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# On quasiconformal mappings of open Riemann surfaces

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## Introduction

Let $f$ be a quasiconformal mapping of the disk $D:|z|<1$ onto $D':|w|<1$. It induces a certain boundary homeomorphism $f \mid \partial D$. We denote by $\mathcal{F}$ the family of all quasiconformal mappings of $D$ onto $D'$ which agree with $f$ on $\partial D$. Let $f_0$ be an extremal mapping in $\mathcal{F}$, i.e. a mapping with smallest maximal dilatation $K_0$. Then, the following inequality was proved by E. Reich and K. Strebel (see [2], p. 376):

$$\frac{1}{K_0} \leq \iint_D |\varphi(z)| \frac{\left|1-\kappa(z)\dfrac{\varphi(z)}{|\varphi(z)|}\right|^2}{1-|\kappa(z)|^2}\, dx\, dy. \tag{1}$$

Here, $\kappa$ is the complex dilatation of $f$, and $\varphi$ is an arbitrary holomorphic quadratic differential of norm

$$\|\varphi\| = \iint_D |\varphi(z)|\, dx\, dy = 1$$

in $D$.

This inequality has a number of consequences, one of them being the converse, for the disk, of a theorem proved by R. S. Hamilton in [1] for arbitrary hyperbolic Riemann surfaces. It says: Let $\mathcal{F}$ be a class of quasiconformal mappings of a hyperbolic Riemann surface $R$ onto a surface $R'$ which are homotopic modulo the boundary, and let $f_0$ with complex dilatation $\kappa_0$ be extremal in $\mathcal{F}$. Then

$$\sup_{\|\varphi\|=1} \left| \iint_R \kappa_0 \varphi\, dx\, dy \right| = k_0, \tag{2}$$

where

$$k_0 = \|\kappa_0\|_\infty = \frac{K_0 - 1}{K_0 + 1}$$

and where the supremum is taken over all holomorphic quadratic differentials on $R$ normed by $\|\varphi\| = 1$.

The theorem by E. Reich and the present author says: Let $f$ be a quasiconformal selfmapping of the disk $D$, with complex dilatation $\kappa$, and let

$$\sup_{\|\varphi\|=1} \left| \iint_D \kappa\varphi \, dx \, dy \right| = k = \|\kappa\|_\infty,$$

where $\varphi$ ranges over the normed holomorphic quadratic differentials of the disk. Then $f$ is extremal in the class $\mathcal{F}$ defined by the boundary values of $f$.

The two theorems combined also provide an existence theorem for extremal mappings of Teichmüller type (see [7], theorem 2 and [8], theorem 1) which goes as follows: If there is, in the class $\mathcal{F}$, a mapping $g$, and a compact set $C \subset D$, such that the maximal dilatation of the restriction of $g$ to $D \setminus C$ (i.e. the maximal dilatation of $g$ in a neighborhood of $\partial D$) is smaller than $K_0$, then there is a Teichmüller extremal in $\mathcal{F}$, associated with a quadratic differential of finite norm (and hence also unique extremal, [3] p. 82).

The inequality (1) is proved in the following way: Let $f$ and $\tilde{f}$ be two mappings in the class $\mathcal{F}$. Then $f_1 \circ f$ with $f_1 = \tilde{f}^{-1}$ is a $qc$ selfmapping of $D$ which keeps the boundary pointwise fixed. Now up to a negligible set of trajectories every trajectory $\alpha$ of $\varphi$ is a cross cut, and it is the shortest line, in the metric of the line element $|\varphi(z)|^{1/2}|dz|$, between its two endpoints. Therefore, the length inequality

$$\int_\alpha |\varphi(z)|^{1/2}|dz| \leq \int_{f_1 \circ f(\alpha)} |\varphi(z)|^{1/2}|dz| \tag{3}$$

holds. By means of a decomposition of $D$ into horizontal strips (see [2], p. 82–87) this inequality can be integrated over $D$ and an application of the length area method eventually leads to (1).

In [5], the procedure was generalized to compact bordered Riemann surfaces. The crucial part consisted in a detailed study of the trajectory structure of a holomorphic quadratic differential of finite norm on such a surface. There are essentially three kinds of trajectories:

Closed trajectories, cross cuts and recurrent trajectories. It is then possible to partition $R$, for any given $\varphi$, into certain subsets characterised by the trajectory structure of $\varphi$, and to carry over the length area method of the original proof.

It is the purpose of this paper to generalize inequality (1) to arbitrary hyperbolic Riemann surfaces. The main object is of course the trajectory structure of a holomorphic quadratic differential. It turns out, that for a differential of finite norm the "bad" trajectories only cover a set of measure zero. The "good" ones then can be partitioned into subsets of the three characteristic types. Then, of course, the length inequality must be proved for cross cuts, whereas it is well known for closed trajectories as well as for closed curves composed of horizontal and vertical arcs.

The applications, like the converse of Hamiltons theorem for arbitrary hyperbolic Riemann surfaces, unique extremality of Teichmüller mappings associated with quadratic differentials of finite norm in their homotopy class etc. then follow in the same way as in the case of the disk: The proofs can be taken over almost word by word.

## §1. Exceptional trajectories

*1. Some definitions.* Let $\varphi$ be a holomorphic quadratic differential on an arbitrary Riemann surface $R$. The zeroes of $\varphi$ are called the critical points, all the others are the regular points of $\varphi$. In the neighborhood of a regular point $P_0$ we can introduce the natural parameter $\zeta = \Phi(z) = \int \sqrt{\varphi(z)}\, dz$. It is determined up to the sign and an arbitrary additive constant, which we can choose such that $\Phi(P_0) = 0$. The analytic continuation of the inverse $\Phi^{-1}$ along the real axis defines the trajectory $\alpha$ through $P_0$ in its natural parametrization $\Phi^{-1}(\xi)$, $\zeta = \xi + i\eta$. Unless $\alpha$ is a Jordan curve on $R$ (closed trajectory), it is subdivided by $P_0$ into two rays $\alpha^+$ and $\alpha^-$, the images of maximal subintervals of the positive respectively the negative real axis.

The initial point $P_0$ is always considered to be a point of the ray. Of course the two rays on a non closed trajectory $\alpha$ depend on the point $P_0 \in \alpha$ and on its orientation, which is determined by the choice of the sign in $\Phi$; but the properties to be considered in the sequel will not depend on $P_0$.

Length and area will always be measured in the $\varphi$-metric, i.e. with the length element $|\varphi(z)|^{1/2}\, |dz|$ and the area element, respectively, $|\varphi(z)|\, dx\, dy$. But nullsets then are of course the same as nullsets in the Euclidean measure of the local parameter.

A ray $\alpha^+$ (or $\alpha^-$) is called critical, if it tends to a zero of $\varphi$. A trajectory $\alpha$ is called critical, if it carries at least one critical ray. As there are only $n+2$ trajectories ending in a zero of $\varphi$ of order $n$, the set of critical trajectories is denumerable and hence of zero area.

A ray $\alpha^+$ is called a boundary ray if $P = \Phi^{-1}(\xi)$ tends to the boundary of $R$ for $\xi \to \xi_\infty$, where $(\xi_{-\infty}, \xi_\infty)$ is the interval on the real axis where $\Phi^{-1}$ is defined and locally univalent. Tending to the boundary of course means that the point $P$ eventually leaves every compact set of $R$ for $\xi \to \xi_\infty$. A trajectory, both rays of which are boundary rays is called a cross cut of $R$.

The limit set of a trajectory ray $\alpha^+$ is the set

$$A^+ = \lim_{\xi_1 \to \xi_\infty} \overline{\{P \mid P = \Phi^{-1}(\xi), \xi > \xi_1\}}.$$

It is empty, iff $\alpha^+$ is a boundary ray. For a critical ray $\alpha^+$, $A^+$ is its critical limiting point. In all the other cases $A^+$ is a closed set of trajectories and their limiting critical endpoints. If $P_0 \in A^+$, in which case $A^+ = \bar{\alpha}$, the closure of the trajectory $\alpha$, $\alpha^+$ is called recurrent. A recurrent trajectory ray will be called a spiral ray, and a trajectory both rays of which are recurrent is called a spiral. Any trajectory of $\varphi$ which is neither closed nor a cross cut nor a spiral is called exceptional.

The main theorem of this § will be that for a holomorphic quadratic differential $\varphi$ of finite norm on an arbitrary Riemann surface $R$ the exceptional trajectories cover a set of measure zero.

*2.* From now on we always assume that $\varphi$ is holomorphic on $R$, not identically zero and of finite norm $\|\varphi\| = \iint_R |\varphi(z)| \, dx \, dy$. Consider an oriented open vertical interval (open subinterval of a vertical trajectory) $\beta$ on $R$. The set of points on $R$ which lie on closed trajectories is open, the complement therefore closed. We will always only consider points of $\beta$ which lie in this complement, without further mentioning, even if we just speak of $P \in \beta$. Every such $P$ splits the trajectory $\alpha$ through $P$ into two rays: $\alpha^+ = \alpha^+(P)$, which leaves $\beta$ to the right, and $\alpha^- = \alpha^-(P)$, which leaves it to the left. Of course, any such ray can meet $\beta$ again at some point $P_1$, and therefore contain $\alpha_1^+(P_1)$ or $\alpha_1^-(P_1)$, depending on the direction in which it cuts $\beta$ at $P_1$.

THEOREM 1. *Let $\varphi$ be of finite norm on $R$. Then the set $E$ of all points $P \in \beta$ for which the ray $\alpha^+(P)$ has $\varphi$-length $|\alpha^+| = \infty$ without being recurrent has measure zero.*

*Proof.* Let $a > 0$. If $\alpha_0^+$ with initial point $P_0 \in \beta$ has length greater than $a$, this is true for all rays $\alpha^+$ with initial point $P \in \beta$ in some neighborhood of $P_0$. This follows from the fact, that there is, for some $\delta > 0$, a rectangle $0 \le \xi \le a$, $|\eta| < \delta$ which is mapped by $\Phi^{-1}$ homeomorphically into $R$. The image is a rectangular strip $S$ on $R$ which contains the subinterval of $\alpha_0^+$ of length $a$ with initial point $P_0$ as middle line. $S$ is (for sufficiently small $\delta > 0$) schlicht on $R$.

Therefore the set of all points $P \in \beta$ with $|\alpha^+(P)| > a$ is relatively open, and the set

$$E_\infty = \{P \mid P \in \beta, |\alpha^+(P)| = \infty\}$$

is a $G_\delta$ set (intersection of denumerably many relatively open sets).

Let $P_0 \in E_\infty$ and assume that $\alpha_0^+$ with initial point $P_0$ cuts $\beta$ in at least one other point $P_1$. Then this is (by a similar argument) true for all rays $\alpha^+$ with initial point $P \in E_\infty$ in a neighborhood of $P_0$. Therefore the set of all points $P \in E_\infty$ for which the ray $\alpha^+(P)$ has at least one more point in common with $\beta$ is relatively open, and the set

$$E'_\infty(\beta) \equiv E'_\infty = \{P \mid P \in \beta, |\alpha^+(P)| = \infty,\ \alpha^+(P) \cap \beta = P\}$$

is a relatively closed subset of $E_\infty$.

Let now $a > 0$. On every ray $\alpha^+(P)$ with $P \in E'_\infty$ we mark the subinterval of length $a$, with initial point $P$. This set is evidently schlicht on $R$ and measurable, and its area is $a \cdot |E'_\infty| \leq \|\varphi\|$, with $|E'_\infty|$ the $\varphi$-length of the set $E'_\infty$. As this is true for every $a > 0$, $E'_\infty$ has measure zero.

Now, given $n \in \mathbb{N}$, we cover $\beta$ by finitely many open intervals $\beta_i$, of length $|\beta_i| < 1/n$. For every $\beta_i$ the corresponding set $E'_\infty(\beta_i)$ is a nullset and therefore the union of all these sets, for all $n$, also is a nullset. If $P \in \beta$ is arbitrary with $|\alpha^+(P)| = \infty, \alpha^+(P)$ not recurrent, $P \in E'_\infty(\beta_i)$ for at least one of the subintervals $\beta_i$, for some $n$. Therefore $E$, which is actually equal to the union of the sets $E'_\infty(\beta_i)$ for all $n$, has measure zero.

*3.* We now pass to the trajectories which are composed of a recurrent ray and a boundary ray.

THEOREM 2. *Let $E$ be the set of points $P$ on a vertical interval $\beta$ such that the trajectory $\alpha$ through $P$ is split into a recurrent ray and a boundary ray. Then* $|E| = 0$.

*Proof.* First we consider the set $E_0^+$ consisting of all those points $P \in \beta$, for which $\alpha^+(P)$ is recurrent and $\alpha^-(P)$ is a boundary ray which does not have any other point in common with $\beta$ besides $P$. We want to show that $E_0^+$ is measurable. The set of points $P \in \beta$ for which $|\alpha^+(P)| = \infty$ is a $G_\delta$ set. Every $\alpha^+(P)$ is either recurrent or not. But we have seen that the points $P$ with non recurrent $\alpha^+(P)$ of infinite length form a nullset. Thus the set $E'_0$ of points $P \in \beta$ with $\alpha^+(P)$ recurrent is measurable.

The set of all those points $P \in \beta$ for which $|\alpha^-(P)| > a$ is relatively open for every $a$. Therefore the set of points $P \in \beta$ for which $|\alpha^-(P)| \leq a$ is relatively closed on $\beta$. The set of points $P$ with $|\alpha^-(P)|<\infty$ is therefore an $F_\sigma$ subset of $\beta$ (union of denumerably many relatively closed sets). If we subtract the $P$'s with $\alpha$ $(P)$ critical and add the ones with $\alpha^-(P)$ a boundary ray (not recurrent!) of infinite length, we get the set of all $P\in\beta$ with $\alpha^-(P)$ a boundary ray. This set is therefore measurable. The set $E_0''$ of all points $P\in\beta$ for which $\alpha^-(P)$ is a boundary ray which does not meet $\beta$ again is thus a relatively closed subset (see the earlier argument) of a measurable set, which is measurable. Finally $E_0^+=E_0'\cap E_0''$ is also measurable.

Every $\alpha^+(P)$ with $P\in E_0^+$ has a well defined next intersection $P_1$ with $\beta$, and evidently $P_1\notin E_0^+$. The set $E_1^+$ of next intersections $P_1$ therefore does not intersect $E_0^+$. The mapping $E_0^+\to E_1^+$ which is induced by walking along the rays $\alpha^+(P)$ evidently is measure preserving in the small, and therefore in the large. Thus $|E_0^+|=|E_1^+|$. Continuing with the next intersections $P_2$ we get a set $E_2^+$ a.s.f., and all the sets

$$E_0^+, E_1^+, E_2^+, \ldots$$

are disjoint subsets of $\beta$ and have the same measure. Thus $|E_0^+|=|E_1^+|=|E_2^+|=\cdots=0$, and hence $|E_0^+\cup E_1^+\cup E_2^+\cup\cdots|=0$.

Similarly, the set $E_0^-$ of points $P\in\beta$ for which $\alpha^-(P)$ is recurrent and $\alpha^+(P)$ is a boundary ray which does not meet $\beta$ again has measure zero, and so do the sets of next intersections $E_1^-$, $E_2^-, \ldots$.

Finally, $E$ is the union of all the sets $E_i^+$ and $E_i^-$ and therefore also has measure zero.

*4.* Bringing theorems one and two together, we now can show that the trajectory structure of a holomorphic quadratic differential $\varphi$ of finite norm on an arbitrary Riemann surface $R$ is in fact quite simple, apart from a set of measure zero.

THEOREM 3. *Almost all trajectories of a holomorphic quadratic differential $\varphi$ of finite norm on an arbitrary Riemann surface are either closed, cross cuts or spirals.*

*Remember.* A spiral is a non closed trajectory both rays of which are recurrent. "Almost all trajectories" means that the trajectories which are not of the indicated kind (i.e. which are exceptional) cover a set of measure zero on $R$.

*Proof.* Let $\beta$ be a vertical arc and let $E$ be a set of measure zero on $\beta$. Then the set of trajectories $\alpha(P)$ through the points $P\in E$ cover a set of measure zero. For the proof we map the vertical arc $\beta$ by a branch of $\zeta=\Phi(z)$ onto a vertical straight interval $\beta'$ in the $\zeta$-plane. Let $E'$ be the image set of $E$. The holomorphic function $\Phi^{-1}$ is now continued along the horizontals through the points of $E'$ in both directions, as long as it stays locally one–one. Evidently, the set of horizontals through the points of $E'$ has Euclidean area zero. Its image by $\Phi^{-1}$ on $R$ is the set of trajectories $\alpha(P)$, $P \in E$. The $\varphi$-measure of this set is at most equal to the Euclidean measure of the preimage (because of possible overlappings on $R$), hence also zero.

Let now $E \subset \beta$ be the set of points $P$ such that $\alpha(P)$ is neither closed, nor a cross cut nor a spiral. Such a trajectory $\alpha(P)$ is either critical, has a non recurrent ray of infinite length or else a recurrent ray and a boundary ray. Therefore $|E| = 0$ and the set of exceptional trajectories through $\beta$ has area zero.

In order to get the result, we choose an everywhere dense, denumerable set of regular points $P_n \in R$. Let $\beta_n$ be the maximal open vertical interval of $\varphi$-length $\leq 1$ with midpoint $P_n$. Let $\alpha$ be any exceptional trajectory, $P \in \alpha$. In any neighborhood of $P$ there is a point $P_n$, and evidently $\alpha$ is contained in the set of exceptional trajectories of some $\beta_n$. But the union of all these sets has measure zero.

## §2. Strips of horizontal cross cuts

5. DEFINITION. Let $\beta$ be a vertical interval. The set of cross cuts (i.e. horizontal trajectories both rays of which tend to the boundary) which cut $\beta$ exactly once is called the (horizontal) strip $S = S(\beta)$ defined by $\beta$.

$S$ is considered to be a pointset on $R$ rather than a set of individual curves.

Let $E = E(\beta)$ be the set of points $P \in \beta$ such that the trajectory $\alpha = \alpha(P)$ through $P$ is in $S$. Clearly, $E$ is measurable, and it is assumed to be non empty. The strip $S$ is now represented in the following way: We choose a branch of $\Phi$ on $\beta$. It maps $\beta$ onto a vertical interval $\beta'$ in the $\zeta$-plane, taking $E$ into $E'$. We continue the inverse function $\Phi^{-1}$ analytically along each horizontal through the points of $E'$ as long as it stays locally one–one (i.e. its derivative at the points of the horizontal is different from zero). The intervals of definition of $\Phi^{-1}$ define the horizontal strip $S'$. It is of course a measurable set, since $E'$ is measurable and the amplitude of the horizontals, measured from $\beta'$ in both directions, is a lower semicontinuous function. The $\Phi^{-1}$-images of these open horizontal intervals are exactly the trajectories $\alpha$ through the points of $E$. Moreover, $\Phi^{-1}$ is one–one and continuous on $S'$: If $\zeta_0 \in S'$, the continuation of $\Phi^{-1}$ along the horizontal interval between $\beta'$ and $\zeta_0$ can be performed in an open horizontal rectangle containing this interval. This shows, that it is in fact a conformal mapping. Let $\Phi^{-1}(\zeta_0) = \Phi^{-1}(\zeta_1)$ for $\zeta_0 \neq \zeta_1 \in S'$. Then the two horizontals through $\zeta_0$ and $\zeta_1$ correspond to the same trajectory in $S$. As it cuts $\beta$ once, by assumption, we must have $\operatorname{Im} \zeta_0 = \operatorname{Im} \zeta_1$. But then we have a closed trajectory in $S$, which is again impossible.

Clearly, the $\varphi$-measure of $S$ is equal to the Euclidean measure of $S'$.

6. We are now going to partition the set of all horizontal cross cuts on $R$ into horizontal strips. To this end, we pick a denumerable, everywhere dense set of regular points $P_1, P_2, P_3, \ldots$ on $R$. To each $P_n$ we assign the sequence of open vertical intervals $\beta_{nk}$ of length $1/k$, $k \geq k_0(P_n)$, with midpoint $P_n$. Here $k_0(P_n) \in \mathbb{N}$ is chosen to be smallest possible, i.e. we start with the largest possible vertical interval for every $P_n$.

The horizontal strips $S_{nk}$ associated with the intervals $\beta_{nk}$ contain all the horizontal cross cuts. For, let $\alpha$ be such a trajectory and let $P \in \alpha$. Then, there is a neighborhood $U(P)$ through which $\alpha$ passes only once. Evidently, there exists an interval $\beta_{nk}$ in $U(P)$ which is cut exactly once by $\alpha$. Thus, $\alpha \subset S_{nk}$.

Let $\beta_1, \beta_2, \beta_3, \ldots$ be a new numbering of the intervals $\beta_{nk}$, with associated strips $S_1, S_2, S_3$ etc. Let $E_1 = \beta_1 \cap S_1$, $E_2 = \beta_2 \cap S_2$. We want to show that $S_1 \setminus S_2$ is a measurable substrip of $S_1$. Let $P \in S_1 \cap S_2$. Then the trajectory through $P$ belongs to both strips. The set of points $P_2 \in E_2$, for which $\alpha(P_2)$ cuts $\beta_1$ at least twice is relatively open, and so is the set of points for which it cuts $\beta_1$ at least once. Therefore the set of points $P_2 \in E_2$ for which $\alpha(P_2)$ cuts $\beta_1$ exactly once is a difference of relatively open subsets of $E_2$. We conclude that the set $S_2 \setminus S_1$ consists of the trajectories through a measurable subset of $E_2$ and hence is measurable itself. Clearly this argument can be repeated, and we get the following partition of the set of all horizontal cross cuts:

$$S_1 \cup (S_2 \setminus S_1) \cup (S_3 \setminus S_1 \setminus S_2) \cup \cdots,$$

where the general term is the substrip

$$S_n \setminus S_1 \setminus S_2 \setminus \cdots \setminus S_{n-1}$$

of $S_n$.

## §3. Spiral sets

7. DEFINITION. Let $\beta$ be an oriented vertical interval. By the spiral set determined by $\beta$ we mean the set of spirals $\alpha$ which have a point in common with $\beta$. It is denoted by $\mathcal{S}(\beta)$.

We do not always make a clear linguistic distinction between the set of spirals (whose elements are curves) and the set of points of $R$ on these spirals. Usually we mean the latter.

Let $E=\mathscr{S}(\beta)\cap\beta$. We want to show that $E$ is measurable. The set of points $P\in\beta$ such that $\alpha(P)$ is not closed and $\alpha^+(P)$ has infinite length is a relative $G_\delta$-set. Now, every ray $\alpha^+(P)$ of infinite length is either recurrent or not. The set of points $P\in\beta$ where the second case occurs has measure zero, therefore the set $E^+$ of points $P\in\beta$ with recurrent $\alpha^+(P)$ is measurable, and so is the set $E^-$ of $P\in\beta$ with $\alpha^-(P)$ recurrent. As $E=E^+\cap E^-$, it is measurable.

Any spiral $\alpha\subset\mathscr{S}(\beta)$ is subdivided by its intersection points with $\beta$ into denumerably many subintervals $\alpha_n$. We say that $\alpha_n$ is of the first kind, if its two endpoints are on the two different edges of $\beta$; otherwise it is called of the second kind. In the second case we denote it by $\alpha_n^+$, if its endpoints are on the edge $\beta^+$ of $\beta$, otherwise by $\alpha_n^-$. Every interval $\alpha_n$ is contained in a rectangular strip, i.e. the $\Phi^{-1}$-image $S$ of a horizontal rectangle $S'$ of the $\zeta=\xi+i\eta$-plane,

$$S':\xi_0\leq\xi\leq\xi_1=\xi_0+a_n,\qquad \eta_0<\eta<\eta_1,$$

where $a_n=|\alpha_n|$ is the $\varphi$-length of the interval $\alpha_n$. Its vertical sides are subintervals of $\beta$, and the ordinates $\eta_0$, $\eta_1$, can be chosen in such a way that $S$ is schlicht on $R$, except possibly for its sides on $\beta$, which may overlap (they lie on the two different edges $\beta^+$ and $\beta^-$ of $\beta$). There evidently is a largest such rectangular strip: It is called the rectangular strip $S_n$ associated with the interval $\alpha_n$. The same thing is true for the intervals $\alpha_n^+$ and $\alpha_n^-$ of the second kind: They determine strips $S_n^+$ resp. $S_n^-$ which are schlicht on $R$ and which have both their vertical sides on $\beta^+$ or on $\beta^-$.

Let $\alpha_n$ and $\tilde{\alpha}_m$ be two intervals on spirals $\alpha$ and $\tilde{\alpha}$ in $\mathscr{S}(\beta)$. Then their strips $S_n$ and $\tilde{S}_m$ either have no interior point in common or they coincide. We thus get a well determined, denumerable set of non overlapping (except on $\beta$) associated strips of the first and of the second kind.

*8.* Consider an arbitrary strip $S_n$ of this collection. The intersection $S_n\cap\mathscr{S}(\beta)$ consists of all the subintervals of the spirals of $\mathscr{S}(\beta)$ which are in $S_n$. This is the $\Phi^{-1}$ image of a certain set of horizontal straight intervals in $S_n'$. Let the Euclidean measure of the ordinates of these horizontals be denoted by $b_n$. Then $a_n\cdot b_n$ is the $\varphi$-measure of the set $S_n\cap\mathscr{S}(\beta)$ of spiral intervals of $\mathscr{S}(\beta)$ in $S_n$, and the total $\varphi$-measure of $\mathscr{S}(\beta)$ evidently is the sum

$$\|\varphi\|_{\mathscr{S}(\beta)}=\sum a_nb_n+\sum a_n^+b_n^++\sum a_n^-b_n^-.$$

Consider $\beta^+$. Every strip $S_n$ has one of its vertical sides on $\beta^+$, and every strip $S_n^+$ has both its vertical sides on $\beta^+$. Therefore the $\varphi$-measure of $E$ is

$$|E| = \sum b_n + 2\sum b_n^+ = \sum b_n + 2\sum b_n^-.$$

Moreover, as every strip of the first kind has one vertical side on $\beta^+$, the other on $\beta^-$, and the measures of $S_n \cap \mathscr{S}(\beta)$ on these two sides are the same, we have

$$\sum b_n^+ = \sum b_n^-.$$

In other words: If we split $E \subset \beta$ in two identical copies $E^+$ and $E^-$ on $\beta^+$ and $\beta^-$, the total measure of the set of points in $E^+$ which are on the vertical sides of the strips $S_n^+$ is the same as the total measure of the points of the strips $S_n^-$ on $\beta^-$.

This fact allows, after a horizontal subdivision of the strips of the second kind, a pairing

$$S_n^+ \leftrightarrow S_n^-$$

such that

$$b_n^+ = b_n^-.$$

One simply starts with the strips with largest ordinate sets, say (by change of the numbering)

$$S_1^+ : b_1^+, \ S_1^- : b_1^-.$$

If $b_1^+ > b_1^-$, we cut off a part $\tilde{S}_1^+$of $S_1^+$, by a horizontal arc in $S_1^+$, such that

$$\tilde{b}_1^+ = b_1^-.$$

The first pair now is $\tilde{S}_1^+$, $S_1^-$. We continue likewise with the rest.

*9.* What is the intersection of two spiral sets determined by vertical intervals $\beta_1$ and $\beta_2$? Let $E_1$ and $E_2$ be the sets of points on spiral trajectories on $\beta_1$ resp. $\beta_2$. Let $P_1 \in E_1$ be such that the spiral $\alpha_1$ through $P_1$ cuts $\beta_2$, which means $\alpha_1 \subset \mathscr{S}_1 \cap \mathscr{S}_2$. Then the same is true for every spiral $\alpha$ through a point $P \in E_1$ in some neighborhood of $P_1$. Therefore the set $E_1'$ of points $P_1 \in E_1$ with $\alpha_1 \subset$

$\mathscr{S}_1 \cap \mathscr{S}_2$ is relatively open, and the spirals of the difference $\mathscr{S}_1 \setminus \mathscr{S}_2$ cut $\beta_1$ is a relatively closed, hence measurable subset of $E_1$. We can then do the same with the corresponding set of spirals as we did above for the total set: Subdivide it into strips of the first kind and pairs of strips of the second kind. In other words, it does not matter, for the procedure, whether we take the full spiral set determined by $\beta$ or a subset of it.

## §4 Boundary correspondence and the length inequality

*10.* Let $R$ and $R'$ be hyperbolic Riemann surfaces. The universal covering surfaces are the unit disks $D: |z|<1$ and $D': |w|<1$, together with the projection mappings.

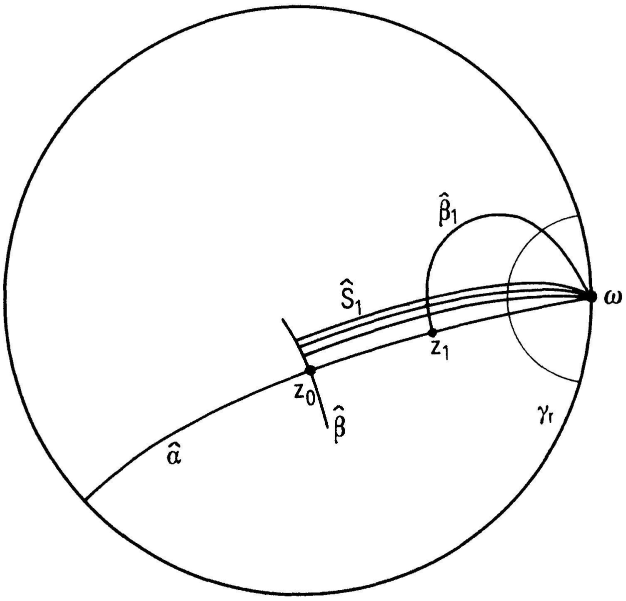
Any quasiconformal mapping $f: R \to R'$ can be lifted to a quasiconformal homeomorphism $\hat{f}: D \to D'$. If a second quasiconformal mapping $g: R \to R'$ is homotopic to $f$, the deformation $h$, which is a one parameter family of mappings of $R$ onto $R'$, can be lifted to a deformation $\hat{h}$ of $\hat{f}$ into a lift $\hat{g}$ of $g$. The mappings $f$ and $g$ are said to be homotopic modulo the boundary, if there is a deformation $h$ such that its lift $\hat{h}$ is constant on $\partial D$. (For an equivalent definition see [2], section 3).

The composition $g^{-1} \circ f$ is a quasiconformal selfmapping of $R$ which is homotopic to the identity modulo the boundary. It is the purpose of this § to show that the trajectories of a holomorphic quadratic differential $\varphi$ of finite norm, in particular the cross cuts of finite length, are mapped by $g^{-1} \circ f$ into curves which have at least the same length.

*11.* Let $S$ be the horizontal strip of cross cuts determined by the open vertical interval $\beta$. It consists of all the cross cuts $\alpha$ of $\varphi$ which have exactly one point in common with $\beta$. Its $\Phi$-image $S'$ is a substrip of the open Euclidean horizontal strip $S_0$ defined by $\beta' = \Phi(\beta)$ in the $\zeta$-plane. $S$ can be lifted to a horizontal strip $\hat{S}$ of the lifted quadratic differential $\hat{\varphi}$ in the unit disk $D: |z|<1$. It should be noted that the $\varphi$-length of an arc is equal to the $\hat{\varphi}$-length of any one of its individual lifts. For a fixed branch of the lifting on $S$ we look at the mapping $\zeta \to z$, composed of $\Phi^{-1}$ and that branch of the lifting, which is of course equal to the corresponding branch of $\hat{\Phi}^{-1}$. We get a 1–1-conformal mapping of the strip $S'$ onto $\hat{S} \subset D$. The Euclidean area $|S'|$ of $S'$ is equal to the $\varphi$-area $|S|_\varphi$ of $S$ (which is in turn equal to the $\hat{\varphi}$-area of $\hat{S}$). We conclude that $|S'|<\infty$. On the other hand, the Euclidean area $|\hat{S}|$ of $\hat{S} \subset D$ is $\leq \pi$. A simple application of the Schwarz inequality (see [3], section 11, p. 86) shows that a.e. cross section $\delta$ in $\hat{S}$ has finite

Euclidean length and therefore converges on its two ends to well determined boundary points $\omega_1$ and $\omega_2$ of $D$.

*12.* Let $E=\beta\cap S$ and let $P_0$ be a point of density of the set $E$. The set $E$ is of course assumed to have positive $\varphi$-length (otherwise $S$ could be neglected). Moreover, we assume that the lift $\hat{\alpha}$ of $\alpha$, passing through the point $z_0\in\hat{\beta}$ above $P_0$, has finite Euclidean length. Both assumptions hold for a.a. points of $E$. Of course, $z_0$ is a point of density of the lift $\hat{E}$ of $E$ on $\hat{\beta}$.




Let $\hat{\beta}_1$ be a vertical trajectory ray of $\hat{\varphi}$ with initial point $z_1\in\hat{\alpha}$. We want to show, by contradiction, that $\beta_1$ cannot converge towards an endpoint $\omega$ of $\hat{\alpha}$. For, let it converge to $\omega$ (Fig.). As $\hat{\beta}_1$ has only the point $z_1$ in common with $\hat{\alpha}$, it bounds, together with a subinterval of $\hat{\alpha}$ and the point $\omega$, a Jordan domain $\hat{G}\subset D$. There is a set $\hat{S}_1$ of horizontal rays, starting at a subset $\hat{E}_1\subset\hat{E}\subset\hat{\beta}$, which enters $\hat{G}$ through $\hat{\beta}_1$, and for which $|\hat{E}_1|_{\hat{\varphi}}=d>0$. All the rays of $\hat{S}_1$ necessarily tend to $\omega$.

We now consider the subarcs $\delta_r$ of the circles $|z-\omega|=r$ which lie in $D$. Let $\hat{E}_1(r)$ be the intersection of $\delta_r$ with $\hat{S}_1$. Clearly, $\hat{E}_1(r)$ is a measurable set and its $\hat{\varphi}$-length is $\geq d$. This follows readily by an application of the mapping $\hat{\Phi}(z)=\int\sqrt{\hat{\varphi}(z)}\,dz$ to the open horizontal strip $\hat{S}_0$ determined by $\hat{\beta}$: The set $\hat{\Phi}(\delta_r\cap\hat{S}_0)$ consists of at most denumerably many open analytic arcs which pass through the horizontal half lines of $\hat{\Phi}(\hat{S}_1)$, the Euclidean height of which is equal to $d$. For the $\hat{\varphi}$-measure of $\hat{E}_1(r)$ we get

$$0<d\leq|\hat{E}_1(r)|_{\hat{\varphi}}=\int_{\hat{E}_1(r)}|\hat{\varphi}(re^{i\vartheta})|^{1/2}r\,d\vartheta.$$

An application of the Schwarz inequality yields

$$d^2 \leq \left( \int_{\hat{E}_1(r)} |\hat{\varphi}(re^{i\vartheta})|^{1/2}\, r\, d\vartheta \right)^2 \leq r\pi \int_{\hat{E}_1(r)} |\hat{\varphi}(re^{i\vartheta})|\, r\, d\vartheta.$$

Therefore, for any sufficiently small positive $r_0$, we have

$$\int_0^{r_0} \frac{d^2}{r}\, dr \leq \pi \int_0^{r_0} \int_{\hat{E}_1(r)} |\hat{\varphi}(re^{i\vartheta})|\, r dr d\vartheta \leq |\hat{S}_1|_{\hat{\varphi}} \leq |\hat{S}|_{\hat{\varphi}} = |S|_{\varphi} \leq \|\varphi\| < \infty,$$

which evidently is impossible.

We conclude, in particular, that the two endpoints $\omega_1$ and $\omega_2$ of $\hat{\alpha}$ are distinct. For, if $\omega_1 = \omega_2 = \omega$, then the vertical rays entering the domain $\hat{G}$ bounded by $\hat{\alpha} \cup \{\omega\}$ would necessarily tend to $\omega$.

*13.* For an arbitrary cross cut $\alpha$ which satisfies the conditions of section 12, let $\gamma = g^{-1} \circ f(\alpha)$. Let $\hat{\gamma}$ be a lift of $\gamma$ which has common endpoints with the lift $\hat{\alpha}$ of $\alpha$. Such a lift exists because of the assumptions about the boundary correspondence (section 10). Pick two arbitrary points $z_1$ and $z_2$ of $\hat{\alpha}$ which lie on non critical vertical trajectories $\hat{\beta}_1$ and $\hat{\beta}_2$. Then, since none of the four rays of $\hat{\beta}_1$ and $\hat{\beta}_2$ converges to an endpoint of $\hat{\alpha}$, the curve $\hat{\gamma}$ cuts both vertical trajectories $\hat{\beta}_1$ and $\hat{\beta}_2$. Therefore, by the divergence principle ([6] I, p. 2), $|\hat{\gamma}|_{\hat{\varphi}} \geq |[z_1, z_2]|_{\hat{\varphi}}$, where $[z_1, z_2]$ is the subinterval of $\hat{\alpha}$ with endpoints $z_1$ and $z_2$. As these two points are arbitrary, $|\hat{\gamma}|_{\hat{\varphi}} \geq |\hat{\alpha}|_{\hat{\varphi}}$, and with $|\gamma|_{\varphi} = |\hat{\gamma}|_{\hat{\varphi}}$, $|\alpha|_{\varphi} = |\hat{\alpha}|_{\hat{\varphi}}$, we get the desired inequality.

## §5 Proof of the main inequality

*14.* The proof of the main inequality consists in an integration of the length inequality over the whole surface. The proof is, in principle, the same as in [2] and [5], but because of the spiral sets it is a bit more complicated and therefore repeated here in full detail for this general situation. We also keep considering horizontal trajectories instead of switching to vertical ones.

Let $f$ and $\tilde{f}$ be quasiconformal mappings of $R$ onto $R'$ which are homotopic modulo the boundary. The differential of $f$ in terms of local parameters $z$ and $w$ near $P \in R$ and $P' = f(P) \in R'$ is denoted by

$$dw = p(z)\, dz + q(z)\, d\bar{z}.$$

Let $f_1 = \tilde{f}^{-1}$ and denote its differential in terms of a local parameter $\tilde{z}$ near $f_1(P')$ by

$$d\tilde{z} = p_1(w)\,dw + q_1(w)\,d\bar{w}.$$

We get

$$d\tilde{z} = (pp_1 + \bar{q}q_1)\,dz + (qp_1 + \bar{p}q_1)\,d\bar{z}.$$

Let now $\varphi$ be a holomorphic quadratic differential of norm $\|\varphi\| = 1$ on $R$, and let $S$ be the horizontal strip defined by the vertical interval $\beta$. We map $S$ by $\Phi$ onto a horizontal strip $S'$ in the $\zeta$-plane and we use $\zeta = \xi + i\eta$ as the parameter of integration in $S$. The length inequality for a trajectory $\alpha$ in $S$ reads:

$$\int_{f_1 \circ f(\alpha)} |d\tilde{\zeta}| \geq \int_\alpha |d\zeta| = \int_{\alpha'} d\xi, \tag{3'}$$

with $\zeta = \Phi(z)$, $d\zeta = \Phi'(z)\,dz$, $d\tilde{\zeta} = \Phi'(\tilde{z})\,d\tilde{z}$. The $\int_{\alpha'} d\xi$ is meant to extend over the horizontal interval $\alpha' = \Phi(\alpha)$; it is of course nothing but the length of this interval. As the inequality holds for a.a. $\eta$ in $S'$, integration over $\eta$ gives

$$\iint_{S'} d\xi\,d\eta \leq \iint_{S'} \left|\frac{\varphi(\tilde{z})}{\varphi(z)}\right|^{1/2} \left|pp_1 + \bar{q}q_1 + (qp_1 + \bar{p}q_1)\frac{\varphi(z)}{|\varphi(z)|}\right| d\xi\,d\eta \tag{4}$$

The integrals can be transplanted to $S \subset R$ by the mapping $\Phi$, the Jacobian being $d\xi\,d\eta = |\varphi(z)|\,dx\,dy$. We get

$$\|\varphi\|_S = \iint_S |\varphi(z)|\,dx\,dy \tag{5}$$

$$\leq \iint_S |\varphi(\tilde{z})|^{1/2}\,|\varphi(z)|^{1/2} \left|pp_1 + \bar{q}q_1 + (qp_1 + \bar{p}q_1)\frac{\varphi(z)}{|\varphi(z)|}\right| dx\,dy.$$

By the exhaustion given in section 6 we can sum up this inequality over denumerably many non overlapping horizontal strips $S_n$ which contain all horizontal cross sections up to a null set. We get

$$\|\varphi\|_{\cup S_n} \leq \iint_{\cup S_n} \cdots\,dx\,dy \tag{6}$$

with the same integrand as in (5).

*15.* To deal with the spiral sets, let $\varepsilon > 0$ and choose a compact set $C \subset R$ which does not contain any critical points of $\varphi$ (cut out little holes around the zeroes of $\varphi$) and such that $\|\varphi\|_{R \setminus C} < \varepsilon$. Every point $P \in C$ is the center of an open $\varphi$-square (the 1–1-image of a Euclidean square with sides parallel to the axes by $\Phi^{-1}$). $C$ is then covered by finitely many of these squares. Let their centers be the points $P_i$ and denote the vertical diameters of the squares with midpoints $P_i$ by $\beta_i$. These arcs are now kept fixed. Let $B = \sum b_i$, with $b_i = |\beta_i|$, the $\varphi$-length of $\beta_i$. Evidently, every spiral which meets $C$ also cuts one of the intervals $\beta_i$ and therefore belongs to the spiral set of one at least of these intervals.

We can now subdivide the vertical intervals $\beta_i$ into intervals $\beta_{ik}$ such that the image $f_1 \circ f(\beta_{ik}) = \beta'_{ik}$ lies in some neighborhood of $\varphi$-radius $\varepsilon$, for every $i$ and $k$. (The $\beta_{ik}$ can be choosen open and the finitely many points of separation discarded.)

Let now $\beta$ be any one of these intervals, with $\mathcal{S}(\beta)$ the spiral set determined by $\beta$ (or any measurable subset of it, see final remark of section 9). We apply to it the procedure developed in sections 7 and 8, using the length inequality for closed curves composed of horizontal and vertical arcs (see [4], p. 364).

Let $S_n$ be a horizontal strip of the first kind. Any horizontal interval $\alpha_n$ of $S_n$ can be closed, by a subinterval of $\beta$, to become a Jordan curve. The length of its $f_1 \circ f$-image, being homotopic to it, is at least equal to $a_n = |\alpha_n|$. As the endpoints of $f_1 \circ f(\alpha_n)$ can be joined by arcs of length $< 2\varepsilon$, we get

$$a_n - 2\varepsilon \leq \int\limits_{f_1 \circ f(\alpha_n)} |\varphi(\tilde{z})|^{1/2} \, |d\tilde{z}|. \tag{7}$$

Introducing, as in (3′), $\zeta = \Phi(z)$ as parameter in $S_n$ and then integrating over $\eta$ in $S'_n$ we get

$$a_n b_n - 2\varepsilon b_n \leq \iint\limits_{S'_n} \cdots \, d\xi \, d\eta = \iint\limits_{S_n} \cdots \, dx \, dy \tag{8}$$

with the same integrands as in (4) respectively (5).

We now do the same for a pair of strips of the second kind $S_n^+$, $S_n^-$. Any pair of arcs $\alpha_n^+ \subset S_n^+ \cap \mathcal{S}(\beta)$, $\alpha_n^- \subset S_n^- \cap \mathcal{S}(\beta)$, can be completed with subintervals of $\beta$ to form a closed curve which is homotopic to a Jordan curve on $R$. Therefore, we get, by the same argument, the length inequality

$$a_n^+ + a_n^- - 4\varepsilon \leq \int\limits_{f_1 \circ f(\alpha_n^+)} |\varphi(\tilde{z})|^{1/2} \, |d\tilde{z}| + \int\limits_{f_1 \circ f(\alpha_n^-)} |\varphi(\tilde{z})|^{1/2} \, |d\tilde{z}| \tag{9}$$

This still holds if we replace the last integral by its infimum over all intervals $\alpha_n^- \subset S_n^- \cap \mathcal{S}(\beta)$. We then get, introducing the $\zeta$-parameter in $S_n^+$ and integrating over the ordinates

$$(a_n^+ + a_n^- - 4\varepsilon)b_n^+ \leq \iint\limits_{(S_n^+)'} \cdots d\xi\, d\eta + b_n^+ \inf \int\limits_{f_1 \circ f(\alpha_n^-)} |\varphi(\tilde{z})|^{1/2}\, |d\tilde{z}| \tag{10}$$

But, because $b_n^+ = b_n^-$, the last member is not greater than the integral

$$\iint\limits_{(S_n^-)'} \cdots d\xi\, d\eta \tag{11}$$

with the same integrand as in (4).

We can now sum up over all the strips of the first and the second kind determined by $\beta$ to get

$$\|\varphi\|_{\mathscr{S}(\beta)} - 2\varepsilon \sum b_n - 4\varepsilon \sum b_n^+ \leq \iint\limits_{\mathscr{S}(\beta)} \cdots dx\, dy \tag{12}$$

with the same integrand as in (5). But

$$2\varepsilon \sum b_n + 4\varepsilon \sum b_n^+ \leq 2\varepsilon b,$$

with $b = |\beta|$ the $\varphi$-length of $\beta$.

Inequality (12) is applied to all the spiral sets of the intervals $\beta_{ik}$, or rather to the differences, in some fixed ordering, as for the horizontal strips (but of course, here we only have finitely many of them). We get for the integral over the set $\cup\, \mathscr{S}(\beta_i)$:

$$\|\varphi\|_{\cup \mathscr{S}(\beta_i)} - 2\varepsilon B \leq \iint\limits_{\cup \mathscr{S}(\beta_i)} \cdots dx\, dy \tag{13}$$

with the same integrand as in (5). Now, for fixed $C$, the $\varepsilon$ in this inequality can be made arbitrarily small, by proper subdivision of the fixed vertical intervals $\beta_i$. Therefore, the $\varepsilon$-term in (13) can be cancelled.

*16.* We still have to do the same thing for the annuli $R_i$ of the trajectory structure. But this offers no difficulty at all, and we get (see e.g. [9], theorem 1, p. 535).

$$\|\varphi\|_{\cup R_i} \leq \iint\limits_{\cup R_i} \cdots dx\, dy. \tag{14}$$

Summing up (6), (13) (without the $\varepsilon$-term) and (14) we get, as the annuli, horizontal strips and spiral sets together cover $C$ up to at most a null set, the inequality

$$1-\varepsilon \le \iint_R |\varphi(\tilde z)|^{1/2}\,|\varphi(z)|^{1/2}\left|pp_1+\bar q q_1+(qp_1+\bar p q_1)\frac{\varphi(z)}{|\varphi(z)|}\right| dx\,dy, \tag{15}$$

which is now true for arbitrary $\varepsilon>0$. Therefore, the $\varepsilon$-term in (15) can be dropped.

*17.* We have to remember that $p$ and $q$ have to be evaluated at the point $z$, whereas $p_1$ and $q_1$ at the point $w=f(z)$, and that $\tilde z=f_1(w)$. In order to compute the integral in (15) we introduce the Jacobians

$$J(w/z)=|p(z)|^2-|q(z)|^2, \qquad J(\tilde z/w)=|p_1(w)|^2-|q_1(w)|^2.$$

We get, as in ([2], p. 378), the inequality

$$1\le \iint_R |\varphi(\tilde z)|^{1/2}J^{1/2}(\tilde z/w)\,J^{1/2}(w/z)\,|\varphi(z)|^{1/2}\,\frac{\left|pp_1+\bar q q_1+(qp_1+\bar p q_1)\dfrac{\varphi(z)}{|\varphi(z)|}\right|}{J(\tilde z/w)^{1/2}J(w/z)^{1/2}}\,dx\,dy$$

Applying the Schwarz inequality and using

$$\iint_R |\varphi(\tilde z)|\,J(\tilde z/w)J(w/z)\,dx\,dy=\iint_R |\varphi(\tilde z)|\,d\tilde x\,d\tilde y=1$$

we get

$$1\le \iint_R |\varphi(z)|\,\frac{\left|pp_1+\bar q q_1+(qp_1+\bar p q_1)\dfrac{\varphi(z)}{|\varphi(z)|}\right|^2}{(|p|^2-|q|^2)(|p_1|^2-|q_1|^2)}\,dx\,dy.$$

Replacing $\varphi$ by $-\varphi$ (which is equivalent to working with the vertical trajectories of $\varphi$ rather than with the horizontal ones) gives the inequality (1.2.7), p. 378 of [2]. Inequalities (1.2.8), (1.5.1) and (1.2.11) are simple transformations.

## §6 Applications

*18.* The main inequality has some almost immediate consequences, proved in [2], [7] and [8], which can now be generalized to arbitrary hyperbolic Riemann surfaces. The first one, where one needs the unabbreviated form of the inequality, is the generalization of the unique extremality of Teichmüller mappings associated with quadratic differentials of finite norm (see [3], where the theorem was first proved for the disk).

THEOREM 4. [Unique extremality]. *Let $f: R \to R'$ be a Teichmüller mapping, i.e. a mapping with a complex dilatation of the form $\kappa = k\,\bar{\varphi}/|\varphi|$, $0 < k < 1$, and let the quadratic differential $\varphi$ be holomorphic and of finite norm. Then, every quasiconformal mapping $\tilde{f}: R \to R'$, $\tilde{f} \neq f$, which is homotopic to $f$ modulo the boundary has a maximal dilatation $\tilde{K} > K = (1+k)/(1-k)$.*

The proof is as in [2], p. 380.

The second consequence is the sufficiency of R. S. Hamiltons necessary condition for a quasiconformal mapping to be extremal.

THEOREM 5. [Sufficient condition for extremality]. *Let $R$ and $R'$ be hyperbolic Riemann surfaces. Let $f: R \to R'$ be a quasiconformal homeomorphism of $R$ onto $R'$, with complex dilatation $\kappa$, $k = \|\kappa\|_\infty$. If*

$$\sup_{\|\varphi\|=1} \left| \iint_R \kappa\varphi \, dx\, dy \right| = k,$$

*then every quasiconformal mapping $\tilde{f}: R \to R'$ which is homotopic to $f$ modulo the boundary has a maximal dilatation $\tilde{K} \geq K = (1+k)/(1-k)$, i.e. $f$ is extremal in its class.*

*Proof.* (See also [2], p. 381): For the proof we only need the main inequality in the form (1), where $K_0$ is now the smallest maximal dilatation in the class $\mathscr{F}$ defined by $f$. We define a Hamilton sequence for the complex dilatation $\kappa$ to be a sequence of holomorphic quadratic differentials $\varphi_n$ on $R$ of norm $\|\varphi_n\| = 1$ and such that

$$\lim_{n\to\infty} \iint_R \kappa\varphi_n \, dx\, dy = k. \tag{16}$$

By assumption there exists such a sequence for the complex dilatation $\kappa$ of $f$. We get

$$\frac{1}{K_0} \leq \frac{1}{1-k^2} \iint_R |\varphi_n| \left(1 + |\kappa|^2 - 2\,\mathrm{Re}\,\kappa \frac{\varphi_n}{|\varphi_n|}\right) dx\, dy \leq \frac{1+k^2}{1-k^2} - \frac{2\,\mathrm{Re}}{1-k^2} \iint \kappa\varphi_n \, dx\, dy.$$

As $|\iint_R \kappa\varphi_n \, dx\, dy| \leq k$, it follows from (16) that the last term of the inequality tends to $2k/(1-k^2)$. Thus

$$\frac{1}{K_0} \leq \frac{1-k}{1+k} = \frac{1}{K},$$

proving the theorem.

A Hamilton sequence $(\varphi_n)$ is called degenerating if it tends to zero locally uniformly in $R$. For such a sequence, there is a variant of Theorem 5, where only the dilatation in the neighborhood of the boundary comes in.

THEOREM 6. *Let there exist a degenerating Hamilton sequence* $(\varphi_n)$ *for the complex dilatation* $\kappa$ *of* $f$. *Let* $\tilde{f} \in \mathcal{F}$ *and* $C \subset R$ *compact. Then* $K \leq \tilde{H}$, *where* $\tilde{H}$ *is the maximal dilatation of* $\tilde{f} \mid (R \setminus C)$. *As this is true for every* $\tilde{f} \in \mathcal{F}$ *and every compact subset* $C$ *of* $R$, *it is also true for the infimum*

$$H = \inf\{\tilde{H};\, \tilde{f} \in \mathcal{F},\, C \subset R,\, C \text{ compact}\}.$$

The proof is the same as in [8], p. 467, following an idea due to E. Reich.

*19.* The next theorems need both, the necessity and sufficiency part of theorem 5.

THEOREM 7. [Existence of Teichmüller mappings]. *Let* $\mathcal{F}$ *be a family of quasiconformal homeomorphisms of* $R$ *onto* $R'$ *which are homotopic modulo the boundary. Let* $f_0$ *with maximal dilatation* $K_0 > 1$ *be extremal in* $\mathcal{F}$ *(there always exists an extremal mapping in* $\mathcal{F}$*). If there exists a quasiconformal mapping* $\tilde{f} \in \mathcal{F}$ *which has a maximal dilatation* $\tilde{H} < K_0$ *in some neighborhood of* $\partial R$, *then* $f_0$ *is a Teichmüller mapping associated with a holomorphic quadratic differential* $\varphi$ *of finite norm and hence is unique extremal.*

*Proof.* The complex dilatation $\kappa_0$ of $f_0$ does not admit any degenerating Hamilton sequence. It is therefore of the form $\kappa_0 = k_0\, \bar{\varphi}/|\varphi|$, $k_0 = (K_0-1)/(K_0+1)$. (For details see [8], theorem 1.)

*Corollary. Let there exist, to every* $\varepsilon > 0$, *a quasiconformal homeomorphism* $\tilde{f} \in \mathcal{F}$ *and a compact set* $C \subset R$ *such that the maximal dilatation* $\tilde{H}$ *of the restriction of* $\tilde{f}$ *to* $R \setminus C$ *is smaller than* $1+\varepsilon$. *Then the family* $\mathcal{F}$ *contains a unique extremal element, which is either conformal or a Teichmüller mapping associated with a holomorphic quadratic differential of finite norm.*

*Proof.* There is an extremal element $f_0 \in \mathscr{F}$. If $K_0 = 1$, $f_0$ is conformal and of course unique. If $K_0 > 1$, theorem 7 applies.

The next theorem shows that there are, in every class $\mathscr{F}$, Teichmüller mappings associated with quadratic differentials of finite norm with at most one (first order) pole and with a dilatation which is arbitrarily close to the maximal dilatation $K_0$ of the extremal mappings in $\mathscr{F}$ (see also [8], theorem 7, p. 479).

To this end, let $P$ be an arbitrary fixed point of $R$. We call the set $V(P) \subset R'$ of images of $P$ by the extremal mappings $f \in \mathscr{F}$ the set of variation of $P$:

$$V(P) = \{P' = f(P); f \text{ extremal in } \mathscr{F}\}.$$

The set $V(P)$ is compact, because the subfamily of all extremal mappings of $\mathscr{F}$ is normal. Pick an extremal mapping $f_0 \in \mathscr{F}$ which maps $P$ onto a boundary point $P_0'$ of $V(P)$. Choose a neighborhood $U$ of $P$. There are points $P' \notin V(P)$ in $U' = f_0(U)$, arbitrarily close to $P_0'$. Let $\tilde{f}$ be a $qc$ mapping which agrees with $f_0$ outside of $U$ and takes $P$ into $P'$. It defines a certain homotopy class $\tilde{\mathscr{F}}$ of mappings of the surface $R = R \setminus \{P\}$ onto the surface $R' = R' \setminus \{P'\}$. Clearly, the smallest maximal dilatation $\tilde{K}_0$ of the mappings in $\tilde{\mathscr{F}}$ is greater than $K_0$, for all points sufficiently close to $P_0'$, and tends to $K_0$ for $P' \to P_0'$. On the other hand, we can find a mapping $g \in \tilde{\mathscr{F}}$ which is conformal in a small neighborhood $U_0 \subset U$ of $P$ and agrees with $f_0$ outside of $U$. The set $C = \bar{U} \setminus U_0$ is compact on $R$ and the maximal dilatation of $g$ on $R \setminus C$ is not greater than $K_0$. Applying theorem 7 to the class $\tilde{\mathscr{F}}$ we get the desired result:

THEOREM 8. *Let $\mathscr{F}$ be a class of qc mappings of $R$ onto $R'$ which are homotopic modulo the boundary. Let $K_0$ be the smallest maximal dilatation of the mappings in $\mathscr{F}$. Then there are Teichmüller mappings in $\mathscr{F}$, associated with quadratic differentials of finite norm and with at most one simple pole, the dilatation of which is arbitrarily close to $K_0$.*

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