On co-H-spaces.

Autor(en): **Hilton, Peter / Mislin, G. / Roitberg, J.**

Objekttyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **53 (1978)**

PDF erstellt am: **31.05.2024**

Persistenter Link: https://doi.org/10.5169/seals-40751

Comment. Math. Helvetici 53 (1978) 1–14 Birkhäuser Verlag, Basel

On co-H-spaces

Peter Hilton, Guido Mislin and Joseph Roitberg

Dedicated to Beno Eckmann on the Occasion of his Sixtieth Birthday

1. Introduction

This paper is concerned with two aspects of the theory of co-H-spaces, which we interrelate in our final result. First, let W be a finite connected complex and let X be a nilpotent space of finite type. It was proved in [HMR] that if W is a suspension or a 1-connected co-H-space, then there exists a cofinite set of primes Q such that the rationalizing map $X_S \to X_0$ induces an injection of homotopy sets $[W, X_S] \to [W, X_0]$, provided that $S \subseteq Q$. We show in Section 3 that the class $\mathcal{W}$ of finite connected complexes W for which this conclusion holds is much broader than the result above indicates, and indeed that it properly contains all finite connected co-H-spaces.

A set M with a binary operation, written additively, is called a *loop* if it admits a two-sided zero, and if the equations

$$a + x = b, \qquad y + a = b$$

have unique solutions x, y in M for all a, b in M. We show in Section 2 that if W is any connected co-H-space and X is any nilpotent space, then the co-H-structure $\mu : W \to W \vee W$ induces a loop structure in $[W, X]$, which is, of course, natural with respect to X. We use this fact, together with Theorem 2.4 of [HMR], to obtain the results referred to above.

Sections 4–6 are concerned with the *Ganea conjecture* (Problem 10 of [G1]) that a connected co-H-space Y is of the homotopy type of $Z \vee B$, where Z is 1-connected and B is a bunch of circles. Considerable progress was made in this direction by Berstein and Dror [BD], who showed that the result was true if the co-H-structure on Y is (homotopy) co-associative. In fact, they proved more and, with regard to connected co-H-spaces Y, they established the following. With any such Y we may associate the classifying map $u : Y \to B$ for the universal cover $\tilde{Y}$, where B is a bunch of circles, and then the Ganea conjecture holds if, for any space A, the binary operation in the set $[Y, A]$ induced by the co-H-structure in

Y satisfies

$$(r+su)+tu=r+(su+tu),\ r:Y\to A,\ s,t:B\to A. \tag{1.1}$$

They describe the condition (1.1) by saying that B *co-operates co-associatively* on Y. Of course, if Y is a co-associative co-H-space then $[Y,A]$ is associative, so (1.1) certainly holds.

We show in Section 4 that the Ganea conjecture holds if Y is a *coloop*;[1] we give the explicit definition of a coloop in Section 2, but, in fact, coloops Y are characterized by the property that $[Y,A]$ is a loop for all spaces A. Then the next two sections are devoted to obtaining a common generalization of the Berstein–Dror condition and the coloop condition. We show that with every connected space X with free fundamental group we may associate a canonical idempotent e, characterized by the property that $\pi_1 e=1$ and $\tilde{e}:\tilde{X}\to\tilde{X}$ is nullhomotopic, where $\tilde{X}$ is the universal cover of X (indeed e is characterized by weaker properties). In the pointed homotopy category any idempotent splits; that is, we have a space $\operatorname{im} e$ and maps

$$p_e:X\to \operatorname{im} e,\ i_e:\operatorname{im} e\to X,\quad \text{with}\quad i_e p_e=e,\qquad p_e i_e=1. \tag{1.2}$$

Then Theorem 6.1 gives conditions under which an idempotent e splits a connected co-H-space Y in the sense that $Y\simeq Z\vee \operatorname{im} e$, for some space Z. We obtain our generalization by showing that these conditions are satisfied by the canonical idempotent e if the equation $x+e=a$ in $[Y,Y]$ has a unique solution for all a in $[Y,Y]$, and that then $\operatorname{im} e=B$ and Z is 1-connected. It is immediate that e has this property if Y is a coloop, or, more generally, if $[Y,Y]$ is a loop; and we adapt arguments of [BD] to show that e has this property if the Berstein–Dror condition is satisfied. Finally we bring together the two parts of the paper to show that the only connected but not 1-connected nilpotent co-H-space is S^1. It is interesting to remark that none of the proper localizations of S^1 can be co-H-spaces; but, of course, the localizations of 1-connected co-H-spaces are again co-H-spaces.

We frequently confuse maps and homotopy classes in what follows (as also in (1.2)); however, we remind the reader, in the text, of this convention.

2. The loop $[W,X]$

Let W be a connected co-H-space with structure map $\mu:W\to W\vee W$. Then μ induces in the set $[W,X]$, for any space X, a binary operation, +, natural in X,

[1] Of course, conversely, if $Y\simeq Z\vee B$, with Z 1-connected and B a bunch of circles, then Y admits a coloop structure.

with 2-sided zero, the class of the constant map. We will show that $[W, X]$ is, in fact, a loop if X is nilpotent.

PROPOSITION 2.1. *The map* $\phi_1 = (1 \vee \nabla) \circ (\mu \vee 1): W \vee W \to W \vee W$ *is a homology equivalence. So, too, is the map* $\phi_2 = (\nabla \vee 1) \circ (1 \vee \mu)$.

Proof. Now, if $n > 0$, $H_n(W \vee W) = H_n W \oplus H_n W$ and $\phi_{1*}: H_n(W \vee W) \to H_n(W \vee W)$ is given by $\phi_{1*}(a, b) = (a, a+b)$, $a, b \in H_n W$. Similarly, $\phi_{2*}(a, b) = (a+b, b)$.

We say that (W, μ) or, simply, W is a (homotopy) *coloop* if ϕ_1 and ϕ_2 are homotopy equivalences.

COROLLARY 2.2. *If W is* 1-*connected, then* ϕ_1, ϕ_2 *are homotopy equivalences, and so W is a coloop.*

THEOREM 2.3. *Let W be a connected co-H-space and X a space. Then* $[W, X]$ *is a loop provided that* (*i*) *W is* 1-*connected, or* (*ii*) *X is nilpotent.*

Proof. Note first that $[W_1 \vee W_2, X] = [W_1, X] \times [W_2, X]$. Then it is easy to see that ϕ_1 induces

$$\phi_1^*: [W, X] \times [W, X] \to [W, X] \times [W, X], \quad \text{given by } \phi_1^*(\alpha, \beta) = (\alpha + \beta, \beta), \tag{2.1}$$

while ϕ_2 induces

$$\phi_2^*: [W, X] \times [W, X] \to [W, X] \times [W, X], \quad \text{given by } \phi_2^*(\alpha, \beta) = (\alpha, \alpha + \beta). \tag{2.2}$$

Thus $[W, X]$ is a loop precisely when ϕ_1^* and ϕ_2^* are bijective. Now if W is 1-connected, ϕ_1, ϕ_2 are homotopy equivalences, so ϕ_1^*, ϕ_2^* are bijective; and if X is nilpotent then any homology equivalence $\phi: A \to B$ induces a bijection $\phi^*: [B, X] \to [A, X]$ (Dror's Theorem).

Remarks. (i) It is known that there are connected spaces W (we may even take $W = S^1 \vee S^1$, according to M. G. Barratt[2]) which admit co-H-structures

---

[2] Indeed, it is not difficult to see that if π is the free group on generators x, y, then the comultiplication $\pi \to \pi * \pi$, given by

$$x \mapsto x'x'', \; y \mapsto y'y''[y', x'']$$

admits no left inverse. Here we write a', a'' for the element $a \in \pi$ regarded as an element of the first, second copy of π in $\pi * \pi$, respectively.

which are not coloop structures. It is not known whether there are connected spaces W which admit co-H-structures but admit no coloop structures.

(ii) The significant fact we use in Theorem 2.3 (ii) is that a nilpotent space is $H_*(-;\mathbf{Z})$-local in the sense of Bousfield [B]. Thus, of course, the conclusion of Theorem 2.3 holds if X is $H_*(-;\mathbf{Z})$-local.

(iii) If W is a nilpotent connected co-H-space, then $[W, W]$ is a loop. This does not immediately guarantee that the co-H-structure on W is a coloop structure. However, as indicated in the Introduction, this is an essential step in our proof below (Theorem 6.7) that W is then either 1-connected or S^1. It is easy to see that any co-H-structure on S^1 admits a 2-sided co-inverse, but this condition is apparently weaker than that of being a coloop.

3. Injectivity of $[W, X_S] \to [W, X_0]$

In this section we study the family $\mathcal{W}$ of finite connected complexes W such that, for all nilpotent X of finite type, there exists a cofinite set of primes Q such that the rationalizing map $r: X_S \to X_0$ induces an injection of homotopy sets

$$r_*: [W, X_S] \rightarrowtail [W, X_0] \quad \text{for all} \quad S \subseteq Q. \tag{3.1}$$

We know, from [HMR], (a) that there are finite connected complexes not in $\mathcal{W}$ and (b) that, if we replaced, in (3.1), the requirement of injectivity by that of *weak* injectivity (that is, $r_*^{-1}(0)=0$), then all finite connected complexes would have the given property. Naturally we will exploit observation (b) in studying the family $\mathcal{W}$, reinforcing it with the following elementary proposition.

PROPOSITION 3.1. *A loop-homomorphism is injective if it is weakly injective.*

We now proceed to the study of $\mathcal{W}$, as a subfamily of the family of all finite connected complexes.

PROPOSITION 3.2. *If W is a co-H-space, then $W \in \mathcal{W}$.*

Proof. This follows from Theorem 2.3(ii), observation (b), and Proposition 3.1.

PROPOSITION 3.3. *If W is a 1-connected rational co-H-space, then $W \in \mathcal{W}$.*

Proof. We are given that W is 1-connected and that there is a co-H-structure $\mu_0: W_0 \to W_0 \vee W_0$. If j_0 embeds $W_0 \vee W_0$ in $W_0 \times W_0$, then $j_0\mu_0 \simeq \Delta_0$, the diagonal map. Consider the map

$$\mu_0 r: W \to W_0 \vee W_0,$$

where $r: W \to W_0$ rationalizes. By Theorem 2.10 of [HMR], we know that there exists a cofinite set of primes Q_1 such that $\mu_0 r$ and $j_0\mu_0 r$ lift uniquely into $W_R \vee W_R$ and $W_R \times W_R$ respectively, for all $R \subseteq Q_1$. Let $\bar{\mu}: W \to W_R \vee W_R$ be the lift of $\mu_0 r$ and let $\bar{\mu}$ induce $\mu_R: W_R \to W_R \vee W_R$. Then it is clear that μ_R is a co-H-structure on W_R.

Now choose Q_2 cofinite, so that $[W, X_S] \to [W, X_0]$ is weakly injective for all $S \subseteq Q_2$ and let $Q = Q_1 \cap Q_2$. We have the commutative diagram, for $S \subseteq Q$,

$$\begin{array}{ccc} [W, X_S] & \xrightarrow{r_*} & [W, X_0] \\ \uparrow{\scriptstyle e^*} & & \uparrow{\scriptstyle e^*} \\ [W_{Q_1}, X_S] & \xrightarrow{r_{**}} & [W_{Q_1}, X_0] \end{array}$$

where $e: W \to W_{Q_1}$ Q_1-localizes. Then each e^* is bijective and r_* is weakly injective. It follows that r_{**} is weakly injective; but r_{**} is a loop-homomorphism by Theorem 2.3, so that r_{**} is injective; so, too, therefore is r_*.

PROPOSITION 3.4. *If $W_1, W_2 \in \mathcal{W}$, so does $W_1 \vee W_2$.*

Proof. This follows immediately from the relation $[W_1 \vee W_2, X] = [W_1, X] \times [W_2, X]$.

PROPOSITION 3.5. *Let $f: W \to W'$ be a map of finite connected complexes inducing a rational homology isomorphism. Then if one of W, W' belongs to $\mathcal{W}$, so does the other.*

Proof. Since W, W' are finite and $f_*: H_*(W; \mathbf{Q}) \cong H_*(W'; \mathbf{Q})$, it follows that there exists a cofinite Q_1 such that, if G is a Q_1-local abelian group,

$$f_*: H_*(W; G) \cong H_*(W'; G). \tag{3.2}$$

Now suppose that there exists a cofinite Q_2 such that $r_*: [W, X_S] \rightarrowtail [W, X_0]$ for $S \subseteq Q_2$ and let $Q = Q_1 \cap Q_2$. Since X_S, X_0 are $H_*(-; \mathbf{Z}_{Q_1})$-local if $S \subseteq Q_1$, it follows that, in the diagram below, with $S \subseteq Q$,

$$\begin{array}{ccc} [W, X_S] & \overset{r_*}{\rightarrowtail} & [W, X_0] \\ \uparrow{\scriptstyle f^*} & & \uparrow{\scriptstyle f^*} \\ [W', X_S] & \xrightarrow{r_*'} & [W', X_0] \end{array},$$

the vertical arrows f^* are bijective. Thus r_*' is also injective. In very similar fashion we infer that, if $W' \in \mathcal{W}$, then so does W.

COROLLARY 3.6. *Let $W \sim W'$ be the equivalence relation generated by the relation $W \rightsquigarrow W'$ which asserts the existence of a rational homology equivalence from W to W'. Then if one of W, W' belongs to $\mathcal{W}$, so does the other.*

From Propositions 3.2, 3.3, 3.4 and Corollary 3.6 we infer

THEOREM 3.7. *Let $W \sim A \vee B$, where A is a (finite) co-H-space and B is a (finite) 1-connected rational co-H-space. Then $W \in \mathcal{W}$.*

Remark. Suppose that W is a (finite) nilpotent rational co-H-space. Then W_0 is (by definition) a co-H-space, so that W_0 is 1-connected (its fundamental group is free and 0-local). Thus $\pi_1 W$ is a finite nilpotent group operating nilpotently on the homology groups of $\tilde{W}$, the universal cover of W. It is easy to see that $\tilde{W}$ is a finite 1-connected rational co-H-space, since $\pi_1 W$ is finite and $\tilde{W}_0 = W_0$. Thus the covering map $\tilde{W} \to W$ is a rational homology equivalence and so $W \in \mathcal{W}$ by Theorem 3.7. We immediately infer that $\mathcal{W}$ contains more than just the co-H-spaces; for example it contains the real projective spaces P^n, for n odd. Of course, we could have inferred from Proposition 3.3 that $\mathcal{W}$ contains spaces which are not co-H-spaces; thus, $\mathcal{W}$ contains $S^3 \cup_\alpha e^7$, where α generates $\pi_6(S^3)$.

4. Co-H-maps

Let (X, μ), (Y, μ) be co-H-spaces and let $f: X \to Y$ be a map making the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu} \\ X \vee X & \xrightarrow{f \vee f} & Y \vee Y \end{array} \tag{4.1}$$

homotopy-commutative. We then say that f is a *co-H-map*. Let

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{\Sigma f} \cdots \tag{4.2}$$

be the Puppe sequence of f. We prove

PROPOSITION 4.1. *Let $f: X \to Y$ be a co-H-map. Then, in (4.2), we may give Z the structure of a co-H-space in such a way that g is a co-H-map. If, further, f is corectractile,*[3] *Y is a coloop, and Z is 1-connected, then the co-H-structure on Z is determined by the requirement that g be a co-H-map.*

Proof. We will deliberately confuse maps and homotopy classes; thus we will write equality in place of the homotopy relation. We consider the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"] & Y \arrow[r, "g"] \arrow[d, "u"] & Z \\ X \vee X \arrow[r, "f\vee f"] \arrow[d, "i"] & Y \vee Y \arrow[r, "g \vee g"] \arrow[d, "i"] & Z \vee Z \arrow[d, "i"] \\ X \times X \arrow[r, "f\times f"] & Y \times Y \arrow[r, "g\times g"] & Z \times Z \end{tikzcd}$$

where $j\mu = \Delta$. Since $(g\vee g)\mu f = (g \vee g)(f\vee f)\mu = (gf \vee gf)\mu = 0$, there exists $\bar{\mu}: Z \to Z \vee Z$ with $\bar{\mu} g = (g \vee g)\mu$. Consider Δ, $j\bar{\mu}: Z \to Z \times Z$. Then $\Delta g = (g \times g)\Delta = (g\times g)j\mu = j(g \vee g)\mu = j\bar{\mu} g$. Now the group $[\sum X, A]$ operates on the set $[Z, A]$, for any space A, and the relation $\Delta g = j\bar{\mu} g$ guarantees $s \in [\sum X, Z \times Z]$, such that $\Delta = (j\bar{\mu})^s$. But $j_*: [\sum X, Z \vee Z] \to [\sum X, Z\times Z]$ is surjective, so that there exists $r \in [\sum X, Z \vee Z]$ with $jr = s$. Set $\mu = \bar{\mu}^r$. Then $\Delta = (j\bar{\mu})^{jr} = j(\bar{\mu}^r) = j\mu$. Moreover $\mu g = \bar{\mu} g$, so that μ is a co-H-structure on Z with respect to which g is a co-H-map.

If f is coretractile, that is, if $\sum f$ has a left inverse, then it is obvious from (4.2) that $h = 0$. Thus $[Z, A] \xrightarrow{g^*} [Y, A]$ is weakly injective. However if we use the co-H-structure μ on Z of the first part, then g is a co-H-map, so that g^* is a homomorphism. But since Z is 1-connected, (Z, μ) is a coloop (Corollary 2.2), so that g^* is a weakly injective homomorphism of loops and therefore (Proposition 3.1) injective. It follows that $\mu: Z \to Z \vee Z$ is uniquely determined by μg, and hence by the relation $\mu g = (g \vee g)\mu$.

Remark. Note that nowhere in the argument do we require that $\mu: X \to X \vee X$ be a co-H-structure.

THEOREM 4.2. *Let $f: X \to Y$ be a co-H-map of coloops with the mapping cone Z 1-connected. If f has a left inverse then $Y \simeq Z \vee X$.*

Proof. Let $uf = 1$ and consider the equation $t + fu = 1$ in $[Y, Y]$. Then $f = (t + fu)f = tf + fuf$, since f is a co-H-map, $= tf + f$. Since X is a coloop, $tf = 0$, so that $t = vg$, $v: Z \to Y$. Consider the maps

$$Y \underset{\langle v, f\rangle}{\overset{i_Z g + i_X u}{\rightleftarrows}} Z \vee X, \quad \text{where} \quad i_Z, i_X \quad \text{embed} \quad Z, X \quad \text{in} \quad Z \vee X.$$

[3] In the sense of James; that is, $\sum f$ has a left (homotopy) inverse.

We will prove that these maps are mutual (homotopy) inverses. First,

$$\langle v, f\rangle(i_Z g+i_X u)=\langle v, f\rangle i_Z g+\langle v, f\rangle i_X u=vg+fu=1.$$

Next $(i_Z g+i_X u)f=i_Z gf+i_X uf$, since f is a co-H-map, $=i_X$, so it remains to show that $(i_Z g+i_X u)v=i_Z$. To see this, observe that

$$\begin{aligned} i_Z g+i_X u &= (i_Z g+i_X u)(vg+fu)=(i_Z g+i_X u)vg+(i_Z g+i_X u)fu \\ &= (i_Z g+i_X u)vg+i_X u. \end{aligned}$$

Since Y is a coloop, we infer that

$$i_Z g=(i_Z g+i_X u)vg.$$

But we saw in the proof of the second part of Proposition 4.1 that g is an epimorphism, so that $i_Z=(i_Z g+i_X u)v$, as required.

Remark. In fact, g has the right inverse v; for $g=g(vg+fu)=gvg$ and g is an epimorphism.

We may illustrate Theorem 4.2 as follows. Let Y be a connected coloop. There is then a bunch of circles B and a map $f:B\to Y$ inducing an isomorphism of fundamental groups. Moreover, since B is an Eilenberg–MacLane space, there is plainly a map $u:Y\to B$ inducing f_*^{-1} on π_1 and $uf=1$. There will be a unique map $\mu:B\to B\vee B$ realizing $(f\vee f)_*^{-1}\mu_* f_*$ on π_1 and this μ will be a co-H-structure on B such that f is a co-H-map. Since $(u\vee u)\mu$, $\mu u:Y\to B\vee B$ induce the same homomorphism of π_1, they are homotopic, so that u is also a co-H-map. It follows that f embeds B as a retract of Y so that B is also a coloop. Thus, if Z is the mapping cone of f, then Z is 1-connected, and we conclude

COROLLARY 4.3. *Let Y be a connected coloop. Then $Y\simeq Z\vee B$, where Z is a 1-connected co-H-space and B is a bunch of circles.*

We now proceed to generalize Corollary 4.3; our generalization will also comprehend the Berstein–Dror condition.

5. Homotopy idempotents

DEFINITION 5.1. Let $d:X\to X$ be an idempotent homotopy class (i.e., $d^2\simeq d$). Then we define the *image* of d by

$$\operatorname{im} d=\xrightarrow{\text{ho lim}}(X\xrightarrow{d}X\xrightarrow{d}X\longrightarrow\cdots).$$

We continue to 'confuse' maps and homotopy classes. Then the diagram

$$\begin{tikzcd}
X \arrow[r, "1"] \arrow[d, "d"] & X \arrow[r, "1"] \arrow[d, "d"] & \cdots \\
X \arrow[r, "d"] \arrow[d, "d"] & X \arrow[r, "d"] \arrow[d, "d"] & \cdots \\
X \arrow[r, "1"] & X \arrow[r, "1"] & \cdots
\end{tikzcd}$$

gives rise to maps $p = p_d : X \to \operatorname{im} d$, $i = i_d : \operatorname{im} d \to X$ such that $d = ip$. We call this the *canonical factorization* of d. We assume henceforth that X is connected.

LEMMA 5.1. $pi = 1 : \operatorname{im} d \to \operatorname{im} d$.

Proof. The space $\operatorname{im} d$ represents the functor $A \mapsto \operatorname{im}([A, X] \xrightarrow{d_*} [A, X])$, from the category of all connected pointed complexes to sets and this functor satisfies the Brown axioms. Explicitly,

$$[A, \operatorname{im} d] = d_*[A, X]. \tag{5.1}$$

Thus $i_* : [A, \operatorname{im} d] \to [A, X]$ corresponds in (5.1) to the embedding of $d_*[A, X]$ in $[A, X]$, and $p_* : [A, X] \to [A, \operatorname{im} d]$ corresponds to $d_* : [A, X] \to d_*[A, X]$. But d_* is the identity on $d_*[A, X]$ so that $p_* i_* = 1$, whence $pi = 1$.

LEMMA 5.2. $\pi_n \operatorname{im} d \cong \operatorname{im} \pi_n d$, $H_n \operatorname{im} d \cong \operatorname{im} H_n d$.

Proof. The first result follows immediately from (5.1); the second follows from the fact that homology commutes with direct limits.

Let H stand for reduced homology; recall that X is connected. We then have

PROPOSITION 5.3. *Let $d, e : X \to X$ be idempotents such that $Hd + He = 1 : HX \to HX$. Then*

$$\{Hp_d, Hp_e\} : HX \cong H(\operatorname{im} d) \oplus H(\operatorname{im} e), \qquad \langle Hi_d, Hi_e \rangle : H(\operatorname{im} \operatorname{im} e) \cong HX.$$

Proof. Since $Hd + He = 1$ it follows that Hd, He are orthogonal idempotents of HX. The result now follows immediately from Lemma 5.2.

We come now to one of our principal results on co-H-spaces.

THEOREM 5.4. *Let X be a connected space with free fundamental group. Then there exists a unique idempotent $e:X\to X$ such that $\pi_1 e=1$ and $\tilde{e}=0:\tilde{X}\to\tilde{X}$. Moreover $p_e:X\to \operatorname{im} e$ is the classifying map for $\tilde{X}$ and e is a co-H-map if X is a co-H-space.*

Proof. We argue as in Section 4 that there is a bunch of circles B and there are maps $f:B\to X$, $u:X\to B$ such that $uf=1$. We set $e=fu:X\to X$. Plainly, from Lemma 5.2, $\pi_n(\operatorname{im} e)\cong\pi_n B$, $n\geq 1$, so that $e=fu$ is the canonical factorization of e, that is, $B=\operatorname{im} e$, $p_e=u$, $i_e=f$. Certainly $u:X\to B$ is the classifying map for $\tilde{X}$. Since $\tilde{e}$ factors through $\tilde{B}$ it is plain that $\tilde{e}=0$. We again refer to the argument in Section 4 showing that, if X is a co-H-space, then B may be given the structure of a co-H-space such that f, u are co-H-maps; this shows that e will be a co-H-map.

Now let $e':X\to X$ be an idempotent such that $\pi_1 e'=1$ and $\tilde{e}'=0$. Then certainly $\operatorname{im} e'$ is a $K(\pi_1X, 1)$ and, since π_1X is free, there will be a homotopy equivalence $\theta:K(\pi_1X, 1)\to B$ giving rise to a (homotopy) commutative diagram

$$\begin{tikzcd}
 & K(\pi_1X, 1) \arrow[dr, "i_{e'}"] \arrow[dd, "\theta"] & \\
X \arrow[ur, "p_{e'}"] \arrow[dr, "u"'] & & X \\
 & B \arrow[ur, "f"'] &
\end{tikzcd}$$

proving the theorem.

6. The main theorem

We come now to the promised generalization of Corollary 4.3. We first need a definition.

DEFINITION 6.1. Let M denote a set with a binary operation, written additively. We say that $e\in M$ is *loop-like on the right* if the equation $x+e=a$ has a unique solution x in M for each $a\in M$.

We now state the main theorem.

THEOREM 6.1. *Let Y be a connected co-H-space and $e:Y\to Y$ an idempotent co-H-map. If e is loop-like on the right (in the set $[Y, Y]$ with binary operation induced by the co-H-structure in in Y), then there exists a unique idempotent $d:Y\to Y$ such that $d+e=1$ and*

$$Y\simeq \operatorname{im} d\vee \operatorname{im} e.$$

We first need a topological and an algebraic lemma. These lemmas, together with Lemma 6.4, are modelled on arguments in [BD].

LEMMA 6.2. *Let Y be a connected co-H-space and k a field. Then $H_n(\tilde{Y}; k)$ is a free $k[\pi_1 Y]$-module, $n \geq 1$, and*

$$H_n(Y; k) = H_n(\tilde{Y}; k) \otimes_{\pi_1 Y} k, \qquad n > 1. \tag{6.1}$$

Proof. Ganea [G2] has shown that Y is a retract of $V = \sum \Omega Y$. By Corollary 4.3 or [BD], $\sum \Omega Y \simeq Z \vee B$ with Z 1-connected and B a bunch of circles, so that, according to Lemma 1.11 of [BD], $H_n(\tilde{V}; k), n \geq 1$, is a free $k[\pi_1 V]$-module. Since Y is a retract of V, $\pi_1 Y$ is a retract of $\pi_1 V$, and so $H_n(\tilde{V}; k)$, $n \geq 1$, is a free $k[\pi_1 Y]$-module. Now $H_n(\tilde{Y}; k)$ is a $\pi_1 Y$-retract of $H_n(\tilde{V}; k)$, so that $H_n(\tilde{Y}; k), n \geq 1$, is a projective $k[\pi_1 Y]$-module. But $\pi_1 Y$ is free, so that, by a result of Cohn–Seshadri, $H_n(\tilde{Y}; k)$ is a free $k[\pi_1 Y]$-module. The relation (6.1) now follows by appealing to the Cartan–Leray spectral sequence of the universal covering $\tilde{Y} \to Y \to K(\pi_1 Y, 1)$, using the fact that $H_n(\tilde{Y}; k)$ is a free $k[\pi_1 Y]$-module, $n \geq 1$, to infer that

$$E^2_{pq} = 0, \qquad p > 0, \quad \text{unless} \quad p = 1, \qquad q = 0;$$
$$E^2_{oq} = H_q(\tilde{Y}; k) \otimes_{\pi_1 Y} k, \qquad q > 0.$$

LEMMA 6.3. *Let π denote a free group and k a field. Let F be a free $k[\pi]$-module and let $e: F \to F$ be an idempotent such that the induced idempotent $\bar{e}: F \otimes_\pi k \to F \otimes_\pi k$ is the identity. Then e is the identity.*

Proof. Since e is an idempotent, $\operatorname{im} e \subseteq F$ is a direct summand, hence projective. Thus the short exact sequence

$$\ker e \rightarrowtail F \twoheadrightarrow \operatorname{im} e$$

splits and remains exact on tensoring with k; moreover, $\ker e$ is also projective. It follows that $\ker e \otimes_\pi k = 0$, since $\bar{e} = 1$. But $\ker e$ is projective and hence free (by the Cohn–Seshadri result), so that $\ker e = 0$ and e is injective. The result follows since an injective idempotent is necessarily the identity.

Remark. Lemma 6.3 remains true if the field k is replaced by a principal ideal domain D.

We now focus on Theorem 6.1, but we prefer to state our argument in the form of a further lemma, since it seems to have some independent interest.

LEMMA 6.4. *Given* $Y \xrightarrow{a} X \xrightarrow{b} Y$ *with* $ba = 1$ *and* X *a connected co-H-space, assume that* a (*or* b) *induces isomorphisms of* π_1 *and* H. *Then we also have* $ab = 1$ *so that* $Y \simeq X$.

Proof. Since $ba = 1$ it follows that if a or b induces isomorphisms of π_1 and H, then a and b induce mutually inverse isomorphisms. Thus if $j = ab$ then $j: X \to X$ is idempotent with $\pi_1 j = 1$, $Hj = 1$. Let k be a field, set $F = H_n(\tilde{X}; k)$, $n > 1$, and let j induce $e: F \to F$. By Lemma 6.2, F is free, and, by (6.1), $\bar{e}$ is just Hj so that $\bar{e} = 1$. By Lemma 6.3, $e = 1$. Since k was an arbitrary field, it follows that j is a homotopy equivalence. But since $ba = 1$, this implies that $ab = 1$.

Proof of Theorem 6.1. Since e is loop-like on the right, there exists a unique $d: Y \to Y$ with $d + e = 1$. We prove that d is idempotent. First $e = (d+e)e = de + e$, since e is a co-H-map. Since e is loop-like, $de = 0$. Thus $d = d(d+e) = d^2 + de = d^2$, so d is idempotent.

Now (compare the proof of Theorem 4.2) we have, with

$$X = \operatorname{im} d \vee \operatorname{im} e,\ Y \xrightarrow{a = i_1 p_d + i_2 p_e} X \xrightarrow{b = \langle i_d, i_e \rangle} Y,$$

where i_1, i_2 embed im d, im e, respectively, in and $ba = i_d p_d + i_e p_e = d + e = 1$. Now im d, im e are retracts of Y, hence co-H-spaces. Thus X is a (connected) co-H-space. Also b induces an isomorphism $HX \cong HY$ by Proposition 5.3. Since $\pi_1 X$, $\pi_1 Y$ are free and $H_1 b$ is an isomorphism, $\pi_1 b$ is injective (by the Stallings–Stammbach Theorem). Since $ba = 1$, $\pi_1 b$ is surjective, so that $\pi_1 b$ is an isomorphism. We may thus apply Lemma 6.4 to infer that

$$Y \simeq \operatorname{im} d \vee \operatorname{im} e.$$

Remark. We could, of course, have reached the same conclusion simply by assuming that, in $[Y, Y]$, we have two idempotents d, e such that $d + e = 1$. However, this condition would be very hard to verify. On the other hand, as we now show, there are very accessible conditions which guarantee that the canonical idempotent e of Theorem 5.4 satisfies the hypotheses of Theorem 6.1.

COROLLARY 6.5. *If the canonical idempotent* $e: Y \to Y$ *of Theorem* 5.4 *is loop-like on the right then we have the Ganea decomposition of the co-H-space* Y,

$$Y \simeq Z \vee B$$

where Z *is* 1-*connected and* B *is a bunch of circles.*

Proof. In the notation of Theorem 6.1, we have only to show that $\operatorname{im} d$ is 1-connected if $\pi_1 e = 1$. Now $H_1 e = 1$ and $H_1 d + H_1 e = 1$ so $H_1 d = 0$. Thus $H_1 \operatorname{im} d = \operatorname{im} H_1 d = 0$. It follows that $\pi_1 \operatorname{im} d$ is a free group whose abelianization is trivial, so that it is itself trivial and the corollary is proved.

Remark. We obviously get the same conclusion if e is loop-like on the left.

THEOREM 6.6. *The canonical idempotent $e: Y \to Y$ of Theorem 5.4 is loop-like on the right if (i) $[Y, Y]$ is a loop or (ii) the induced co-operation of $B = \operatorname{im} e$ on Y is co-associative in the sense of* [BD].

Proof. (i) is obvious and is included in order to show explicitly that we have here a generalization of Corollary 4.3. To prove[4] (ii) note that the condition asserts that, in the set $[Y, A]$ with binary operation $+$ induced from the co-H-structure in Y, we have

$$(r + su) + tu = r + (su + tu), \qquad r: Y \to A, \qquad s, t: B \to A, \tag{6.2}$$

where $u: Y \to B$ is the classifying map for $\tilde{Y}$ (see the proof of Theorem 5.4). Now (6.2) immediately implies that, if we give B the co-H-structure making $u: Y \to B$ and $f: B \to Y$ co-H-maps ($e = fu$), then $[B, A]$ is associative, so that the co-H-structure on B is co-associative. However, this implies (see [EH] or [K]) that the co-H-structure on B is a (homotopy) cogroup structure so that $[B, A]$ is a group. Now recall that $e = fu$. Set $\bar{e} = (-f)u: Y \to Y$. Then, since u is a co-H-map, $e + \bar{e} = \bar{e} + e = 0$. Thus, by (6.2), we have, in $[Y, Y]$, $(a + \bar{e}) + e = a + (\bar{e} + e) = a$, and, if $x + e = y + e$ then $x = x + (e + \bar{e}) = (x + e) + \bar{e} = (y + e) + \bar{e} = y + (e + \bar{e}) = y$. Thus e is loop-like on the right as claimed.

We may apply Corollary 6.5 and Theorem 6.6 to prove the following.[5]

THEOREM 6.7. *Let Y be a nilpotent co-H-space. Then Y is 1-connected or $Y \simeq S^1$.*

Proof. By Theorem 2.3 $[Y, Y]$ is a loop. Thus we know that $Y \simeq Z \vee B$ where Z is 1-connected and B is a bunch of circles. Since $\pi_1 Y$ is nilpotent, $B = 0$ or $B = S^1$. If $B = 0$, Y is 1-connected. If $B = S^1$ and Z is not contractible, let $H_n(Z; k) \neq 0$ for some $n > 0$ and some field k. Then $H_n(\tilde{Y}; k)$ is a non-trivial free $k[\mathbf{Z}]$-module (Lemma 6.2) so that $\pi_1 Y$ does not act nilpotently on all $H_i \tilde{Y}$. This contradicts the nilpotency of Y and shows that, if $B = S^1$, then $Y \simeq S^1$.

---

[4] Much of this argument is contained in [BD].

[5] Mislin gave a (more involved) proof of this result in a letter to R. Held.

REFERENCES

[B] A. K. BOUSFIELD, *The localization of spaces with respect to homology*, Topology *14* (1975), 133–150.

[BD] I. BERSTEIN and E. DROR, *On the homotopy type of non-simply-connected co-H-spaces*, Ill. Journ. Math. *20* (1976), 528–534.

[EH] B. ECKMANN and P. J. HILTON, *Structure maps in group theory*, Fund. Math. *50* (1961), 207–221.

[G1] T. GANEA, *Some problems on numerical homotopy invariants*, Lecture Notes in Mathematics *249*, Springer, 1971, 13–22.

[G2] ——, *Cogroups and suspensions*, Invent. Math. *9* (1970), 185–197.

[HMR] P. HILTON, G. MISLIN and J. ROITBERG, *On maps of finite complexes into nilpotent spaces of finite type: a correction to 'Homotopical Localization'*, Proc. London Math. Soc. (1977) (to appear).

[K] D. M. KAN, *On monoids and their dual*, Bol. Soc. Math. Mex. *3* (1958), 52–61.

Battelle Seattle Research Center and Case Western Reserve University, Cleveland
Eidgenössische Technische Hochschule, Zürich
Hunter College and Graduate Center of CUNY, New York

Received March 25, 1977.