen, E.K.**

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Whitehead transfers for S^1-bundles, an algebraic description

HANS J. MUNKHOLM and ERIK KJAER PEDERSEN

§0. Introduction

Let $F \to E \xrightarrow{f} B$ be a bundle of finite complexes. D. Anderson [1] defines a geometric transfer homomorphism $f^*_{\mathrm{Wh}}: \mathrm{Wh}(\pi_1 B) \to \mathrm{Wh}(\pi_1 E)$ which maps the Whitehead torsion of a homotopy equivalence $g: K \to B$ to the Whitehead torsion of the induced homotopy equivalence $g^*(E) \to E$. (Actually Anderson works with PL bundles, but by topological invariance of Whitehead torsion [4] this is no longer necessary.) Similarly there is a homomorphism $f^*_{K_0}: \tilde{K}_0(\mathbf{Z}\pi_1 B) \to \tilde{K}_0(\mathbf{Z}\pi_1 E)$ [6] relating the finiteness obstruction of a complex dominated by B to the finiteness obstruction of the total space of the pullback. The homomorphisms f^*_{Wh} and $f^*_{K_0}$ are known in some cases e.g. if the bundle is trival [9], [8] or when some assumptions are made on the behaviour of fundamental groups [7], [11], [12]. The case $F = S^1$ turns out to be of particular interest since knowing the geometric transfer homomorphisms in this case enables one to compute it in a number of other cases (see §7). In this paper we give a complete algebraic description of the geometric transfers in the case $F = S^1$ and also compute it for other fibres such as lens spaces, $S^3/Q(8)$, tori, see Theorems 7.1, 7.2, 7.3.

We prove

THEOREM A. *Let $S^1 \to E \to B$ be an orientable S^1-bundle with fundamental group sequence $\mathbf{Z} \to \pi \to \rho \to 1$ and let $x \in \mathrm{Wh}(\rho)$ be represented by an $n \times n$ invertible matrix $\bar{A}$ over $\mathbf{Z}\rho$. Choose A and B, matrices over $\mathbf{Z}\pi$, reducing to $\bar{A}$ and $\bar{A}^{-1}$ respectively, and let $t \in \pi$ be the image of $1 \in \mathbf{Z}$. Then $AB = I - (t-1)C$ for some $n \times n$ matrix C over $\mathbf{Z}\pi$ and $f^*_{\mathrm{Wh}}(x)$ is given by the $2n \times 2n$ matrix*

$$\begin{pmatrix} A & -C \\ t-1 & B \end{pmatrix}.$$

which determines a well defined element of $\mathrm{Wh}(\pi)$.

We also compute f^*_{Wh} in case the bundle is not orientable, see Theorem 2.1 and Theorem 3.1 for the exact formulation.

The theorem has various corollaries (Corollary 5.2):

THEOREM B. *Let $S^1 \to E \to B$ be an orientable fibration with $\pi_1 B$ finite and B finitely dominated. Then E is homotopy equivalent to a finite complex.*

This theorem was known in case of a product S^1-bundle or with the assumption that the exact sequence $0 \to \mathbf{Z}/\ker i_* \to \pi_1 E \xrightarrow{f_*} \pi_1 B$ is the reduction of an integral extension $0 \to \mathbf{Z} \to \pi \to \pi_1 B$.

A corollary of Theorem B is the following (Theorem 7.2):

THEOREM C. *Let M be a differentialbe manifold with 0 Euler characteristic and $M \to E \to B$ a bundle with structure group a compact connected Lie group. If $\pi_1 B$ is finite and B is finitely dominated then E is homotopy equivalent to a finite complex.*

Given a bundle of manifolds $S^1 \to E \to B$ various authors have defined transfers $K_{-i}(\mathbf{Z}\pi_1 B) \to K_{-i}(\mathbf{Z}\pi_1 E)$, $\mathrm{Wh}(\pi_1 B) \to \mathrm{Wh}(\pi_1 E)$ using concordances. It is easy to see [14] that these transfers agree with the ones considered here, and thus our computations extend to computations of concordance transfers.

§1. The algebraic S^1-transfer map

Let S be a ring (with unit, not necessarily commutative). Assume given an automorphism $s \to s^t$ of S and an element $\sigma \in S$ with $\sigma^t = \sigma$, $\sigma s = s^t \sigma$, for all $s \in S$. Then the right ideal (σ) generated by σ is a twosided ideal. Let $p: S \to S/(\sigma)$ be the projection. If $M_n(S)$ is the ring of $n \times n$ matrices over S then $s \to s^t$ extends to an automorphism, $A \to A^t$, of $M_n(S)$ with $\sigma A = A^t \sigma$. Also $M_n(S/(\sigma)) = M_n(S)/(\sigma)$.

THEOREM 1.1. *In the above situation there is a well defined homomorphism $p^\#: K_1(S/(\sigma)) \to K_1(S)$ given as follows: If $\bar{A} \in \mathrm{Gl}(n, S/(\sigma))$ then*

$$p^\#([\bar{A}]) = \left[\begin{pmatrix} A & -C \\ \sigma I_n & B^t \end{pmatrix}\right]$$

where $A, B, C \in M_n(S)$ are chosen to satisfy

(i) $p(A) = \bar{A}$
(ii) $AB = I_n - C\sigma$

Moreover,

(iii) $p^{\#}$ *is natural, i.e. if* $f: S \to S'$ *maps* σ *to* σ' *and* $f(s^t) = f(s)^t$ *then*

$$\begin{array}{ccc} K_1(S) & \xrightarrow{f_*} & K_1(S') \\ \uparrow{\scriptstyle p^{\#}} & & \uparrow{\scriptstyle (p')^{\#}} \\ K_1(S/(\sigma)) & \xrightarrow[\bar{f}_*]{} & K_1(S'/(\sigma')) \end{array}$$

commutes

(iv) $p^{\#}p_*(x) = x - x^t$, $x \in K_1(S)$
where $x \to x^t$ *is induced by* $s \to s^t$

(v) *Let* $\bar{s} \to \bar{s}^t = \overline{(s^t)}$ *be the induced automorphism of* $S/(\sigma)$, *and let* $y \to y^t$ *be the induced automorphism on* $K_1(S/(\sigma))$. *Then* $p_*p^{\#}(y) = y - y^t$ *for all* $y \in K_1(S/(\sigma))$.

(vi) *If* σ *is not a zero divisor then* $S/(\sigma)$ *has a free, finitely generated resolution over* S *and the resulting classically defined transfer map* $K_1(S/(\sigma)) \to K_1(S)$ *coincides with* $p^{\#}$.

Warning. Note that $p^{\#}$ depends on the element σ and the automorphism $s \to s^t$, not just the projection p.

Proof. Let $M(A, B, C) = \begin{pmatrix} A & -C \\ \sigma I_n & B^t \end{pmatrix}$. If we have $AB = I_n - C\sigma$ then $BA = I_n - D\sigma$ for suitable D and

$$M(A, B, C)M(-B, -A, D) = \begin{pmatrix} -I_n & -AD \quad +CA^t \\ 0 & -I_n \end{pmatrix}.$$

Hence $M(A, B, C)$ is invertible.

If, for fixed A, B one chooses a different C, say C_1, then $(C - C_1)\sigma = 0$. The identity

$$\begin{pmatrix} I_n & (C - C_1)A^t \\ 0 & I_n \end{pmatrix}\begin{pmatrix} A & -C \\ \sigma I_n & B^t \end{pmatrix} = \begin{pmatrix} A & -C_1 \\ \sigma I_n & B^t \end{pmatrix}$$

shows that $[M(A, B, C)]$ is independent of the choice of C.

Any change in A and B has the form $A_1 = A + A_2\sigma$, $B_1 = B + B_2\sigma$ for arbitrary $A_2, B_2 \in M_n(S)$. It may be compensated for by letting $C_1 = C - A_2B^t - AB_2 - A_2\sigma B_2$. The identity

$$\begin{pmatrix} I_n & A_2 \\ 0 & I_n \end{pmatrix}\begin{pmatrix} A & -C \\ \sigma I_n & B^t \end{pmatrix}\begin{pmatrix} I_n & B_2 \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} A_1 & -C_1 \\ \sigma I_n & B_1^t \end{pmatrix}$$

shows that $[M(A, B, C)]$ is independent of the choice of A and B. If $\bar{E} \in \mathrm{Gl}\,(n, S/(\sigma))$ is an elementary matrix then $\bar{E} = p(E)$ for some elementary matrix $E \in \mathrm{Gl}\,(n, S)$. Also $p(AE) = \overline{AE}$ and $(AE)(E^{-1}B) = I_n - C\sigma$. The identity

$$\begin{pmatrix} I_n & 0 \\ 0 & (E^{-1})^t \end{pmatrix}\begin{pmatrix} A & -C \\ \sigma I_n & B^t \end{pmatrix}\begin{pmatrix} E & 0 \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} AE & -C \\ \sigma I_n & (E^{-1}B)^t \end{pmatrix}$$

shows that $\bar{A}$ and $\overline{AE}$ give rise to the same element in $K_1(S)$. Similarly one treats $\overline{EA}$ and $\bar{A}$.

Finally, it is easy to deal with stabilisation. Altogether we have shown that $p^{\#}: K_1(S/(\sigma)) \to K_1(S)$ is a well defined map.

The homomorphism property is obvious when one uses

$$[\bar{A}_1] + [\bar{A}_2] = \left[\begin{pmatrix} \bar{A}_1 & 0 \\ 0 & \bar{A}_2 \end{pmatrix}\right]$$

to define the sum.

The naturality property is obvious. If $A \in \mathrm{Gl}\,(n, S)$ and $\bar{A} = p(A)$ then one may take $B = A^{-1}$, $C = 0$. Hence the formula in (iv). Finally (v) holds because

$$p_*([M(A, B, C)]) = \left[\begin{pmatrix} \bar{A} & -\bar{C} \\ 0 & (\bar{A}^{-1})^t \end{pmatrix}\right] = [\bar{A}] - [\bar{A}]^t.$$

In case σ is not a zero divisor then $S/(\sigma)$ has $S \xrightarrow{r_\sigma} S \xrightarrow{p} S/(\sigma) \to 0$ as a resolution by left S modules, so p gives rise to a classical transfer map $\mathrm{tr}_p : K_1(S/(\sigma)) \to K_1(S)$, see e.g. Bass [17] p. 451. The commutative diagram

$$\begin{tikzcd}
S^n \oplus S^n \arrow[r, "\begin{pmatrix} \sigma I_n & 0 \\ 0 & I_n \end{pmatrix}"] \arrow[d, "\begin{pmatrix} A^t & -\sigma C \\ I_n & B^t \end{pmatrix}"] & S^n \oplus S^n \arrow[r, "\begin{pmatrix} pI_n \\ 0 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} A & -C \\ \sigma I_n & B^t \end{pmatrix}"] & (S/(\sigma))^n \arrow[d, "\bar{A}"] \\
S^n \oplus S^n \arrow[r, "\begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix}"'] & S^n \oplus S^n \arrow[r, "\begin{pmatrix} pI_n \\ 0 \end{pmatrix}"'] & (S/(\sigma))^n
\end{tikzcd}$$

shows that

$$\mathrm{tr}_p([\bar{A}]) = p^{\#}([\bar{A}]).$$

In fact the identity

$$\begin{pmatrix} I & 0 \\ B^t & I \end{pmatrix}\begin{pmatrix} I & -A^t \\ 0 & I \end{pmatrix}\begin{pmatrix} A^t & -\sigma C \\ I & B^t \end{pmatrix} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

shows that

$$\left[\begin{pmatrix} A^t & -\sigma C \\ I_n & B^t \end{pmatrix}\right]$$

vanishes. (Note: When we deal with left modules we have matrices on the right of row vectors in the description of homomorphisms.)

We proceed to introduce the example we are most interested in.

EXAMPLE 1.2. Let

$$\mathbf{Z} \xrightarrow{i} \pi \xrightarrow{p} \rho \longrightarrow 1 \qquad (\mathscr{E})$$

be an exact sequence of groups with $t = i(1) \in \pi$. Let $\omega : \rho \to \{\pm 1\}$ be a homomorphism with $gtg^{-1} = t^{\omega(p(g))}$ for all $g \in \pi$. Also let R be any commutative ring. By abuse of language we write $p : R\pi \to R\rho$ for the homomorphism of group rings induced by $p : \pi \to \rho$. Now $S = R\pi$ and $\sigma = t - 1$ together with the automorphism given on generators $g \in \pi$, by

$$g^t = \begin{cases} -gt^{-1} & \text{if } \omega(p(g)) = -1 \\ g & \text{if } \omega(p(g)) = 1 \end{cases}$$

is an example of the situation described in Theorem 1.1.

The resulting homomorphism $p^{\#} : K_1(R\rho) \to K_1(R\pi)$ induces (when $R = \mathbf{Z}$) a homomorphism (still called) $p^{\#} : \mathrm{Wh}\,(\rho) \to \mathrm{Wh}\,(\pi)$ which still satisfies (iv) and (v).

In this case the naturality shows that the following diagrams commute

$$\begin{array}{ccc} K_1(R\pi) & \xrightarrow{f_*} & K_1(R'\pi) \\ \uparrow{\scriptstyle p^{\#}} & & \uparrow{\scriptstyle p^{\#}} \\ K_1(R\rho) & \xrightarrow[f_*]{} & K_1(R'\rho) \end{array} \qquad \begin{array}{ccc} K_1(R\pi) & \xrightarrow[\varphi_*]{} & K_1(R\pi_1) \\ \uparrow{\scriptstyle p^{\#}} & & \uparrow{\scriptstyle p^{\#}} \\ K_1(R\rho) & \xrightarrow[\psi_*]{} & K_1(R\rho_1) \end{array}$$

Here $f:R\to R'$ is a ring homomorphism and φ, ψ are group homomorphisms making

$$\begin{array}{ccccccc} \mathbf{Z} & \longrightarrow & \pi & \longrightarrow & \rho & \longrightarrow & 1 \\ \| & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} & & \\ \mathbf{Z} & \longrightarrow & \pi_1 & \longrightarrow & \rho_1 & \longrightarrow & 1 \end{array}$$

commute and having $\omega_1(\psi(g))=\omega(g)$, $g\in\rho$. Note that the element $t=i(1)\in\pi$ can be of finite or infinite order. Call the order k. If $k>2$, then the extension $(\mathscr{E})$ determines ω uniquely. If $k=1$ or 2 then $p^{\#}$ depends also on ω though this fact is suppressed in the notation. The choice of a generator for $\mathbf{Z}$ is immaterial. In fact, replacing t by t^{-1} can be compensated for by changing A to At^{-1}, B to tB and C to $-Ct$. The identity

$$\begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}\begin{pmatrix} At^{-1} & Ct \\ (t^{-1}-1)I_n & (tB)^{t^{-1}} \end{pmatrix}\begin{pmatrix} tI_n & 0 \\ 0 & -t^{-1}I_n \end{pmatrix}=\begin{pmatrix} A & -C \\ (t-1)I_n & B^t \end{pmatrix}$$

shows the desired.

In case $k<\infty$ one may change t to t^r where $(r,k)=1$. This gives a new exact sequence

$$\mathbf{Z} \xrightarrow[ir]{} \pi \xrightarrow[p]{} \rho \longrightarrow 1 \qquad (\mathscr{E}_r)$$

and an S^1-transfer map which we shall call $p^{\#}_{(r)}:K_1(\mathbf{Z}\rho)\to K_1(\mathbf{Z}\pi)$.

§2. Whitehead torsion and linear S^1-bundles

Let $f:E\to B$ be a locally trivial S^1-bundle with structure group O(2) (for short: a linear S^1-bundle) over a connected, finite CW complex B. There results a pair $(\mathscr{E},\omega)$

$$\mathbf{Z}=\pi_1(S^1)\xrightarrow{i}\pi=\pi_1(E,e_0)\xrightarrow{p}\rho=\pi_1(B,b_0)\longrightarrow 1 \qquad (\mathscr{E})$$

$$\omega:\pi_1(B,b_0)=\rho\to\{\pm 1\}=\mathrm{Aut}\,(\pi_1(S^1))$$

where p is induced by f. Thus, from Section 1, we get $p^{\#}:\mathrm{Wh}\,(\rho)\to\mathrm{Wh}\,(\pi)$.

Next, let X be a finite CW complex containing B as a deformation retract, with retraction r, say. Then E is a deformation retract of $Y=r^*(E)$. The pairs (Y, E) and (X, B) have Whitehead torsions $\tau(Y,E)\in \mathrm{Wh}(\pi)$, $\tau(X,B)\in \mathrm{Wh}(\rho)$. We shall prove

THEOREM 2.1. *In the above situation one has*

$$\tau(Y,E)=p^{\#}\tau(X,B).$$

Proof. First consider the case where (X, B) is in simplified form ((4.7) of Cohen, [5]), i.e.

$$X=B\cup\left(\bigcup_{j=1}^{n} e_j^r\right)\cup\left(\bigcup_{j=1}^{n} e_j^{r+1}\right). \tag{2.1}$$

Here we take r to be an *even* integer $>\dim B$. The cells e_j^ν have characteristic maps $\varphi_j^\nu : I^\nu \to X$ with $\varphi_j^r(\partial I^r)=b_0$. There is a commutative diagram

$$\begin{tikzcd}
S^1 \arrow[d] \arrow[r, equal] & S^1 \arrow[d] & \bar S^1 \arrow[l] \arrow[d] & & \\
Y \arrow[d, "f"] & \bar Y \arrow[l] \arrow[d, "\bar f"] & \tilde Y \arrow[l] \arrow[d, "\tilde f"] & & \\
X & \bar X \arrow[l, "q"'] \arrow[r, equal] & \tilde X & & \\
& & & I^\nu \arrow[lllu, "\varphi_j^\nu"] \arrow[lu, "\tilde\varphi_j^\nu"'] \arrow[luu, "\Phi_j^\nu"] \arrow[r, "i_0"'] & \bar S^1\times I^\nu \arrow[lluu, "\varphi_j^\nu"']
\end{tikzcd} \tag{2.2}$$

which we proceed to explain. The horizontal maps are coverings, universal ones being indicated by a $\sim$. The map $\bar f$ is the pull back of f. If $X\to \mathrm{BO}(2)$ classifies f then $\tilde X \xrightarrow[q]{} X \longrightarrow \mathrm{BO}(2)$ deforms into BS^1. Therefore, we can assume that $\bar f : \bar Y \to \bar X$ is a *principal* S^1 bundle. Let $\mu : S^1\times \bar Y \to \bar Y$ be the corresponding action. If, as in Section 1, k is the order of $t=i(1)\in\pi$ then $\tilde Y\to \bar Y$ is a principal $\mathbf{Z}/k\mathbf{Z}$ bundle ($\mathbf{Z}/k\mathbf{Z}=\mathbf{Z}$, if $k=\infty$) The action μ lifts to an action $\bar\mu : \bar S^1\times \tilde Y\to\tilde Y$ where $\bar S^1=\mathbf{R}/k\mathbf{Z}$ ($=\mathbf{R}$ if $k=\infty$). And $\tilde f$ is the corresponding principal $\bar S^1$-bundle. Finally $\tilde\varphi_j^\nu$ and Φ_j^ν lift φ_j^ν, and ψ_j^ν extends Φ_j^ν $\bar S^1$-equivariantly, i_0 being given by $i_0(x)=(0,x)$. Also note that $\tilde E=\tilde Y\mid\tilde B$ is an $\bar S^1$-invariant subspace of $\tilde Y$.

We pick basepoints $\tilde{e}_0 \in \tilde{E} \subseteq \tilde{Y}$ and $\tilde{b}_0 \in \tilde{B} \subseteq \tilde{X}$, compatibly, and above e_0 and b_0. We use them to identify π and ρ with the covering transformation groups on $\tilde{Y}$ and $\tilde{B}$ in the standard way (see e.g. §3 of Cohen, [5]).

We go on to describe the cellular chain complexes $C_*(\tilde{X}, \tilde{B})$ and $C_*(\tilde{Y}, \tilde{E})$.

2.A. *The description of* $C_*(\tilde{X}, \tilde{B})$

The cell e_j^ν lifts to a cell of $\tilde{X}$, say $\tilde{e}_j^\nu$, with characteristic map $\tilde{\varphi}_j^\nu$ (compare (2.1) and (2.2)). The translates under ρ, $g\tilde{e}_j^\nu$, form a cell decomposition of $(\tilde{X}, \tilde{B})$, i.e.

$$\tilde{X} = \tilde{B} \cup \Big(\bigcup_{j,g} g\tilde{e}_j^r\Big) \cup \Big(\bigcup_{j,g} g\tilde{e}_j^{r+1}\Big) \tag{2.3}$$

where the cell $g\tilde{e}_j^\nu$ has characteristic maps $g\tilde{\varphi}_j^\nu : I^\nu \to \tilde{X}$.

Now let $\iota_\nu \in H_\nu(I^\nu, \partial I^\nu)$ be the standard generator. Then $C_*(\tilde{X}, \tilde{B})$ has a $\mathbf{Z}\rho$-basis consisting of

$$x_j^\nu = (\tilde{\varphi}_j^\nu)_*(\iota_\nu) \in C_\nu(\tilde{X}, \tilde{B}), \qquad j = 1, 2, \ldots, n; \qquad \nu = r, r+1. \tag{2.4}$$

Using this basis we see that $C_*(\tilde{X}, \tilde{B})$ takes the form

$$(\mathbf{Z}\rho)^n \xleftarrow[\tilde{A}]{} (\mathbf{Z}\rho)^n, \quad \text{where} \quad \tilde{A} \in \mathrm{Gl}\,(n, \mathbf{Z}\rho).$$

Of course, $\tilde{A}$ is invertible because $\tilde{X}$ deforms to $\tilde{B}$.

2.B. *The cells of* $(\tilde{Y}, \tilde{E})$

Let us fix the following ranges for the various indices used

j and l range over $1, 2, \ldots, n$,
g and h range over ρ,
α and β range over $\mathbf{Z}/k\mathbf{Z}$,
ν ranges over $r, r+1$

Also choose a set map $\sigma : \rho \to \pi$ splitting $p : \pi \to \rho$ and having $\sigma(1) = 1$.

The map $\sigma(g)\psi_j^\nu : \bar{S}^1 \times \mathrm{int}\,(I^\nu) \to \tilde{Y}$ then trivializes the restriction of $\tilde{Y}$ to the cell $g\tilde{e}_j^\nu \subseteq \tilde{X}$. We give $\bar{S}^1$ the obvious cellular structure with vertices α and 1-cells $(\alpha, \alpha+1)$, $\alpha \in \mathbf{Z}/k\mathbf{Z}$. The product structure on $\bar{S}^1 \times \mathrm{int}\,(I^\nu)$ can now be transported into $\tilde{Y}$ by means of $\sigma(g)\psi_j^\nu$. Thus we get the cells

$$e^\nu(j, g, \alpha) = \sigma(g)\psi_j^\nu(\{\alpha\}\times \mathrm{int}\,(I^\nu)) \tag{2.6}$$

$$e^{\nu+1}(j, g, [\alpha, \alpha+1]) = \sigma(g)\psi_j^\nu((\alpha, \alpha+1)\times \mathrm{int}\,(I^\nu)) \tag{2.7}$$

with characteristic maps

$$\eta^\nu(j, g, \alpha) = \sigma(g)\psi_j^\nu(i_\alpha \times I^\nu) : I^\nu \to \tilde{Y} \tag{2.8}$$

$$\eta^{\nu+1}(j, g, [\alpha, \alpha+1]) = \sigma(g)\psi_j^\nu(i_{[\alpha,\alpha+1]}\times I^\nu) : I^{\nu+1} \to \tilde{Y}. \tag{2.9}$$

Here $i_\alpha : * \to \bar{S}^1$ and $i_{[\alpha,\alpha+1]} : I^1 \to \bar{S}^1$ are given by $i_\alpha(*) = \alpha$, $i_{[\alpha,\alpha+1]}(s) = s + \alpha (\in \mathbf{R}/k\mathbf{Z})$. It is clear that the cells of (2.6–7) form a cellular structure on $(\tilde{Y}, \tilde{E})$.

2.C. *The π action on the cells of $(\tilde{Y}, \tilde{E})$*

Recall, e.g. from MacLane [10], that each element of π is uniquely of the form $\sigma(g)t^\alpha$ (g, α ranging as in 2B). And the group structure is given by

$$\sigma(h)\sigma(g) = \sigma(hg)t^{\gamma(h,g)} \tag{2.10}$$

$$t\sigma(g) = \sigma(g)t^{\omega(g)} \tag{2.11}$$

where $\gamma : \rho \times \rho \to \mathbf{Z}/k\mathbf{Z}$ is a 2-cocycle representing the characteristic class of the extension $\mathbf{Z}/k\mathbf{Z} \to \pi \to \rho$.

Also, note that under the action $\bar{\mu} : \bar{S}^1 \times \tilde{Y} \to \tilde{Y}$ one has $\bar{\mu}(1, -) = t : \tilde{Y} \to \tilde{Y}$. It follows that

$$t\psi_j^\nu(s, -) = \psi_j^\nu(s+1, -), \qquad s \in \bar{S}^1 \tag{2.12}$$

and from (2.12) together with (2.10–11) and (2.8–9) one easily computes the action of the general element $\sigma(h)t^\beta \in \pi$ on the cells of $(\tilde{Y}, \tilde{E})$. One gets

$$\sigma(h)t^\beta \cdot \eta^\nu(j, g, \alpha) = \eta^\nu(j, hg, \alpha + \omega(g)\beta + \gamma(h, g)) \tag{2.13}$$

$$\begin{aligned}\sigma(h)t^\beta \cdot \eta^{\nu+1}(j, g, [\alpha, \alpha+1]) = \eta^{\nu+1}(j, hg, [&\alpha + \omega(g)\beta + \gamma(h, g),\\ &\alpha + 1 + \omega(g)\beta + \gamma(h, g)]).\end{aligned} \tag{2.14}$$

Especially

$$\sigma(h)t^{\beta} \cdot \eta^{\nu}(j, 1, 0) = \eta^{\nu}(j, h, \beta) \tag{2.15}$$

$$\sigma(h)t^{\beta} \cdot \eta^{\nu+1}(j, 1, [0, 1]) = \eta^{\nu+1}(j, h, [\beta, \beta+1]). \tag{2.16}$$

Thus there is a $\mathbf{Z}\pi$ basis for $C_*(\tilde{Y}, \tilde{E})$ consisting of the elements

$$y^{\nu}_{j,0} = \eta^{\nu}(j, 1, 0)_*(\iota_{\nu}), \quad \text{and} \tag{2.17}$$

$$y^{\nu+1}_{j,[0,1]} = \eta^{\nu+1}(j, 1, [0, 1])_*(\iota_{\nu+1}) \tag{2.18}$$

2.D. *The boundary in* $C_*(\tilde{Y}, \tilde{E})$

We want to prove

PROPOSITION 2.2. *Using the* $\mathbf{Z}\pi$ *basis given in* (2.17–18) $C_*(\tilde{Y}, \tilde{E})$ *takes the form*

$$(\mathbf{Z}\pi)^n \xleftarrow[\binom{A}{t-1}]{} (\mathbf{Z}\pi)^n \oplus (\mathbf{Z}\pi)^n \xleftarrow[(t-1,\,-A^t)]{} (\mathbf{Z}\pi)^n$$

where $A \in M_n(\mathbf{Z}\pi)$ *has* $p(A) = \bar{A} \in \mathrm{Gl}\,(n, \mathbf{Z}\rho)$ (cfr. (2.5)).

Proof. Let us use superscript $\langle i \rangle$ to denote the i-skeleton of any CW complex. Also, if b_κ is a given R-basis for a free R-module M, and if $m \in M$ let us write $\langle m, b_\kappa \rangle$ for the coordinates of m relative to b_κ (here R will be $\mathbf{Z}$ or $\mathbf{Z}\pi$). The commutative diagram

$$\begin{tikzcd}
H_{r+1}(I^1 \times I^r, \partial(I^1 \times I^r)) \arrow[r, "\eta^{r+1}(j,1,[0,1])_*"] \arrow[d, "\partial"] & H_{r+1}(\tilde{Y}^{\langle r+1 \rangle}, \tilde{Y}^{\langle r \rangle}) = C_{r+1}(\tilde{Y}, \tilde{E}) \arrow[d, "\partial"] \\
H_r(\partial(I^1 \times I^r), (I^1 \times I^r)^{\langle r-1 \rangle} \cup I^1 \times \partial I^r \arrow[r] \arrow[d, "(\mathrm{incl}_*)^{-1}", "\simeq"'] & H_r(\tilde{Y}^{\langle r \rangle}, \tilde{E}) = C_r(\tilde{Y}, \tilde{E}) \\
H_r(\{0, 1\} \times I^r, \{0, 1\} \times \partial I^r) \arrow[ur, "(\eta^r(j,1,0) \cup \eta^r(j,1,1))_*"'] &
\end{tikzcd}$$

together with (2.15) shows that

$$\partial \dot{y}^{r+1}_{j,[0,1]} = (t-1)\dot{y}^{r}_{j,0}. \tag{2.19}$$

Similarly one proves

$$\langle \partial y^{r+2}_{j,[0,1]}, y^{r+1}_{l,0} \rangle = \delta_{jl}(t-1). \tag{2.20}$$

Let $A = (A_{jl}) \in M_n(\mathbf{Z}\pi)$ be given by

$$\partial y^{r+1}_{j,0} = \sum_l A_{jl} y^r_{l,0}. \tag{2.21}$$

Under $\tilde{f}_*$ $y^\nu_{j,0}$ maps to x^ν_j so a comparison with (2.5) reveals that

$$p(A) = \bar{A}. \tag{2.22}$$

Thus all that is now left to prove is

$$\langle \partial y^{r+2}_{j,[0,1]}, y^{r+1}_{l,[0,1]} \rangle = -(A_{jl})^t \in \mathbf{Z}\pi. \tag{2.23}$$

Let

$$A_{jl} = \sum_{h,\beta} a^{h\beta}_{jl} \sigma(h) t^\beta, \qquad a^{h\beta}_{jl} \in \mathbf{Z}. \tag{2.24}$$

Then

$$(A_{jl})^t = \sum\nolimits^+ a^{h\beta}_{jl} \sigma(h) t^\beta - \sum\nolimits^- a^{h\beta}_{jl} \sigma(h) t^{\beta-1} \tag{2.25}$$

where $\sum^\pm$ means the sum over all h, β with $\omega(h) = \pm 1$.

In order to prove (2.23) we *change to integral bases*. For $C_*(\tilde{Y}, \tilde{E})$ an integral basis consists of

$$y^\nu(j, g, \alpha) = \eta^\nu(j, g, \alpha)_*(\iota_\nu), \quad \text{and} \tag{2.26}$$

$$y^{\nu+1}(j, g, [\alpha, \alpha+1]) = \eta^{\nu+1}(j, g, [\alpha, \alpha+1])_*(\iota_{\nu+1}). \tag{2.27}$$

In terms of these the known formula, (2.21), becomes

$$\langle \partial y^{r+1}(j, 1, 0), y^r(l, h, \beta) \rangle = a^{h\beta}_{jl}. \tag{2.28}$$

And the desired one, (2.23), becomes

$$\langle \partial y^{r+2}(j, 1, [0, 1]), y^{r+1}(l, h, \beta)\rangle = \begin{cases} -a_{jl}^{h\beta} & \text{if} \quad \omega(h)=1 \\ a_{jl}^{h,\beta+1} & \text{if} \quad \omega(h)=-1. \end{cases} \tag{2.29}$$

We shall prove (2.29) for the case $\omega(h)=-1$ (the other case being slightly easier). Consider the diagram (2.30).

By (2.28) the left hand vertical composition maps ι_{r+1} to $a_{jl}^{h,\beta+1}\iota_r$. Similarly (2.29) means that the right hand vertical composition maps ι_{r+2} to $a_{jl}^{h,\beta+1}\iota_{r+1}$. Thus, if we can fill in dotted arrows so that the squares commute up to the sign indicated then we are through. The action $\bar{\mu}: \bar{S}^1 \times \tilde{Y} \to \tilde{Y}$ induces a map

$$a:(I^1, \partial I^1)\wedge(\tilde{Y}^{\langle r+1\rangle}, \tilde{Y}^{\langle r\rangle}) \to (\tilde{Y}, \tilde{Y}^{\langle r+1\rangle})$$

where we include $(I^1, \partial I^1)$ into $(\bar{S}^1, \{0, 1\})$ via $i_{[0,1]}$. We put $\Sigma_1(x)=a_*(\iota_1\wedge x)$. Since

$$\begin{array}{ccc} I^1\times I^{r+1} & \xrightarrow{I^1\times\eta^{r+1}(j,1,0)} & I^1\times\tilde{Y} \\ \Big\| & & \Big\downarrow{\scriptstyle\bar{\mu}} \\ I^{r+2} & \xrightarrow[\eta^{r+2}(j,1,[0,1])]{} & \tilde{Y} \end{array}$$

commutes, so does the upper square in (2.30).

$$\begin{array}{ccc} H_{r+1}(I^{r+1}, \partial I^{r+1}) & \xrightarrow{\iota_1\wedge -} & H_{r+2}(I^{r+2}, \partial I^{r+2}) \\ \Big\downarrow{\scriptstyle\eta^{r+1}(j,1,0)_*} & (-1) & \Big\downarrow{\scriptstyle\eta^{r+2}(j,1,[0,1])_*} \\ H_{r+1}(\tilde{Y}^{\langle r+1\rangle}, \tilde{Y}^{\langle r\rangle}) & \xrightarrow{\Sigma_1} & H_{r+2}(\tilde{Y}^{\langle r+2\rangle}, \tilde{Y}^{\langle r+1\rangle}) \\ \Big\downarrow{\scriptstyle\partial} & (-1) & \Big\downarrow{\scriptstyle\partial} \\ H_r(\tilde{Y}^{\langle r\rangle}, \tilde{Y}^{\langle r\rangle}-e^r(l,h,\beta+1)) & \xrightarrow{\Sigma_2} & H_{r+1}(\tilde{Y}^{\langle r+1\rangle}, \tilde{Y}^{\langle r+1\rangle}-e^{r+1}(l,h,[\beta,\beta+1])) \\ \Big\downarrow{\scriptstyle\eta^r(l,h,\beta+1)_*^{-1}} & (-1) & \Big\downarrow{\scriptstyle\eta^{r+1}(l,h,[\beta,\beta+1])_*^{-1}} \\ H_r(I^r, \partial I^r) & \xrightarrow[\iota_1\wedge -]{} & H_{r+1}(I^{r+1}, \partial I^{r+1}) \end{array}$$

Diagram 2.30.

If we let $T_1(s)=1-s$ for $s\in I^1\subset \bar{S}^1$ then

$$\begin{tikzcd} I^1\times\tilde{Y} \arrow[r, "\bar{\mu}"] \arrow[d, "T_1\times\sigma(h)t^{\beta+1}"'] & \tilde{Y} \arrow[d, "t\sigma(h)t^{\beta+1}=\sigma(h)t^{\beta}"] \\ I^1\times\tilde{Y} \arrow[r, "\bar{\mu}"'] & \tilde{Y} \end{tikzcd}$$

commutes. In fact

$$\begin{aligned}\bar{\mu}(1-s,\sigma(h)t^{\beta+1}x)&=\sigma(h)t^{\beta+1}\bar{\mu}(s-1,x)\\ &=\sigma(h)t^{\beta}\bar{\mu}(s,x).\end{aligned}$$

Hence

$$\begin{tikzcd} I^1\times I^r \arrow[r, phantom, "="] \arrow[d, "T_1\times\eta^r(l,h,\beta+1)"'] & I^{r+1} \arrow[d, "\eta^{r+1}(l,h,[\beta,\beta+1])"] \\ I^1\times\tilde{Y} \arrow[r, "\bar{\mu}"'] & \tilde{Y} \end{tikzcd} \tag{2.31}$$

also commutes. Consequently $\bar{\mu}$ induces a map

$$b:(I^1,\partial I^1)\wedge(\tilde{Y}^{\langle r\rangle},\tilde{Y}^{\langle r\rangle}-e^r(l,h,\beta+1))\to(\tilde{Y}^{\langle r+1\rangle},\tilde{Y}^{\langle r+1\rangle}-e^{r+1}(l,h,[\beta,\beta+1])).$$

We let $\Sigma_2(x)=b_*(\iota_1\wedge x)$. Then the middle square in (2.30) commutes up to the sign -1 because

$$\partial(\iota_1\wedge x)=-\iota_1\wedge\partial x\in H_{r+1}(\tilde{Y}^{\langle r+1\rangle},\tilde{Y}^{\langle r+1\rangle}-e^{r+1}(l,h,[\beta,\beta+1])).$$

And the bottom square in (2.30) commutes up to the sign -1 because of (2.31). This finishes the proof of Proposition 2.2.

2.E. *Completion of the proof of Theorem* 2.1

As in Section 1 we may choose $B, C\in M_n(\mathbf{Z}\pi)$ with $AB=I_n-C(t-1)$. Then $(t-1,-A^t)$ is split by $\begin{pmatrix} C\\ -B^t\end{pmatrix}$. Thus (from the definitions in §§15, 19 of Cohen,

[5]) we see that

$$\tau(Y, E) = \left[\begin{pmatrix} A & C \\ t-1 & -B^t \end{pmatrix}\right], \qquad \tau(B, X) = [\bar{A}].$$

This finishes the proof in the special case considered so far.

Using the special case repeatedly we see that

In the situation of Theorem 2.1, if X is a CW expansion of B then $\tau(Y, E) = 0.$ (2.32)

Also, if the pairs (X, B) and (X_1, B) have equal torsions then X and X_1 both expand to a common expansion of B. Therefore, arguing as in Anderson's proof of 2.1 of [1], we see that

For fixed $f: E \to B$ $\tau(Y, E)$ depends only on $\tau(X, B)$. (2.33)

Since it is known (see e.g. (7.4) of Cohen's book [5]) that any $\tau \in \mathrm{Wh}(\rho)$ is realized by a pair in simplified form the proof is now complete.

§3. Relation to Anderson's geometric transfer

In this section we consider PL fibre bundles $f: E \to B$ with fiber S^1, as defined by Anderson [2]. Anderson, [1] proves that any such gives rise to a map $f^*: \mathrm{Wh}(\pi_1(B)) \to \mathrm{Wh}(\pi_1(E))$ – we shall call it a geometric transfer map – such that

For any homotopy equivalence $h: B_1 \to B$ (B_1 a finite polyhedron) one has $f^*\tau((h)) = \tau(\bar{h})$ where $\bar{h}: h^*E \to E$ covers h. (3.1)

In this context we also have $p = f_*: \pi_1(E) \to \pi_1(B)$ and the algebraic S^1-transfer map $p^{\#}: \mathrm{Wh}(\pi_1(B)) \to \mathrm{Wh}(\pi_1(E))$. As a corollary of Theorem 2.1 we get

THEOREM 3.1. *For any* PL *fiber bundle $f: E \to B$ with fiber S^1 one has*

$$f^* = p^{\#}: \mathrm{Wh}(\pi_1(B)) \to \mathrm{Wh}(\pi_1(E)).$$

In fact, Pedersen, [15], proves that f^* depends only on the exact sequence

$$\mathbf{Z} = \pi_1(S^1) \longrightarrow \pi_1(E) \xrightarrow{p} \pi_1(B) \longrightarrow 1 \qquad (\mathscr{E})$$

and the orientation map

$$\omega : \pi_1(B) \to \{\pm 1\}.$$

Now any pair $(\mathscr{E}, \omega)$ can be realized by a PL fiber bundle (fiber S^1) with structure group O (2). Thus we may assume that Theorem 2.1 applies to $f: E \to B$. Also, any $\tau \in \mathrm{Wh}(\pi_1(B))$ has the form $\tau(X, B)$ for some $X \supseteq B$, and if $i: B \to X$, $r: X \to B$ are the inclusion and retraction respectively then $\tau = \tau(X, B) = i_*^{-1}\tau(i) = r_*\tau(i) = -\tau(r)$.

Thus

$$f^*\tau = -f^*\tau(r) = -\tau(\bar{r}) = \tau(Y, E) = p^{\#}\tau(X, B) = p^{\#}\tau.$$

Remark 3.2. Instead of comparing $p^{\#}$ with Anderson's f^* one might also redo Anderson's work, [1], in the context of locally trivial S^1-bundles $f: E \to B$ over finite CW complexes. One proves that any such gives a well defined map $f^*: \mathrm{Wh}(\pi_1(B)) \to \mathrm{Wh}(\pi_1(E))$, and that f^* depends only on the fundamental group sequence and the orientation map (as in Pedersen, [15]). The above argument then proves that $f^* = p^{\#}$ also in such a context.

§4. The Bass–Heller–Swan homomorphism

Let ρ be a group. In [3], Bass, Heller and Swan define a split epimorphism $\varphi: K_1(\mathbf{Z}(\rho \times \mathbf{Z})) \to K_0(\mathbf{Z}\rho)$. The splitting $h: K_0(\mathbf{Z}\rho) \to K_1(\mathbf{Z}(\rho \times \mathbf{Z}))$ is given by $h([P]) = [P[s, s^{-1}], l_s]$. Here s is the generator of $\mathbf{Z}$, P is any finitely generated projective $\mathbf{Z}\rho$ module, $P[s, s^{-1}]$ its extension to a $\mathbf{Z}(\rho \times \mathbf{Z}) = \mathbf{Z}(\rho)[s, s^{-1}]$ module, and l_s is left multiplication by s. It is easily seen that there are "reduced editions" forming a commutative diagram

$$\begin{tikzcd} \tilde{K}_0(\mathbf{Z}\rho) \arrow[r, "h"] \arrow[dr, "\mathrm{id}"'] & \mathrm{Wh}(\rho \times \mathbf{Z}) \arrow[d, "\varphi"] \\ & \tilde{K}_0(\mathbf{Z}\rho) \end{tikzcd}$$

In this section we shall give the following homotopy theoretic interpretation of φ.

THEOREM 4.1. *Let X be a finite, connected* CW *complex with fundamental group ρ. Any $\tau \in \mathrm{Wh}(\rho \times \mathbf{Z})$ is the Whitehead torsion, $\tau(g)$, of some map $g: Y \to$*

$X \times S^1$ *where*

(i) Y *is a connected, finite* CW *complex with* $Z = g^{-1}(X \times 1)$ *a connected subcomplex*

(ii) $g \mid Z : Z \to X \times 1$ *induces an isomorphism on fundamental groups.*

For such a g *let*

$$\begin{tikzcd} \bar{Y} \arrow[r, "g"] \arrow[d, "q"'] & X \times \mathbf{R} \arrow[d, "X \times \exp"] \\ Y \arrow[r, "g"'] & X \times S^1 \end{tikzcd}$$

be a pull-back, and put $\bar{Y}_+ = \bar{g}^{-1}(X \times [0, \infty))$. *Then*

(iii) $\bar{g}_+ : \bar{Y}_+ \to X \times [0, \infty) \to X$ *induces an isomorphism* $\bar{g}_{+*} : \pi_1(\bar{Y}_+) \to \rho$,

(iv) $\bar{Y}_+$ *is finitely dominated with finiteness obstruction given by* $\bar{g}_{+*}(w(\bar{Y}_+)) = \varphi(\tau) \in \tilde{K}_0(\mathbf{Z}\rho)$.

Proof. Certainly $\tau = \tau(g)$ for some homotopy equivalence $g : Y \to X \times S^1$ where Y is a finite complex. Replacing Y by a regular neighborhood of Y in some $\mathbf{R}^N$ we have Y a compact manifold. Then make $\mathrm{prog} : Y \to X \times S^1 \to S^1$ smooth and transverse to 1 to ensure that $Z = g^{-1}(X \times 1)$ is a submanifold. Finally do "ambient surgery" to ensure that Z be connected and that (ii) holds, and triangulate the pair (X, Y) to obtain (i).

It follows easily from (i) and (ii) that $\bar{Y}_+$ and $\bar{Y}_- = \bar{g}^{-1}(X \times (-\infty, 0])$ are subcomplexes with $g : \bar{Y}_+ \cap \bar{Y}_- \to Z$ a homeomorphism. An application of van Kampens theorem then gives the following pushout diagram of groups

$$\begin{tikzcd} & \pi_1(\bar{Y}_+) \arrow[dr, "j_+"] & \\ \pi_1(\bar{Y}_+ \cap \bar{Y}_-) = \rho \arrow[ur, "i_+"] \arrow[dr, "i_-"'] & & \rho = \pi_1(\bar{Y}). \\ & \pi_1(\bar{Y}_-) \arrow[ur, "j_-"'] & \end{tikzcd}$$

Here $j_+ i_+ = j_- i_- = \mathrm{id}$ with the identification indicated. Since this implies that $\pi_1(\bar{Y}_\pm)$ is a semi direct product $K_\pm \times_{\alpha_\pm} \rho$ it follows that the inclusion $\rho \to (K_+^* K_-) \times_{\alpha_+ * \alpha_-} \rho$ is an isomorphism. Thus $K_\pm = \{1\}$ and $j_+ : \pi_1(\bar{Y}_+) \to \rho$ is an isomorphism. This proves (iii).

To prove (iv) we consider the cellular complexes of the universal coverings of $X\times\mathbf{R}$ and $\bar{Y}$ as well as their restrictions to $X\times[0,\infty)$ and $\bar{Y}_+$. They fit into a diagram with short exact rows

$$\begin{tikzcd} C_*(\tilde{X}\times[0,\infty)) \arrow[r] \arrow[d] & M(\gamma_+) \arrow[r] \arrow[d] & C_{*-1}(\bar{Y}_+) \arrow[d] \\ C_*(\tilde{X}\times\mathbf{R}) \arrow[r] & M(\gamma) \arrow[r] & C_{*-1}(\tilde{Y}). \end{tikzcd}$$

Here $M(\gamma)$ is the mapping cone of $\gamma=\tilde{g}_*:C_*(\tilde{Y})\to C_*(\tilde{X}\times\mathbf{R})$, and $M(\gamma_+)$ arises similarly. By definition τ is the Whitehead torsion, $\tau(M(\gamma))$, of the acyclic simplex $M(\gamma)$ with its $\mathbf{Z}(\rho\times\mathbf{Z})$ basis coming from the cells of Y and $X\times S^1$ lifted to $\tilde{Y}$ and $\tilde{X}\times\mathbf{R}$. Also $X\times[0,\infty)$ has the homotopy type of X so $C_*(\tilde{X}\times[0,\infty))$ is equivalent (over $\mathbf{Z}\rho$) to a finite free chain complex. Hence it suffices to prove that $M(\gamma_+)$ is dominated (over $\mathbf{Z}\rho$) by a finite, free chain complex, and that its finiteness obstruction is $\varphi(\tau)$.

Now we may choose the lifted cells in $\tilde{Y}$ and $\widetilde{X\times S^1}=\tilde{X}\times\mathbf{R}$ so that:

If $b_{i,1},\ldots,b_{i,k(i)}$ is the preferred basis for $M_i(\gamma)$ over $\mathbf{Z}(\rho\times\mathbf{Z})$ and $s\in\mathbf{Z}$ is the generator for $\mathbf{Z}$ (written multiplicatively) then

$$\{b_{ij}s^k \mid j=1,2,\ldots,k(i);\ k=0,1,2,\ldots\} \tag{4.1}$$

is a basis for $M_i(\gamma_+)$ over $\mathbf{Z}(\rho)$.

In fact, in each $\rho\times\mathbf{Z}$ orbit of cells in $\tilde{Y}$ or $X\times S^1$ one chooses a cell e so that $e\subseteq\tilde{Y}$ or $\tilde{X}\times[0,\infty)$ but $s^{-1}e$ is not. After this observation the following proposition finishes the proof of Theorem 4.1.

PROPOSITION 4.2. *Let*

$$D=\left(D_n \xrightarrow{d_n} D_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} D_0\right)$$

be an acyclic chain complex of finitely based $\mathbf{Z}(\rho\times\mathbf{Z})$ modules. Let $b_i=\{b_{i,1},\ldots,b_{i,k(i)}\}$ be a preferred basis for D_i and let $D_i^{(+)}$ be the $\mathbf{Z}(\rho)$ module generated by all $b_{i,j}s^k$, $(j=1,2,\ldots,k(i);\ k=0,1,2,\ldots)$. Choose an integer N so big that $s^Nd_i(D_i^{(+)})\subseteq D_{i-1}^{(+)}$ for all i, and let

$$D^{(N)}=\left(D_n^{(+)} \xrightarrow{s^Nd_n} D_{n-1}^{(+)} \longrightarrow \cdots \xrightarrow{s^Nd_1} D_0^{(+)}\right)$$

be the resulting chain complex of $\mathbf{Z}\rho$ *modules. Then* $D^{(N)}$ *is finitely dominated with finiteness obstruction* $w(D^{(N)}) = \varphi(\tau(D))$ *in* $\tilde{K}_0(\mathbf{Z}\rho)$.

Proof. We use induction on n. For $n=1$ Bass, Heller and Swan show that $\operatorname{Cok}(s^N d_1)$ is a projective over $\mathbf{Z}\rho$, and they define φ by letting $\varphi(\tau(D)) = [\operatorname{Cok}(s^N d_1)]$. Since $D^{(N)}$ is equivalent to the trivial complex $\operatorname{Cok}(s^N d_1)$ we have the desired conclusion. Let $n>1$ and write E for the chain complex $D_0 \xrightarrow{1} D_0$ concentrated in degrees 1 and 0, F for $D_0 \xrightarrow{-1} D_0$ in degrees 2 and 1. Also choose a splitting σ of $d_1 : D_1 \to D_0$ and let

$$\bar{D} = \left(D_n \xrightarrow{\quad d_n \quad} \cdots \xrightarrow{\binom{d_3}{0}} D_2 \oplus D_0 \xrightarrow{\begin{pmatrix} d_2 & \sigma \\ 0 & -1 \end{pmatrix}} D_1 \oplus D_0 \xrightarrow{(d_1, 1)} D_0 \right)$$

$$\hat{D} = \left(D_n \xrightarrow{\quad d_n \quad} \cdots \xrightarrow{\binom{d_3}{0}} D_2 \oplus D_0 \xrightarrow{(d_2, \sigma)} D_1 \right).$$

The obvious short sequences

$$0 \to D \to \bar{D} \to F \to 0, \qquad 0 \to E \to \bar{D} \to \hat{D} \to 0$$

(which are compatibly based over $\mathbf{Z}(\rho \times \mathbf{Z})$) show that

$$\tau(D) = \tau(\bar{D}) = \tau(\hat{D}).$$

Also, for N large enough, we have short exact sequences

$$0 \to D^{(N)} \to \bar{D}^{(N)} \to F^{(N)} \to 0, \qquad 0 \to E^{(N)} \to \bar{D}^{(N)} \to \hat{D}^{(N)} \to 0$$

of $\mathbf{Z}(\rho)$ chain complexes. Now $F^{(N)}$ and $E^{(N)}$ are finitely dominated with vanishing finiteness obstruction. Also, by the inductive hypothesis, $\hat{D}^{(N)}$ is finitely dominated with $w(\hat{D}^{(N)}) = \varphi\tau(\hat{D})$. It follows that $\bar{D}$, and then D, are finitely dominated with $w(D^{(N)}) = w(\bar{D}^{(N)}) = w(\hat{D}^{(N)})$, which finishes the proof.

§5. The $\tilde{K}_0$- and Wh transfer maps related by the Bass–Heller–Swan homorphism

Let $f : E \to B$ be a Hurewicz fibration with fibre S^1 and connected, finitely dominated base B. Let

$$\mathbf{Z} = \pi_1(S^1) \to \pi = \pi_1(E) \to \rho = \pi_1(B) \to 1 \qquad (\mathscr{E})$$

$$\omega : \pi_1(B) \to \{\pm 1\}$$

be the associated pair as in Section 1. In [6], Ehrlich showed that f gives rise to a homomorphism $f^*: \tilde{K}_0(\mathbf{Z}\rho) \to \tilde{K}_0(\mathbf{Z}\pi)$ taking $w(B)$ to $w(E)$. And in [15], Pedersen proved that f^* depends only on the pair $(\mathscr{E}, \omega)$. We shall give the following algebraic description of f^*.

THEOREM 5.1. *For any Hurewicz fibration as above, the diagram commutes.*

$$\begin{tikzcd} \tilde{K}_0(\mathbf{Z}\pi) & \mathrm{Wh}\,(\pi \times \mathbf{Z}) \arrow[l, "\varphi"'] \\ \tilde{K}_0(\mathbf{Z}\rho) \arrow[u, "f^*"] \arrow[r, "h"'] & \mathrm{Wh}\,(\rho \times \mathbf{Z}) \arrow[u, "(f^* \times 1)^\#"'] \end{tikzcd}$$

COROLLARY 5.2. *Let $f: E \to B$ be an orientable S^1-fibration with B finitely dominated and connected. If $\pi_1(B)$ is finite then E has the homotopy type of a finite complex.*

Proof. If $\pi_1(E)$ is infinite, then $\ker(p) = \mathbf{Z}$ so $\mathscr{E}$ is pseudoabelian, in the sense of Ehrlich, [7]. Hence the result follows from Ehlich, [7] (or see Munkholm and Pedersen, [12]). Compare also with the proof of Proposition 6.1.

Now let $\pi = \pi_1(E)$ be finite. Then an easy application of Milnor's Mayer–Victoris sequence shows that $p_*: \tilde{K}_0(\mathbf{Z}\pi) \to \tilde{K}_0(\mathbf{Z}\rho)$ is onto. Thus the general element of $\tilde{K}_0(\mathbf{Z}\rho)$ has the form $[p_*P]$ for some $\mathbf{Z}\pi$ module P which admits a stable inverse Q, i.e. $P \oplus Q \cong (\mathbf{Z}\pi)^n$. Now

$$\begin{aligned} h([p_*P]) &= [(p_*P)[s, s^{-1}], l_s] \\ &= (p \times 1)_*[P[s, s^{-1}], l_s]. \end{aligned}$$

But by (iv) of Theorem 1.1 $(p \times 1)^\#(p \times 1)_* = 0$ in the orientable case. It follows that $f^* = 0$ as claimed.

Proof of Theorem 5.1. Since the given data $(\mathscr{E}, \omega)$ can be realized by a PL fibre bundle, and since f^* depends only on $(\mathscr{E}, \omega)$ we may assume that f *is* a PL fibre bundle. By Theorem 3.1 it then suffices to show that

$$\begin{tikzcd} \tilde{K}_0(\mathbf{Z}\pi) & \mathrm{Wh}\,(\pi \times \mathbf{Z}) \arrow[l, "\varphi"'] \\ \tilde{K}_0(\mathbf{Z}\rho) \arrow[u, "f^*"] & \mathrm{Wh}\,(\rho \times \mathbf{Z}) \arrow[u, "f^*"'] \arrow[l, "\varphi"] \end{tikzcd}$$

the righthand vertical arrow is associated with the bundle crossed with S^1

commutes. And that is an easy consequence of the description of φ given in Theorem 4.1.

§6. Some computations

We have so far been unable to answer the following

Basic question. Does there exist an orientable extension $\mathbf{Z} \to \pi \xrightarrow{p} \rho \to 1$ (with ρ finitely presented, say) for which $0 \neq p^{\#}: \mathrm{Wh}(\rho) \to \mathrm{Wh}(\pi)$?

For the unoriented case the examples of Pedersen and Taylor, [16], together with Theorem 5.1 shows that one can have $(p \times 1)^{\#} \neq 0$. Since essentially nothing is known about Wh (ρ) when ρ is infinite we have concentrated on the case when ρ is finite.

PROPOSITION 6.1. *If* $\mathbf{Z} \to \pi \xrightarrow{p} \rho \to 1$ *is orientable and* π *is infinite,* ρ *is finite, then* $p^{\#} = 0: K_1(\mathbf{Z}\rho) \to K_1(\mathbf{Z}\pi)$.

Proof. If $|\rho| = m$ then there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{Z} \arrow[r] \arrow[d, "m"] & \pi \arrow[r, "p"] \arrow[d, "i"] & \rho \arrow[r] \arrow[d, equal] & 1 \\
0 \arrow[r] & \mathbf{Z} \arrow[r] & \rho \times \mathbf{Z} \arrow[r, "p_1"'] & \rho \arrow[r] & 1
\end{tikzcd}$$

where m is multiplication by m and i is an inclusion. One may realize this diagram by

$$\begin{tikzcd}
S^1 \arrow[r] \arrow[d] & E \arrow[r, "f"] \arrow[d] & B \arrow[d, equal] \\
S^1/(\mathbf{Z}/m\mathbf{Z}) \arrow[r] & E/(\mathbf{Z}/m\mathbf{Z}) \arrow[r, "f_1"'] & B
\end{tikzcd}$$

where the map f is a PL fibre bundle with fibre S^1, hence a principal S^1-bundle, and the $\mathbf{Z}/m\mathbf{Z}$-action comes from the inclusion $\mathbf{Z}/m\mathbf{Z} \to S^1$. It then follows that $f^* = i^* f_1^*$, i.e. – in view of Theorem 3.1 – $p^{\#} = i^* p_1^{\#}$. Since p_{1*} is onto a reference to (iv) in Theorem 1.1 finishes the proof.

For the rest of the paragraph assume that π is finite. Recall that $Cl_1(\mathbf{Z}\pi)$ is defined to be the kernel of

$$SK_1(\mathbf{Z}\pi) \to \bigoplus_q SK_1(\hat{\mathbf{Z}}_q\pi) \hookrightarrow \bigoplus_q K_1(\hat{\mathbf{Z}}_q\pi)$$

where $\hat{\mathbf{Z}}_q$ denotes the q-adic completion of $\mathbf{Z}$ (see Oliver, [13]).

PROPOSITION 6.2. *If $\mathbf{Z} \to \pi \to \rho \to 1$ is orientable, and π is finite then $p^{\#}(K_1(\mathbf{Z}\rho)) \subseteq Cl_1(\mathbf{Z}\pi)$.*

Proof. Since $Q\pi \to Q\rho$ splits as a map of $Q\pi$ modules, $p^{\#} = 0 : K_1(Q\rho) \to K_1(Q\pi)$. Hence, from Theorem 1.1 $p^{\#}(K_1(\mathbf{Z}\rho)) \subseteq SK_1(\mathbf{Z}\pi)$ $(= \operatorname{Ker}(K_1(\mathbf{Z}\pi) \to K_1(Q\pi)))$. Similarly $p_* : K_1(\hat{\mathbf{Z}}_q\pi) \to K_1(\hat{\mathbf{Z}}_q\rho)$ is onto so the same argument finishes the proof.

Now $Cl_1(\mathbf{Z}\pi)$ is pretty well understood when π is abelian, so one might hope to detect nontriviality of $p^{\#}$ by projecting into an abelian group. However one has

PROPOSITION 6.3. *Let $\mathbf{Z} \to \pi \to \rho \to 1$ be orientable and π finite. If $h : \pi \to \pi_1$ is a homomorphism into an abelian group then $h_* p^{\#} = 0 : K_1(\mathbf{Z}\rho) \to K_1(\mathbf{Z}\pi_1)$.*

Proof. Clearly we can assume that h is onto. Then, by the naturality in Theorem 1.1(iii), we can assume that π is abelian and h is the identity. But any such extension is the reduction of an integral extension so we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{Z} \arrow[r, "i_1"] \arrow[d, equal] & \pi' \arrow[r, "p_1"] \arrow[d, "\varphi"] & \rho \arrow[r] \arrow[d, equal] & 0 \\
 & \mathbf{Z} \arrow[r, "i"'] & \pi \arrow[r, "p"'] & \rho \arrow[r] & 0
\end{tikzcd}$$

of abelian groups and with exact rows, and we only have to refer to Theorem 1.1(iii) and Proposition 6.1.

§7. Examples

In this section we infer consequences for the geometrically defined transfers in bundles with other fibres than S^1. We remind the reader of some facts from [15]: Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a bundle (structure group Homeo (F)) with B and F finite

CW-complexes. Anderson [1] then gives a homomorphism $f^*: \mathrm{Wh}(\pi_1 B) \to \mathrm{Wh}(\pi_1 E)$ that relates Whitehead torsion at total and basespace level. By the fundamental group data of the bundle we understand the exact sequence $\pi_1 F \to \pi_1 E \to \pi_1 B$ and the orientation homomorphism $\pi_1 E \to \pi_0$ (Homeo $(F, *)$) (Homeo $(F, *)$ = basepoint preserving homeomorphisms). The existence of universal examples with given fundamental group data [15] implies that f^* above only depends on F and the fundamental group data. This means that we compute f^* in general if we can compute f^* for enough examples of bundles with F as fibre. The results of [15] also tell us how many examples we need to compute with a given fundamental group ρ in the base and a given orientation homomorphism $\rho \to \pi_0(\mathrm{Homeo}(F))$: Let Homeo $(F) \to F$ be the evaluation map and $G_1^{\mathrm{Top}}(F) \subset \pi_1(F)$ the image of the induced map. If it is possible to realize some fundamental group data $\pi_1 F \to \pi \to \rho$ compatible with the above orientation homomorphism and with $A = \ker(\pi_1 F \to \pi)$, then all such fundamental group data are classified by $H^2(\rho; G_1(F)/A)$ (local coefficients) in the sense that this group acts (transitively and faithfully) on the set of realizable fundamental group data. Also the image in $H^2(\rho; C(\pi_1(F)/A))$ gives the corresponding action on the exact sequence [10]. So if we are considering an orientable case the image in $H^2(\rho; C(\pi_1(F)/A))$ will be the characteristic class of the extension $\pi_1(F)/A \to \pi \to \rho$. We now restrict ourselves to orientable bundles. (Note: this does not imply that $\pi_1 E \to \pi_0(\mathrm{Homeo}(F, *))$ is trivial.) We use these observations to compute some examples:

Let $L = L(m; a_1, \ldots, a_n)$ be a $2n-1$ dimensional lens space given as $S^{2n-1} = \{(z_1, \ldots, z_n) \in \mathbf{C}^n \mid \sum |z_i|^2 = 1\}$ divided out by $\mathbf{Z}/m\mathbf{Z}$ thought of as m'th roots of unity and $\zeta \in \mathbf{Z}/m\mathbf{Z}$ acting as $\zeta(z_1, \ldots, z_n) = (\zeta^{a_1} z_1, \ldots, \zeta^{a_n} z_n)$ (the a_i's are relatively prime and prime to m). S^1 acts on L by the formulae

$$z \cdot [(z_1, \ldots, z_n)] = [(\xi^{a_1} z_1, \ldots, \xi^{a_n} z_n)] \quad \text{where} \quad \xi^m = z.$$

We let $s \in \pi_1(L)$ be the element obtained by letting S^1 act on the basepoint. Let $L \to E \to B$ be an orientable bundle with fundamental group data $\mathbf{Z}/m\mathbf{Z} \xrightarrow{i} \pi \xrightarrow{p} \rho \to 1$. Consider the exact sequence

$$\mathbf{Z} \to \pi \to \rho \to 1$$

where $\mathbf{Z} \to \pi$ sends 1 to $i(s)$. We then have homomorphisms $p^{\#}_{(b)}: \mathrm{Wh}(\rho) \to$

Wh (π) defined for any b prime to the order of $i(s)$ (compare with the end of Section 1).

THEOREM 7.1. *The geometric* Wh-*transfer f^* associated with the bundle $L \to E \to B$ is given by*

$$f^* = \sum p^{\#}_{(b_i)}$$

where $a_i b_i \equiv 1 \pmod{m}$.

Proof. $G_1^{\mathrm{Top}}(L) = \pi_1(L)$ since we have exhibited an S^1-action on L which applied to the basepoint gives a generator of $\pi_1(L)$. Thus the transfer is completely determined by the exact sequence $\mathbf{Z}/m\mathbf{Z} \to \pi \to \rho \to 1$ and we need to construct appropriate examples. Let $\mathbf{Z} \to \pi \to \rho \to 1$ be the exact sequence above and let $S^1 \to X \to B$ be a principal S^1-bundle realizing this fundamental group data. (This is always possible by [12] or [15].) Clearly $L \to X \times_{S^1} L \to B$ realizes the fundamental group data $\mathbf{Z}/m\mathbf{Z} \to \pi \to \rho$. (Notice we have carefully chosen a generator of $\pi_1(L)$ thus exhibiting an isomorphism to $\mathbf{Z}/m\mathbf{Z}$.) We prove the theorem by induction, using what amounts to an S^1-CW structure on L. Let K be an S^1-equivariant regular neighborhood of $L(m; a_1, \ldots, a_{n-1}) \subset L(m; a_1, \ldots, a_n)$. Then $X \times_{S^1} L \to B$ is the union of two bundles $A = X \times_{S^1} K$ and $C = X \times_{S^1} S^1 \times D^{2n-2}$ (the S^1-action on $S^1 \times D^{2n-2}$ given by $z \cdot (z_n, d) = (z^{a_n} \cdot z_n, d)$, intersecting in a bundle $H = X \times_{S^1} S^1 \times S^{2n-3}$. Siebenmanns sum formulae for Whitehead torsion (see e.g. [5]) implies in obvious notation that

$$f^* = f_A^* + f_C^* - f_H^*$$

(the sums computed in Wh (π) through the natural maps). However the S^1-equivariant map

$$S^1 \times S^{2n-3} \subset S^1 \times D^{2n-2} \to S^1$$

induces a bundle

$$X \times_{S^1} S^1 \times S^{2n-3} \to X \times_{S^1} S^1$$

with fibre S^{2n-3}, so by [11] f_H^* is 0 (because $2n-3$ is odd. Had we considered the attaching of an even S^1-cell we would get $f_H^* = 2f_C^*$ and thus f_C^* would be subtracted leading one to think of an S^1-CW-complex Euler characteristic).

To complete the proof we only need to show $f^*_C = p^{\#}_{(b_n)}$.
Since $X \times_{S^1} S^1 = X/(\mathbf{Z}/a_n\mathbf{Z})$ we get a diagram

$$\begin{array}{ccccc} S^1 & \longrightarrow & S^1/(\mathbf{Z}/a_n\mathbf{Z}) = S^1 & \longrightarrow & L \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & X/(\mathbf{Z}/a_n\mathbf{Z}) & \longrightarrow & X \times_{S^1} L \\ \downarrow & & \downarrow & & \downarrow \\ B & \xrightarrow{1} & B & \longrightarrow & B \end{array}$$

with a corresponding diagram of fundamental groups

$$\begin{array}{ccccc} \mathbf{Z} & \xrightarrow{a_n} & \mathbf{Z} & \longrightarrow & \mathbf{Z}/m\mathbf{Z} \\ \downarrow & & \downarrow & & \downarrow \\ \pi & \longrightarrow & \pi_1(E/(\mathbf{Z}/a_n\mathbf{Z})) & \longrightarrow & \pi \\ \downarrow & & \downarrow & & \downarrow \\ \rho & \longrightarrow & \rho & \longrightarrow & \rho \end{array}$$

The composite $\mathbf{Z} \xrightarrow{a_n} \mathbf{Z} \longrightarrow \mathbf{Z}/m\mathbf{Z}$ sends 1 to 1 so $\mathbf{Z} \longrightarrow \mathbf{Z}/m\mathbf{Z}$ must send 1 to b_n where $a_n \cdot b_n \equiv 1 \pmod m$. It follows by naturality of the algebraic S^1-transfer that $f^*_C = p^{\#}_{(b_n)}$ thus ending the induction step and the proof.

We now consider $F = S^3/Q(8)$, a simple example with nonabelian fundamental group. Let $F \to E \to B$ be an orientable bundle with fundamental group sequence $Q(8) \to \pi \to \rho \to 1$. If the kernel $Q(8) \to \pi$ is nontrivial the sequence is split and it is shown e.g. in [15] that $f^*: \mathrm{Wh}(\rho) \to \mathrm{Wh}(\pi)$ is trivial. Otherwise since $\mathbf{Z}/2\mathbf{Z}$ is the center of $Q(8)$ we may establish a pushout diagram

$$\begin{array}{ccccc} \mathbf{ZZ}/2\mathbf{ZZ} & \longrightarrow & \bar{\pi} & \longrightarrow & \rho \\ \downarrow & & \downarrow & & \| \\ Q(8) & \longrightarrow & \pi & \longrightarrow & \rho \end{array} \qquad (7.1)$$

Let $h_s: \mathbf{Z}/4\mathbf{Z} \to Q(8)$, $s = i, j, k$ be the inclusions sending the generator to i, j and k respectively and consider the pushout diagram

$$\begin{array}{ccccc} \mathbf{Z}/2\mathbf{Z} & \longrightarrow & \bar{\pi} & \longrightarrow & \rho \\ \downarrow{\scriptstyle 2} & & \downarrow & & \| \\ \mathbf{Z}/4\mathbf{Z} & \longrightarrow & \pi' & \xrightarrow{q} & \rho \\ \downarrow{\scriptstyle h_s} & & \downarrow{\scriptstyle \bar{h}_s} & & \| \\ Q(8) & \longrightarrow & \pi & \longrightarrow & \rho \end{array} \qquad (7.2)$$

Let $a:\mathrm{Wh}(\rho)\to\mathrm{Wh}(\pi)$ be the composite $\mathrm{Wh}(\rho)\xrightarrow{p^{\#}}\mathrm{Wh}(\bar{\pi})\to\mathrm{Wh}(\pi')$ and $b_s:\mathrm{Wh}(\rho)\to\mathrm{Wh}(\pi)$, $s=i,j,k$ the composite $\mathrm{Wh}(\rho)\xrightarrow{q^{\#}}\mathrm{Wh}(\pi')\xrightarrow{(\bar{h}_s)_*}\mathrm{Wh}(\pi)$.

THEOREM 7.2. *The geometric* Wh-*transfer associated with the bundle* $S^3/Q(8)\to E\xrightarrow{f} B$ *is given by*

$$f^*=2b_i+2b_j+2b_k-4a$$

whenever $Q(8)\to\pi$ *is monic.*

Remark. Note that the diagram

$$\begin{tikzcd}
\mathrm{Wh}(\rho) \arrow[rr, "q^{\#}"] \arrow[dr, "p^{\#}"'] & & \mathrm{Wh}(\pi'), \arrow[dl, "\mathrm{Res}"] \\
& \mathrm{Wh}(\bar{\pi}) &
\end{tikzcd}$$

where Res is the restriction map, is commutative, so to prove $f^*=0$ in a specific case it would suffice to prove $q^{\#}=0$.

Proof of Theorem 7.2. We let S^1 act on the left of $S^3/Q(8)$ by $z\cdot[z_1,z_2]=[\xi z_1,\xi z_2]$ where $\xi^2=z$. Letting S^1 act on the base-point gives the central element of $\pi_1(S^3/Q(8))$. The action has 6 singular orbits of the type $\mathbf{Z}/2\mathbf{Z}$ and the induced action of $S^1/(\mathbf{Z}/2\mathbf{Z})$ at the singular orbits define elements of $\pi_1(S^3/Q(8))$ up to conjugacy. We get i,j and k respectively at 2 points each. We have an S^1-CW structure on $S^3/Q(8)$ with 6 0-cells, 12 1-cells, and 8 2-cells, all 1 and 2-cells with 0 isotropy subgroup. We construct a principal S^1-bundle $S^1\to X\to B$ with fundamental group data $\mathbf{Z}\to\mathbf{Z}/2\mathbf{Z}\to\bar{\pi}\to\rho$ and consider

$$\begin{tikzcd}
S^1 \arrow[r] \arrow[d] & S^3/Q(8) \arrow[d] \\
X \arrow[r] \arrow[d] & X\times_{S^1} S^3/Q(8) \arrow[d] \\
B \arrow[r] & B
\end{tikzcd}$$

The right hand bundle has the right fundamental group data and can thus be used to compute f^*. The decomposition of $S^3/Q(8)$ as an S^1-CW complex gives a decomposition of the bundle in S^1-bundles which we sum using Siebenmanns sum formulae. The 0-cells give diagrams

$$\begin{tikzcd}
S^1 \arrow[r] \arrow[d] & S^1 \arrow[r] \arrow[d] & S^3/Q(8) \arrow[d] \\
X \arrow[r] \arrow[d] & X \times_{S^1} S^1 = X/(\mathbf{Z}/2\mathbf{Z}) \arrow[r] \arrow[d] & X \times_{S^1} S^3/Q(8) \arrow[d] \\
B \arrow[r, equal] & B \arrow[r, equal] & B
\end{tikzcd}$$

with fundamental group diagram (7.2) which gives the contribution $2b_i + 2b_j + 2b_k$ to f^*. The contributions from the 1-cells are subtracted (see proof of Theorem 7.2) and the 2-cells added to give the result.

It has been essential to these two examples that the center of the fundamental group was "picked up" by an S^1-action so therefore has to be cyclic. To remedy this restriction slightly we consider $F = T^n$, the n-torus. Let $T^n \to E \to B$ be an orientable bundle with fundamental group data $\mathbf{Z}^n \to \pi \to \rho$. Consider $\pi_i = \operatorname{Cok}(\mathbf{Z}^i \to \pi)$ where $\mathbf{Z}^i \subset \mathbf{Z}^n$ as the first i factors. We get exact sequences $\mathbf{Z} \to \pi_{i-1} \xrightarrow{p_i} \pi_i$.

THEOREM 7.3. *The* Wh-*transfer* $f^*: \mathrm{Wh}(\rho) \to \mathrm{Wh}(\pi)$ *associated with the bundle* $T^n \to E \to B$ *is the composition*

$$\mathrm{Wh}(\rho) = \mathrm{Wh}(\pi_n) \xrightarrow{p_1^{\#} \circ p_2^{\#} \circ \cdots \circ p_n^{\#}} \mathrm{Wh}(\pi_0) = \mathrm{Wh}(\pi).$$

Proof. We may replace $T^n \to E \to B$ by a principal T^n-bundle $T^n \to X \to Y$ with the same fundamental group data and use this to compute f^*. The sequence of S^1-bundles

$$S^1 \to X/T^i \to X/T^{i+1}$$

now finishes the proof by referring to the main theorem.

If G is a compact connected Lie group and F a G-CW complex it is clear from the above that we can compute the geometric Wh-transfer for any bundle $F \to E \to B$ with G as structure group, since the inclusion of the maximal torus $T \subset G$ induces an epimorphism of fundamental groups and we thus may produce a bundle with the same fundamental group data and T as structure group, and then may proceed as in Theorems 7.1 and 7.2. This general result we prefer not to state since it uses the actual T-cellular structure of F and thus gets a very complicated form. In case of the K_0-transfer and finite fundamental group of the base however we know the geometric S^1-transfer is 0 so in the inductive argument we keep adding zeroes to obtain e.g.

THEOREM 7.4. *Let M be a differentiable manifold with 0 Euler characteristic and $M \to E \to B$ a smooth bundle with structure group a compact connected Lie group. If $\pi_1 B$ is finite and B is finitely dominated then E is homotopy equivalent to a finite complex.*

Remark. In case $\chi(M) \neq 0$ the K_0-transfer has been computed by Ehrlich [7].

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Dept. of Mathematics
Odense University
Campusvy 55
DK-5230 Odense

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