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The Euler class and periodicity of group cohomology

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We investigate here the cohomology of finite groups with coefficients in modules over group rings. The well-known Cartan-Eilenberg Periodocity Theorem for a finite group acting freely on a sphere is generalized. As a corollary we obtain a criterion for the cohomological triviality of modules over group rings of finite groups. It turns out that a module over a group ring is of trivial cohomology if and only if its restriction to each elementary abelian subgroup of the group is of trivial cohomology. As another application of the Euler class technique we examine the nontriviality of the restriction homomorphism in low dimensions. These algebraic results are deduced from constructions and theorems of equivariant topology and elementary representation theory.

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§1. The Euler class

In this section we define the Euler class of a cohomological sphere with a group action. Our approach is motivated by Atiyah's construction of the Chern classes of complex representations (cf. [1]).

Let Σ^n be an oriented n-dimensional cohomological sphere over the ring R. Assume that a topological group G acts on Σ^n preserving the orientation. Then the Euler class

$$e(G, \Sigma^n) \in H^{n+1}(BG : R)$$

is defined as the Euler class of the sphere bundle $EG \times_G \Sigma^n \to BG$. The Euler class defined above inherits properties of the Euler class of sphere bundle. In particular we have:

1.1. PROPOSITION. *If the action of G on Σ^n has a fixed point, then* $e(G, \Sigma^n) = 0$.

We would like to know when the Euler class of a G-sphere is nonzero. We restrict ourselves to the case of linear actions of compact Lie groups on spheres given by complex representations. The Euler class of a complex representation is defined as the Euler class of its sphere. The obvious necessary condition for the nonvanishing of the Euler class is given by the last proposition. It turns out that only in the case of an elementary abelian group or a torus group the Euler class of an arbitrary complex representation which does not contain the trivial representation is nonzero. Following tom Dieck this result will be formulated in terms of the localization of the cohomology ring with respect to the Euler classes. We denote by $S \subset H^*(BG:R)$ the multiplicative set of the Euler classes of all complex representations of G which do not contain the trivial representation.

1.2. THEOREM. *If G is a compact Lie group, then*

(a) $S^{-1}H^*(BG:Q) \neq 0$ *if and only if G is a torus group.*
(b) $S^{-1}H^*(BG:Z/pZ) \neq 0$ *if and only if G is an elementary abelian p-group,*
(c) $S^{-1}H^*(BG:Z) \neq 0$ *if and only if $G = T \times K$ where T is a torus group and K is an elementary abelian p-group for some prime p.*

Proof. If G is one of the groups listed in the theorem, then a straightforward verification shows that corresponding localization is nonzero.

Conversely: tom Dieck [6] using his Fixed Point Localization Theorem for actions of compact Lie groups proved, that for an arbitrary ring R, $S^{-1}H^*(BG:R) \neq 0$ implies that G is an abelian group. Assume that G is an abelian group. We prove only (c); proofs of (a), (b) are similar. If G is not of the required form, then there exists an epimorphism $G \to Z/nZ$ where $n = pq$, $p, q \neq 1$. Let V be the canonical one-dimensional representation of the group Z/nZ and V^p, V^q be p-fold and q-fold tensor powers of V. The representation $V^p \oplus V^q$ does not contain the trivial representation and its Euler class vanishes. This implies that $S^{-1}H^*(BG:Z) = 0$.

§2. The Periodicity Theorem

The Cartan–Eilenberg Periodicity Theorem (cf. [4] Ch. XVI, §9) asserts that if a finite group G acts freely on a sphere Σ^n then the multiplication by the Euler class

$$e(G, \Sigma^n) \cup \cdot : H^q(G:A) \to H^{q+n+1}(G:A)$$

is an isomorphism for $q>0$ and an arbitrary G-module A. The generalized periodicity theorem formulates conditions for this multiplication to be an isomorphism for the fixed module A.

Let R be the fixed ring. For a given compact Lie group G, $\pi_0(G)$ denotes the group of connected components of G. Let A be a module over the group algebra $R[\pi_0(G)]$. We denote by $H^*(BG:A)$ the cohomology module of the classifying space BG with coefficients in the locally constant sheaf (local coefficient system) defined by the homomorphism $\pi_1(BG)=\pi_0(G)\to \mathrm{Aut}_R(A)$. Recall that for a topological space X we denote by $\mathrm{cd}_R(X)$ the supremum of the integers m such that there exists a sheaf F of R-modules on X with $H^m(X:F)\neq 0$. If the group G acts on X then for every point $x\in X$, G_x denotes its isotropy group.

2.1. THEOREM. *Let (G,Σ^n) be an orientation preserving action of the compact Lie group G on the R-cohomological sphere Σ^n for which $\mathrm{cd}_R(\Sigma^n)=n$. If A is a $R[\pi_0(G)]$-module such that for every point $x\in X$, $H^i(BG_x:A)=0$ for all $i>0$, then the multiplication*

$$e(G,\Sigma^n)\cup\cdot : H^q(BG:A)\to H^{q+n+1}(BG:A)$$

is an isomorphism for every $q>0$.

Proof. Consider a universal G-bundle $EG\to BG$ and two projections

$$BG \xleftarrow{p_1} EG\times_G \Sigma^n \xrightarrow{p_2} \Sigma^n/G$$

defined on the total space of the associated bundle with fibre Σ^n. The map p_1 is the projection of the oriented sphere bundle and therefore we have the Gysin sequence (cf. [10], Ex. 5.J.6.):

$$\cdots\to H^{q+n}(EG\times_G\Sigma^n : p_1^*A)\to H^q(BG:A)\xrightarrow{e(G,\Sigma^n)} H^{q+n+1}(BG:A)$$
$$\to H^{q+n+1}(EG\times_G\Sigma^n : p_1^*A)\to\cdots$$

To prove that the multiplication by the Euler class is an isomorphism in positive dimensions it is enough to show that $H^i(EG\times_G\Sigma^n:p_1^*A)=0$ for $i>n$. To prove this consider the induced homomorphism

$$p_2^* : H^*(\Sigma^n/G : p_{2*}p_1^*A)\to H^*(EG\times_G\Sigma^n : p_1^*A)$$

where $p_{2*}p_1^*A$ denotes the direct image of the locally constant sheaf defined by p_1^*A. We apply the Vietoris–Begle Theorem (cf. [3]) to the map $p_2: EG \times_G \Sigma^n \to \Sigma^n/G$. This map is not closed, but application of the Vietoris–Begle Theorem is justified by the existence of the filtration $\{(EG)^n\}_{n=1}^{\infty}$ of the space EG by the compact n-classifying spaces $(EG)^n$. The cohomology of fibres $H^*(p_2^{-1}([x]): p_1^*A) = H^*(BG_x : A)$ is zero in positive dimensions and therefore the Vietoris–Begle Theorem implies that p_2^* is an isomorphism. From the assumption of the Theorem we have $\mathrm{cd}_R(\Sigma^n) = n$ and therefore (cf. [8], Proposition A.11) $\mathrm{de}_R(\Sigma^n/G) \leq n$ which implies $H^i(EG \times_G \Sigma^n : p_1^*A) = 0$ for $i > n$.

If G is a finite group then $\hat{H}^*(G:A)$ denotes as usual the Tate cohomology of G with coefficients in the G-module A. As a consequence of the last theorem we obtain a criterion for periodicity of the Tate cohomology:

2.2. COROLLARY. *Let (G, Σ^n) be an action satisfying the assumption of Theorem* 2.1. *Assume that G is a finite group and A is a G-module such that $\hat{H}^*(G_x : A) = 0$ for all $x \in \Sigma^n$. Then the multiplication $e(G, \Sigma^n) \cup \cdot : \hat{H}^*(G:A) \to \hat{H}^*(G:A)$ is an isomorphism.*

Proof. We apply the method of dimension shifting. By the Theorem 2.1 the multiplication $e(G, \Sigma^n) \cup \cdot : \hat{H}^*(G:A) \to \hat{H}^*(G:A)$ is an isomorphism in positive dimensions. Consider the multiplication $e(G, \Sigma^n) \cup \ : \hat{H}^{-q}(G:A) \to \hat{H}^{-q+n+1}(G:A)$ for $q \geq 0$. There exists G-module A' and the natural isomorphism $\hat{H}^{-q}(G:A) \cong \hat{H}^1(G:A')$ which is compatible with the multiplication by the Euler class. It is also clear that $\hat{H}^*(G_x : A') = 0$ for all $x \in \Sigma^n$. Applying Theorem *2.1* to the module A' we obtain the required result.

§3. Modules of trivial cohomology

Let G be a finite group. Recall that a G-module A is of trivial cohomology if for every subgroup $K < G$ the Tate cohomology $\hat{H}^*(K:A) = 0$. As an application of our periodicity theorem we prove the following criterion for cohomological triviality of modules over group rings:

3.1. THEOREM. *A G-module A is of trivial cohomology if and only if for every elementary abelian subgroup T of G $\hat{H}^*(T:A) = 0$.*

Proof. The part "only if" follows from the definition of the module of trivial cohomology.

To prove that our condition is sufficient we proceed by induction on the order of the group G. The theorem is trivially true for elementary abelian groups. Assume that it is true for every group K of order $|K|<n$. Let G be a group of order n and let G be not elementary abelian. Now it is enough to prove that $\hat{H}^*(G:A)=0$. From the inductive assumption we know that for every proper subgroup $K<G$, $\hat{H}^*(K:A)=0$. From Theorem 1.2 it follows that there exists a complex representation V of G which does not contain the trivial representation and its Euler class vanishes. On the other hand Corollary 2.2 says that the multiplication $e(V)\cup\cdot:\hat{H}^*(G:A)\to\hat{H}^*(G:A)$ is an isomorphism. Hence $\hat{H}^*(G:A)=0$.

Let us remark that the last theorem remains true if we assume that $\hat{H}^*(T:A)=0$ only for maximal elementary abelian subgroups T of G.

Let R be an arbitrary ring. Theorem 3.1 implies the following characterization of weak projective and projective modules over the group algebra $R[G]$.

3.2. COROLLARY. *A $R[G]$-module A is (weakly) $R[G]$-projective if and only if it is (weakly) $R[T]$-projective for all elementary abelian subgroups T of G.*

Proof. The part "only if" follows from [9].

For any finite group H and any $R[H]$-module A, A is weakly projective if and only if $\hat{H}^*(H:\mathrm{Hom}_R(A,A))=0$ (cf. [9]). Now the corollary follows from Theorem 3.1. Projectivity of an $R[G]$-module is equivalent to R-projectivity and weak $R[G]$-projectivity (cf. [9]).

The last corollary was proved by Chouinard [5] using purely algebraic methods. It is worthwhile to compare his proof with our "topological" arguments. The method of Chouinard can be also applied to a proof of Theorem 3.1.

Theorem 3.1 implies also that theorems of Nakayama, Tadesi and Tate (cf. [2]), Theorems 10.1, 10.2, 10.3) remain true after replacing Sylow subgroups by elementary abelian subgroups.

§4. The nontriviality of the restriction homomorphism

In this section we show an application of the Euler class technique to the problem of nontriviality of the restriction homomorphism. Let G be a compact Lie group and Z be the ring of integers with trivial G-module structure. Venkov [12] and Swan [11] proved that for an arbitrary subgroup $K<G$ the restriction homomorphism $H^*(BG:Z)\to H^*(BK:Z)$ defines on $H^*(BK:Z)$ the structure of a finite $H^*(BG:Z)$-module. Therefore if $K\neq\{e\}$ then the restriction

homomorphism is nonzero in infinitely many dimensions. This result was improved by Evens [7] and Venkov [13]. Evens proved that for an arbitrary subgroup K of a finite group G there exists an integer $k \mid |G|$ for which the restriction homomorphism $H^{2k}(BG:Z) \to H^{2k}(BK:Z)$ is nonzero. Venkov's theorem says that if K is a cyclic subgroup of order two of the group G then the smallest integer for which the restriction homomorphism $H^k(BG:Z/2Z) \to H^k(BK:Z/2Z)$ is nonzero is a power of two. We improve here the result of Evens and we discuss a version of Venkov's theorem for cyclic subgroups of odd prime order.

We consider finite groups. For a given group G and its subgroup K, let $n(K)$ be the smallest positive integer k for which the restriction homomorphism $H^{2k}(BG:Z) \to H^{2k}(BK:Z)$ is non-zero. Let c.d. (G) denote the set of integers which are dimensions of irreducible complex representations of G and let $b(G)$ be the maximum of such integers. From elementary representation theory we know that for every $k \in$ c.d. (G) and normal abelian subgroup A of G, $k \mid (G:A)$. We have also $b(G) \leq \min\{\sqrt{\ }|G|, (G:A) \mid A \text{ abelian} < G\}$.

4.1. THEOREM. *If K is a nontirvial cyclic subgroup of G, then $n(K) \leq b(G)$.*

Proof. It suffices to consider a cyclic subgroup of prime order. Let V be an irreducible complex representation of G which restricted to K in not the trivial representation. It is easy to see that the total Chern class (cf. [1]) $c(V \mid K) \in H^*(BK:Z)$ has nonzero components in positive dimensions. Naturality of the Chern classes implies that there exists an integer m, $0 < m \leq \dim(V)$ such that $\mathrm{res}^G_K(c_m(V)) \neq 0$.

The following simple example shows that if K is a cyclic subgroup of prime order p, then $n(K)$ is not necessarily a power of p.

4.2. *Example.* Let p be an odd prime and D_p the corresponding dihedral group. If $C < D_p$ is the subgroup of rotations then the restrictions homomorphism $H^{2q}(D_p:Z) \to H^{2q}(C:Z)$ is nonzero if and only if q is even.

However for supersolvable groups we have the stronger version of Theorem 4.1:

4.3. THEOREM. *Let K be a nontrivial subgroup of a supersolvable group G. Then there exists $m \in$ c.d. (G) such that the restriction homomorphism $H^{2m}(BG:Z) \to H^{2m}(BK:Z)$ is nonzero.*

The proof of Theorem 4.3 is based on the following lemma from representation theory:

4.4. LEMMA. *Let G be a supersolvable group and $g \in G$ be a nontrivial element of G. Then there exists a representation (V, ρ) of G such that the operator $\rho(g): V \to V$ does not have eigenvalue* 1.

Proof. We proceed by induction on the order of the group G. The lemma is true for the trivial group. Assume, that it is true for groups of order $< n$. Let G be a supersolvable group of order n. It has a nontrivial cyclic normal subgroup S and we have the extension $S \rightarrowtail G \twoheadrightarrow G/S$. If $g \notin S$ then we take the appropriate representation for G/S and the coset gS. It exists by our inductive assumption. Let V be the canonical 1-dimensional representation of the cyclic group S. If $g \in S$ then it does not have eigenvalue 1 on the induced representation $\mathrm{ind}_S^G(V)$.

The following example shows that the last lemma is not true for solvable groups:

4.5. *Example.* Consider the semi-direct product $G = (Z_2 \oplus Z_2 \oplus Z_2) \tilde{x} Z_3$ where Z_3 acts on $Z_2 \oplus Z_2 \oplus Z_2$ by the cyclic permutation of the coordinates. The element $(0, 1, 1, 0)$ has eigenvalue 1 on every representation of G.

Proof of Theorem 4.3. We proceed as in the proof of Theorem 4.1. It suffices to prove the theorem for cyclic subgroups of prime orders. Choose, using Lemma 4.4, a representation V of G such that its restriction to K does not contain the trivial representation. The restriction of the Euler class $e(V) \in H^{2m}(BG:Z)$ is the Euler class $e(V \mid K) \in H^{2m}(BK:Z)$ and therefore is nonzero (cf. Theorem 1.2).

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