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Autor(en): **Bieri, Robert / Strebel, Ralph**

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Almost finitely presented soluble groups

ROBERT BIERI and RALPH STREBEL

Dedicated to Professor B. Eckmann on his sixtieth birthday

1. Introduction

1.1 Finitely presented soluble groups have been investigated by several authors. Roughly speaking, their results deal with two aspects: with the subgroup structure of finitely presented soluble groups [2], [4], [7], [18], and with soluble varieties whose non-cyclic relatively free group are infinitely related [16], [3]. Here we attack the problem of recognizing which finitely generated soluble groups are finitely related in a somewhat more systematic way: We show that all finitely presented soluble groups have a certain structural property which is inherited by homomorphic images (whether this holds for the property of being finitely presented itself is an old problem of P. Hall's and is still open).

1.2 The methods of [2] and [4] made it clear that even in the soluble case the *HNN-construction* is an important tool for obtaining finitely presented groups. Recall that every group B containing a pair of isomorphic subgroups $\vartheta : S \xrightarrow{\sim} T$ is embedded in the HNN-group

$$G = \langle B, t; tst^{-1} = \vartheta(s) \qquad (\text{all } s \in S)\rangle,$$

in which ϑ is induced by an inner automorphism. B is termed the base group, S and T the associated subgroups, and t the stable letter of G. If the base group coincides with one of the associated subgroups G will be called an *ascending HNN-group.* It is only in this special case that G can be soluble (see, e.g., [12]).

Our methods allow to replace the assumption that the groups in question be finitely presented by a (conceivably) weaker condition. Let K be a commutative ring with non-trivial unit. A group G is called *almost finitely presented over* K if there is a presentation $G = F/R$, such that F is a finitely generated free group and the tensor product $R/[R, R] \otimes_{\mathbf{Z}} K$ is finitely generated as a KG-module.

A key result of this paper is

THEOREM A. *Let G be a group containing a normal subgroup N with infinite cyclic quotient G/N, and let $t \in G$ be an element with $gp(t, N) = G$. If G is almost finitely presented over some ring K then G is an HNN-group with stable letter t such that both base group and associated subgroups are finitely generated and contained in N.*

Every finitely presented infinite soluble group G contains a subgroup G_0 of finite index which is, of course, again finitely presented and which does map onto an infinite cyclic group. It thus follows that *G_0 is an ascending HNN-group over a finitely generated base group.* This is the necessary condition for a soluble group G to be finitely presented mentioned in Section 1.1 above.

1.3 A first application of Theorem A concerns relatively free groups or, more generally, groups that arise from relatively free ones by adding "few" relations (see G. Baumslag [3]). In the sequel $\mathfrak{O}$ will denote the variety of all groups and $\mathfrak{A}_p\mathfrak{A}$ the variety of all extensions of elementary Abelian p-groups by Abelian groups.

THEOREM B. *Let $\mathfrak{B} \neq \mathfrak{O}$ be a variety of groups containing $\mathfrak{A}_p\mathfrak{A}$ for some prime p. Let*

$$G = \langle x_1, x_2, \ldots, x_m; r_1, r_2, \ldots, r_n \rangle$$

be a finitely presented group with $m \geq n+2$. Then $G/\mathfrak{B}(G)$ is not almost finitely presented (over any ring K).

COROLLARY B1. *If $\mathfrak{B}$ is a variety of groups with $\mathfrak{A}_p\mathfrak{A} \subseteq \mathfrak{B} \neq \mathfrak{O}$ then every almost finitely presented $\mathfrak{B}$-free group is cyclic.*

Theorem B answers a question of Gilbert Baumslag ([1], Problem 5) who had settled the case $\mathfrak{A}_p\mathfrak{A} \subseteq \mathfrak{B} \subseteq \mathfrak{A}^2$ in [3]. Its corollary generalizes Šmel'kin's result [16] that all non-cyclic free soluble groups are infinitely related.

Another consequence of Theorem B concerns the derived series of the finitely presented group G:

COROLLARY B2. *If $G = \langle x_1, \ldots, x_m; r_1, \ldots, r_n \rangle$ is a finitely presented group with $m \geq n+2$ then none of the factors $G^{(d)}/G^{(d+1)}$ of the derived series is a torsion group or divisible by a prime. In particular, $G^{(d)} \neq G^{(d+1)}$ for all $d \geq 0$.*

1.4 The condition imposed by Theorem A on a finitely generated soluble group of the form $G = N \rtimes C$ to be (almost) finitely presented is, in general, far from being sufficient (e.g., it gives no restriction when N is finitely generated). But it is both necessary and sufficient if N is nilpotent, and in that case we obtain

THEOREM C. *Let $G = N \rtimes C$ be the split extension of a nilpotent group N by an infinite cyclic group C. Then the following conditions* (i)–(iv) *are equivalent:*

(i) *G is finitely presented,*

(ii) *G is almost finitely presented over some ring K,*

(iii) *G is an ascending HNN-group over a finitely generated nilpotent base group,*

(iv) *$N_{ab} = N/[N, N]$ is a finitely generated C-module with the following three properties:*

(a) *the $\mathbf{Z}$-torsion subgroup of N_{ab} is finite,*

(b) *the rational vector space $V = N_{ab} \otimes_{\mathbf{Z}} \mathbf{Q}$ has finite dimension, and*

(c) *there is a generator t of C such that the characteristic polynomial of $t \otimes \mathbf{Q} \in \operatorname{End}(V)$ is integral.*

COROLLARY C1. *Let $G = N \rtimes C$ be as in Theorem C. Then G is finitely presented* (*respectively almost finitely presented over K*) *if and only if $G/[N, N]$ is.*

The class of all finitely generated nilpotent-by-infinite-cyclic groups is admittedly rather small. But it is of some interest because the groups in this class need not satisfy the maximal condition for normal subgroups, contrasting the well-known result of P. Hall on finitely generated metabelian groups [10]. In fact a finitely generated nilpotent-by-infinite-cyclic group can even be isomorphic to a proper quotient of itself ([11], p. 348). These pathologies cannot occur in the finitely presented case.

COROLLARY C2. *Every finitely presented nilpotent-by-cyclic-by-finite group satisfies the maximal condition for normal subgroups.*

1.5 For a final application of Theorem C we go back to the situation of Theorem B but restrict to the case of a single defining relation. Let $\mathfrak{V}$ be a variety with $\mathfrak{A}_p\mathfrak{A} \subseteq \mathfrak{V} \neq \mathfrak{O}$, and let $G = \langle x_1, \ldots, x_m; r \rangle$, If $m > 2$ then the quotient group $Q = G/\mathfrak{V}(G)$ is infinitely related (by Theorem B) and if $m = 1$, Q is cyclic. In the remaining case $m = 2$, Q may or may not be finitely presented depending on r and on $\mathfrak{V}$.

Let $\mathfrak{N}_c$ denote the variety of all nilpotent groups of class c. In case the variety

$\mathfrak{B}$ satisfies $\mathfrak{A}^2 \subseteq \mathfrak{B} \subseteq \mathfrak{N}_c\mathfrak{A}$ and r is not a commutator word, Theorem C applies and shows that the answer depends solely upon the relator r. In Section 5 we derive an algorithm for deciding whether a given relator leads to a finitely presented metabelian quotient G/G'' or not (see Theorem D in Section 6.1).

2. The proof of Theorem A

2.1 Almost finitely presented groups. Throughout the paper K will denote a commutative ring with $1 \neq 0$. Let $R \rightarrowtail F \twoheadrightarrow G$ be a short exact sequence of groups. If R is finitely generated as a normal subgroup of F then $R_{ab} = R/R'$ is a finitely generated G-module. Thus it is immediate from the definition given in Section 1.2 that a finitely presented group G is almost finitely piesented over $\mathbf{Z}$ and this, in turn, implies that G is almost finitely presented over every ring K. Whether the converse holds is an open question. Evidence in favour of such a conjecture is provided by the fact that all these finiteness conditions have much the same properties, such as being inherited by extensions and by subgroups of finite index.

Almost finitely presented groups arise naturally in homology theory: G is almost finitely presented over K if and only if the trivial KG-module K admits a projective resolution which is finitely generated in dimensions 0, 1 and 2 (cf. [6], p. 20). In the proof of Theorem A we shall work with yet another characterization

LEMMA 2.1. *G is almost finitely presented over K if and only if there is a short exact sequence of groups $M \rightarrowtail X \twoheadrightarrow G$ such that X is finitely presented and $M_{ab} \otimes_{\mathbf{Z}} K = 0$.*

Proof. If F is a free group and $S \subseteq R$ are two normal subgroups of F one has the exact sequence of $K(F/R)$-modules

$$SR'/R' \otimes_{\mathbf{Z}} K \xrightarrow{\mu} R_{ab} \otimes_{\mathbf{Z}} K \longrightarrow (R/S)_{ab} \otimes_{\mathbf{Z}} K \longrightarrow 0. \tag{2.1}$$

Suppose $G = F/R$ is almost finitely presented over K. Then $R_{ab} \otimes_{\mathbf{Z}} K$ is generated by a finite set of elements of the form $r_iR' \otimes 1$. Take S to be the normal subgroup of F generated by $\{r_i\}$. Then μ is an epimorphism, hence $(R/S)_{ab} \otimes_{\mathbf{Z}} K = 0$ and $R/S \rightarrowtail F/S \twoheadrightarrow G$ is the required short exact sequence.

Conversely, if we are given the short exact sequence $M \rightarrowtail X \twoheadrightarrow G$ and a finite presentation $X = F/S$, take $R \lhd F$ with $F/R \cong G$ and $R/S \cong M$. Now $M_{ab} \otimes_{\mathbf{Z}} K = 0$ implies that the map μ in (2.1) is epimorphic, whence $R_{ab} \otimes_{\mathbf{Z}} K$ is finitely generated as a KG-module.

2.2 Let G be an almost finitely presented group over K. By Lemma 2.1 there is a short exact sequence of groups $M \rightarrowtail X \overset{\pi}{\twoheadrightarrow} G$, where X is finitely presented and $M_{ab} \otimes_{\mathbf{Z}} K = 0$. Given an element $t \in G$ and a normal subgroup $N \triangleleft G$ such that G/N is infinite cyclic generated by tN, we choose a finite set of generators $\bar{a}_1, \ldots, \bar{a}_m, \bar{x}$ of X such that $\pi(\bar{a}_1), \ldots, \pi(\bar{a}_m) \in N$ and $\pi(\bar{x}) = t$. Then X admits a finite presentation of the form

$$X = F/R = \langle a_1, \ldots, a_m, x; r_1, \ldots, r_n \rangle.$$

with $\bar{a}_i = a_i R$, $1 \leq i \leq m$, $\bar{x} = xR$. By the choice of $a_1, \ldots, a_m$ the relators $r_1, \ldots, r_n$ have zero exponent sum on x. Hence there is a natural number λ such that all relators $r_1, \ldots, r_n$ are contained in the finitely generated subgroup

$$U = gp(a_i^{x^k} \mid 1 \leq i \leq m, -\lambda \leq k \leq \lambda) \subseteq F.$$

Now consider the finitely generated subgroup

$$S = gp(\pi(\bar{a}_i)^{t^k} \mid 1 \leq i \leq m, -\lambda \leq k < \lambda) \subseteq G.$$

S and its conjugate $T = S^t$ are contained in $B = \pi(UR/R)$ and we can define the HNN-group

$$G^* = \langle B, p; psp^{-1} = s^t \qquad (s \in S) \rangle$$

We claim the canonical epimorphism $\psi: G^* \twoheadrightarrow G$ induced by the inclusion $B \leq G$ and $\psi(p) = t$ is an isomorphism. To see this define a homomorphism $\varphi: F \to G^*$ by setting

$$\varphi(a_i) = \pi(\bar{a}_i) \in B \quad \text{and} \quad \varphi(x) = p.$$

φ is epimorphic and, by the choice of B, sends all relators $r_1, \ldots, r_n$ to the unit element. Thus φ induces an epimorphism $\bar{\varphi}: X = F/R \twoheadrightarrow G^*$. The composite $\psi \circ \bar{\varphi}: X \twoheadrightarrow G$ coincides with π, hence $\ker(\psi \circ \bar{\varphi}) = M$. Since $M_{ab} \otimes_{\mathbf{Z}} K = 0$ and $\bar{\varphi}$ maps $\ker(\psi \circ \bar{\varphi})$ onto $\ker \psi$, it follows that $(\ker \psi)_{ab} \otimes_{\mathbf{Z}} K = 0$. On the other hand, $\ker \psi$ is a normal subgroup of the HNN-group G^* missing the base group and hence is free (cf. [12], p. 627). Both facts put together show that $\ker \psi$ is trivial.

3. Ascending HNN-groups

3.1 Let C be an infinite cyclic group. For each generator t of C we define a class $\mathcal{k}_t$ of C-groups as follows: A C-group N is in $\mathcal{k}_t$ *if it contains a finitely* *generated subgroup $B \leq N$ with the two properties*

$$B^t \subseteq B \tag{3.1}$$

and

$$B \text{ generates } N \text{ as a } C\text{-group.} \tag{3.2}$$

Any subgroup $B \leq N$ satisfying (3.1) and (3.2) will be termed a *base group of* N (with respect to t). Notice that if B is a base group of N, so is every transform B^{t^i}, $i \in \mathbf{Z}$, and that N is the ascending union $N = \bigcup_{i=0}^{\infty} B^{t^{-i}}$. From this it follows readily that a C-group N is in the class $\mathfrak{k}_t$ if and only if the split extension $N \rtimes C$ is an ascending HNN-group over a finitely generated base group $B \leq N$, with stable letter t, and such that B coincides with the *first* associated subgroup.

3.2 The goal of this section is a classification of all nilpotent C-groups in $\mathfrak{k}_t$; this will immediately lead to a proof of Theorem C.

LEMMA 3.1. (i) *If $N \in \mathfrak{k}_t$ then N is finitely generated as a C-group.*
(ii) *If $N \in \mathfrak{k}_t \cap \mathfrak{k}_{t^{-1}}$ then N is a finitely generated group.*

Proof. (i) is plain from the definition. So assume $N \in \mathfrak{k}_t \cap \mathfrak{k}_{t^{-1}}$, and let B, B' be finitely generated base groups of N with respect to t and t^{-1}, respectively. By replacing B' by a suitable transform $(B')^{t^i}$ we can achieve that B' contains generators of the finitely generated group B, i.e., $B \subseteq B'$. Then we have $B^{t^{-j}} \subseteq B'^{t^{-j}} \subseteq B'$ for all $j \geq 0$ by (3.2), hence $N = gp_C(B) \subseteq B'$, and $N = B'$ is finitely generated.

3.3 Next we prove that the class $\mathfrak{k}_t$ is closed with respect to homomorphic images and extensions.

LEMMA 3.2. *Let N be a C-group and $M \lhd N$ a C-invariant normal subgroup. Then one has:*

(i) *if N is in $\mathfrak{k}_t$, so is N/M,*
(ii) *if M and N/M are in $\mathfrak{k}_t$, so is N.*

Proof. (i) If B is a finitely generated base group of N then BM/M is a base group of N/M.

(ii) Let U be a finitely generated subgroup of N such that UM/M is a base group of N/M, and pick generators $u_1, u_2, \ldots, u_f$ of U. By (3.1) we can find elements $m_j \in M$, $u_j' \in U$, $j = 1, 2, \ldots, f$, with $u_j^t = m_j u_j'$. Now choose a finitely generated base group V of M with $m_j \in V$ for all $j = 1, 2, \ldots, f$ and let B be the subgroup of N generated by U and V. B is finitely generated, $B^t \subseteq B$, and B generates N as a C-group, whence $N \in \mathscr{k}_t$.

3.4 We now turn to Abelian C-groups (i.e., to C-modules) belonging to $\mathscr{k}_t$. These can be characterized as follows:

PROPOSITION 3.3. *A finitely generated C-module A is in $\mathscr{k}_t$ if and only if the following three conditions hold:*

(a) *the* $\mathbf{Z}$*-torsion subgroup* tor A *of* A *is finite,*
(b) *the rational vector space* $V = A \otimes_{\mathbf{Z}} \mathbf{Q}$ *is of finite dimension,*
(c) *the characteristic polynomial of* $t \otimes \mathbf{Q} \in \operatorname{End}(V)$ *is integral.*

Proof. Let $A \in \mathscr{k}_t$ and let B be a finitely generated base group of A. Since t induces an injective endomorphism $B \rightarrowtail B$ it induces an automorphism on the torsion-subgroup tor B. Hence

$$\operatorname{tor} A = \left(\bigcup_{i=0}^{\infty} B^{t^{-i}}\right) \cap \operatorname{tor} A = \bigcup_{i=0}^{\infty} (B^{t^{-i}} \cap \operatorname{tor} A) = \operatorname{tor} B \text{ is finite.}$$

In view of Lemma 3.2 it thus remains to consider the case where the additive group of A is torsion-free. Again, let $A \in \mathscr{k}_t$ and let B be a finitely generated base group of A. Every transform $B^{t^{-i}}$ is a free Abelian group of the same finite rank n, hence $A = \bigcup B^{t^{-i}}$ is a torsion-free Abelian group of rank n. The endomorphism $t: B \to B$ can be described by an integral matrix, and therefore the characteristic polynomial of $t \otimes \mathbf{Q}: A \otimes \mathbf{Q} \to A \otimes \mathbf{Q}$ has integral coefficients.

Conversely, assume A is a finitely generated C-module, satisfying (b), (c), whose additive group is torsion-free. Let

$$\chi(x) = x^n + z_1 x^{n-1} + z_2 x^{n-2} + \cdots + z_n \in \mathbf{Z}[x]$$

be the characteristic polynomial of $t \otimes \mathbf{Q}$ acting on $V = A \otimes \mathbf{Q}$. As A is embedded in V we have $\chi(t)A = 0$ hence

$$a^{t^n} = -z_1 a^{t^{n-1}} - z_2 a^{t^{n-2}} - \cdots - z_n a$$

for every element $a \in A$. Now let $a_1, a_2, \ldots, a_m$ be a finite set of generators for the C-module A and let B denote the additive subgroup generated by $a_j^{t^i}$, $0 \leq i \leq n-1$, $1 \leq j \leq m$. Then one has $B^t \subseteq B$ and $gp_C(B) = A$, hence B is a finitely generated base group and $A \in \mathscr{k}_t$.

As an immediate consequence of Proposition 3.3 we note that the class of all

C-modules in $\mathcal{k}_t$ is closed with respect to the tensor product (over $\mathbf{Z}$); of course this can also be proved directly.

LEMMA 3.4. *If A_1 and A_2 are two C-modules in $\mathcal{k}_t$ then so is the tensor product (over $\mathbf{Z}$) $A_1 \otimes A_2$ when endowed with the diagonal C-action.*

3.5 Let N be a group and N_m the m^{th} term of its lower central series, i.e., $N_1 = N$ and $N_m = [N, N_{m-1}]$, $m \geq 2$. If N is a C-group then all factors N_{m-1}/N_m are C-modules and there is a C-module epimorphism

$$\pi : N/N_2 \otimes N_{m-1}/N_m \twoheadrightarrow N_m/N_{m+1},$$

given by $\pi(xN_2 \otimes yN_m) = [x, y]N_{m+1}$ (see [15], p. 55). Using Lemmata 3.2 and 3.4 we now have immediately

PROPOSITION 3.5. *Let N be a C-group. If its Abelianization N/N_2 is in $\mathcal{k}_t$ then so is every lower central factor N/N_m, $m \geq 2$.*

COROLLARY 3.6. *If N is a nilpotent C-group then $N \in \mathcal{k}_t$ if and only if $N_{ab} \in \mathcal{k}_t$.*

3.6 Proof of Theorem C. A finitely presented group is almost finitely presented over every ring K. If $G = N \rtimes C$ is nilpotent-by-infinite-cyclic and almost finitely presented then G is an ascending HNN-group over a finitely generated nilpotent base group $B \subseteq N$ by Theorem A. All finitely generated nilpotent groups are finitely presented, and every HNN-group over a finitely presented base group and with finitely generated associated subgroups is finitely presented. These facts establish the cycle (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). The equivalence (iii) $\Leftrightarrow$ (iv) is covered by Corollary 3.6 in conjunction with Proposition 3.3.

3.7 Proof of Corollary C2. Let G be a finitely presented nilpotent-by-cyclic-by-finite group. We have to show that every homomorphic image of G is finitely presented. Since being finitely presented is inherited by extensions and subgroups of finite index we may as well assume that G is nilpotent-by-cyclic. Let $M \lhd G$. If G/M contains a nilpotent subgroup of finite index then G/M is polycyclic, hence finitely presented. Otherwise $G = N \rtimes C$ is nilpotent-by-infinite-cyclic and $M \subseteq N$, whence G/M is finitely presented by Theorem C and Lemma 3.2.

Remark. Note that every almost finitely presented nilpotent-by-cyclic-by-finite group is *constructable* in the sense of [4]; thus Corollary C2 is a special case of [4], Corollary 5.

3.8 Connexion with the Alexander polynomial. Let G be a finitely generated group satisfying the conditions

(a) G/G' is infinite cyclic, and
(b) G'/G'' is torsion-free.

These conditions hold, e.g., for the fundamental group of the complement of a tame knot in $\mathbf{R}^3$ (see e.g. [8]) and, more generally, for any group having a finite presentation of the form

$$\langle x_1, \ldots, x_m, x_{m+1}; w_1x_1, \ldots, w_mx_m \rangle,$$

where $w_1, w_2, \ldots, w_m$ are words with zero exponent sum on all generators $x_1, x_2, \ldots, x_{m+1}$ (see e.g. [17], p. 332). Let t denote a generator of the infinite cyclic group G/G'.

In this situation the Alexander polynomial $\Delta(t)$ is defined, and it has been proved by H. F. Trotter that G/G'' is finitely presented if and only if either the first or the last coefficient of $\Delta(t)$ is a unit ([19], Theorem 3; his formulation is actually slightly different, but the translation is provided by the formula

$$\Delta(t) = \det(tA + (I - A)) = t^n \det A + \cdots \pm \det(I - A)$$

on p. 657). This result is a special case of Theorem C, for, as G/G'' is torsion-free and $\Delta(1) = \pm 1$, the characteristic polynomial of $t \otimes \mathbf{Q} \in \operatorname{End}(G'/G'' \otimes \mathbf{Q})$ is precisely the normalized Alexander polynomial.

4. The proof of Theorem B

4.1 Let

$$G = \langle x_1, x_2, \ldots, x_m; r_1, \ldots, r_n \rangle$$

be a finitely presented group with $m \geq n + 2$, and $\mathfrak{B} \neq \mathfrak{O}$ a variety containing $\mathfrak{A}_p\mathfrak{A}$ for some prime p. Then by the arguments of [3], pp. 307–308, $Q = G/\mathfrak{B}(G)$ contains a normal subgroup N with infinite cyclic factor group $Q/N = \langle tN \rangle$ and with $N/N'N^p$ *infinite* elementary Abelian. No infinite Abelian torsion group can lie in the class $\mathscr{k}_t \cup \mathscr{k}_{t^{-1}}$, hence $N \notin \mathscr{k}_t \cup \mathscr{k}_{t^{-1}}$ by Lemma 3.2. As $\mathfrak{B} \neq \mathfrak{D}$, $Q = G/\mathfrak{B}(G)$ contains no non-Abelian free subgroup. Therefore it follows from Theorem A that Q is not almost finitely presented over any ring K. This proves Theorem B and Corollary B1.

4.2 In order to prove Corollary B2 let first d be an integer ≥ 2 and consider $G^{(d)}/G^{(d+1)}$. If $G^{(d)}/G^{(d+1)} \otimes_{\mathbf{Z}} K$ were trivial for some K then, by Lemma 2.1, $G/G^{(d)}$ would be almost finitely presented over K, contradicting Theorem B when applied to the variety of all soluble groups of derived length d. Thus $G^{(d)}/G^{(d+1)} \otimes_{\mathbf{Z}} K \neq 0$ for every commutative ring K. For $K = \mathbf{Q}$ this means that $G^{(d)}/G^{(d+1)}$ is not a torsion group, and for $K = \mathbf{F}_p$ that $G^{(d)}/G^{(d+1)}$ is not divisible by p.

The case $d = 0$ is obvious, so we are left with $d = 1$. The KG_{ab}-module $G'/G'' \otimes K$ is isomorphic to $H_1(G; KG_{ab})$. Computing this homology group by means of the free resolution associated to the presentation $\langle x_1, \ldots, x_m; r_1, \ldots, r_n \rangle$ we obtain $G'/G'' \otimes K$ as the middle homology group of a complex of the form

$$(KG_{ab})^n \to (KG_{ab})^m \to KG_{ab}. \tag{4.1}$$

Now assume that K is a field and choose a free-Abelian subgroup T of finite index j in G_{ab}. Then (4.1) is a complex of free modules of rank nj, mj and j, respectively, over the integral domain KT, and $G'/G'' \otimes K \neq 0$ follows by counting ranks (since $m \geqslant n+2$).

5. Two generator one relator groups in the variety of all metabelian groups: preparations

5.1 Let G be a two generator group with a single defining relation whose abelianization G_{ab} is the direct product of an infinite cyclic group C and a finite cyclic group C_e of order $e \geq 1$. Then G has a presentation of the form

$$G = \langle x_1, x_2; w(x_1, x_2) x_1^e \rangle, \tag{5.1}$$

where $w = w(x_1, x_2)$ is a word with zero exponent sum on x_1 and x_2 (cf. [14], Theorem 3.5). Clearly G/G'' is finitely related if and only if its subgroup of finite index $gp(x_2, G')/G''$ is. As $C = gp(x_2, G')/G'$ is infinite cyclic generated by $t = x_2 G'$ this in turn is equivalent, by Theorem C, with G'/G'' being in the class $\mathscr{k}_t \cup \mathscr{k}_{t^{-1}}$.

Our aim is to deduce necessary and sufficient conditions, in terms of the relator $r = w x_1^e$, for G'/G'' to be in $\mathscr{k}_t$.

5.2 We begin by recalling some facts about the free differential calculus [9]. Let F be the free group freely generated by x_1 and x_2. A (free) derivation of F is a map $D: F \to \mathbf{Z}F$ satisfying $D(uv) = Du + u(Dv)$. A derivation is uniquely determined by its values on the generators x_1, x_2; and because x_1, x_2 generate F freely they may be assigned arbitrary values. The partial derivations $D_i: F \to \mathbf{Z}F$, $i=1,2$, are defined by $D_i(x_i)=1$ and $D_i(x_j)=0$ if $i \neq j$. They are related by the "fundamental formula"

$$1-u = D_1u \cdot (1-x_1) + D_2u \cdot (1-x_2), \qquad u \in F. \tag{5.2}$$

We shall write ξ_i for the coset $x_i[F,F] \in F_{ab}$ and $\bar{D}_i$ for the partial derivation D_i followed by the canonical ring homomorphism $\mathbf{Z}F \twoheadrightarrow \mathbf{Z}F_{ab}$. The derivations $\bar{D}: F \to \mathbf{Z}F_{ab}$ have a special feature: For $i \neq j$ one has

$$D_j(ux_iu^{-1}) = D_ju - ux_iu^{-1}D_ju = (1-ux_iu^{-1}) \cdot D_ju, \qquad u \in F.$$

From this it follows readily that $\bar{D}_jw \in \mathbf{Z}F_{ab}$ is divisible by $1-\xi_i$ when $w \in F$ is a word with zero exponent sum on x_j and $i \neq j$.

5.3 Let $F = \langle x_1, x_2 \rangle$, G and $r = wx_1^{\varepsilon}$ be as above and let $\pi: \mathbf{Z}F \twoheadrightarrow \mathbf{Z}G$ be the canonical ring homomorphism. The partial derivatives of r can be used to express the second differential in a free resolution of the trivial G-module $\mathbf{Z}$: one has an exact sequence of (left) G-modules

$$\mathbf{Z}G \xrightarrow{\partial_2} \mathbf{Z}G \oplus \mathbf{Z}G \xrightarrow{\partial_1} \mathbf{Z}G \xrightarrow{\varepsilon} \mathbf{Z} \longrightarrow 0, \tag{5.3}$$

where ε is the augmentation, ∂_1 is given by

$$\partial_1(1,0) = 1-\pi(x_1), \qquad \partial_1(0,1) = 1-\pi(x_2),$$

and ∂_2 is given by

$$\partial_2(1) = (\pi(D_1r), \pi(D_2r)).$$

The Shapiro Lemma gives an isomorphism $H_1(G; \mathbf{Z}G_{ab}) \cong H_1(G'; \mathbf{Z}) = G'/G''$. Under this isomorphism the G-action on $H_1(G, \mathbf{Z}G_{ab})$ induced by left multiplication on $\mathbf{Z}G_{ab}$ corresponds to the G-action on G'/G'' induced by (left) conjugation. Using (5.3) to compute $H_1(G; \mathbf{Z}G_{ab})$ yields

$$G'/G'' = \ker(\mathbf{Z}G_{ab} \otimes_G \partial_1)/\mathrm{im}(\mathbf{Z}G_{ab} \otimes_G \partial_2).$$

To find a presentation of G'/G'' we first need one for $\ker(\mathbf{Z}G_{ab} \otimes_G \partial_1)$. This can be done by taking the tensor product over $\mathbf{Z}$ of the C-free resolution

$$0 \longrightarrow \mathbf{Z}C \xrightarrow{1-\pi_*(\xi_2)} \mathbf{Z}C \xrightarrow{\varepsilon} \mathbf{Z} \longrightarrow 0$$

with the standard C_e-free resolution

$$\cdots \longrightarrow \mathbf{Z}C_e \xrightarrow{1-\pi_*(\xi_1)} \mathbf{Z}C_e \xrightarrow{N} \mathbf{Z}C_e \xrightarrow{1-\pi_*(\xi_1)} \mathbf{Z}C_e \xrightarrow{\varepsilon} \mathbf{Z} \longrightarrow 0,$$

where $\pi_*: F_{ab} \to G_{ab}$ is induced by π, and

$$N = \pi_*(1+\xi_1+\cdots+\xi_1^{e-1}).$$

This yields the $(C\times C_e)$-free resolution

$$\cdots \longrightarrow \mathbf{Z}G_{ab} \oplus \mathbf{Z}G_{ab} \xrightarrow{\delta_3} \mathbf{Z}G_{ab} \oplus \mathbf{Z}G_{ab} \xrightarrow{\delta_2} \mathbf{Z}G_{ab} \oplus \mathbf{Z}G_{ab} \xrightarrow{\delta_1} \mathbf{Z}G_{ab} \twoheadrightarrow \mathbf{Z}, \tag{5.4}$$

where δ_1 coincides with $\mathbf{Z}G_{ab} \otimes_G \partial_1$, and δ_2, δ_3 are given by the matrices

$$\Delta_3 = \begin{pmatrix} \pi_*(1-\xi_1) & 0 \\ \pi_*(1-\xi_2) & N \end{pmatrix}, \qquad \Delta_2 = \begin{pmatrix} N & 0 \\ -\pi_*(1-\xi_2) & \pi_*(1-\xi_1) \end{pmatrix}.$$

Exactness of (5.4) yields a presentation of $\ker \delta_1 = \ker(\mathbf{Z}G_{ab} \otimes_G \partial_1)$, name

$$\mathbf{Z}G_{ab} \oplus \mathbf{Z}G_{ab}/\mathrm{im}\, \delta_3 \xrightarrow{\delta_2} \ker(\mathbf{Z}G_{ab} \otimes_G \partial_1).$$

The image of $\mathbf{Z}G_{ab} \otimes_G \partial_2$ in $\ker(\mathbf{Z}G_{ab} \otimes_G \partial_1)$ is generated by $b = (\pi_*(\bar{D}_1 r), \pi_*(\bar{D}_2 r))$. We claim that $(1, \pi_*(\bar{D}_2 r/(1-\xi_1)))$ is a preimage of b under δ_2. To see this notice first that $r = wx_1^e$ has zero exponent sum on x_2, whence $\bar{D}_2 r$ is divisible by $1-\xi_1$ in $\mathbf{Z}F_{ab}$. Applying (5.2) to $r = wx_1^e$ and taking images under $\mathbf{Z}F \twoheadrightarrow \mathbf{Z}F_{ab}$ gives

$$1-\xi_1^e = \bar{D}_1 r \cdot (1-\xi_1) + \bar{D}_2 r \cdot (1-\xi_2),$$

and upon dividing by $(1-\xi_1)$ one obtains

$$1+\xi_1+\xi_1^2+\cdots+\xi_1^{e-1} = \bar{D}_1 r + \bar{D}_2 r \cdot \frac{1-\xi_2}{1-\xi_1}. \tag{5.5}$$

Using (5.5) one has then

$$\delta_2(b) = \left(N - \pi_*\left(\bar{D}_2 r \frac{1}{1-\xi_1}\right) \cdot \pi_*(1-\xi_2), \pi_*\left(\frac{\bar{D}_2 r}{1-\xi_1}\right) \cdot \pi_*(1-\xi_1)\right)$$
$$= (\pi_*(\bar{D}_1 r), \pi_*(\bar{D}_2 r)),$$

as asserted. This shows that G'/G'' is isomorphic to the quotient of $\mathbf{Z}G_{ab} \oplus \mathbf{Z}G_{ab}$ modulo the submodule generated by $(\pi_*(1-\xi_1), 0)$, $(\pi_*(1-\xi_2), N)$ and $(1, \pi_*(\bar{D}_2 r/(1-\xi_1)))$. The third relator can be used to eliminate the generator $(1, 0)$, so that we finally obtain

PROPOSITION 5.1. *Let G be the group* (5.1). *Then the G_{ab}-module G'/G'' is isomorphic to $\mathbf{Z}G_{ab}/I$ where I is the ideal generated by $\pi_*(\bar{D}_1 r)$ and $\pi_*(\bar{D}_2 r)$.*

5.4 Remark. The arguments used to establish Proposition 5.1 have little to do with G being a one relator group. In fact a slight modification yields the following more general result which we record without proof

PROPOSITION 5.2. *Let*

$$G = \langle x_1, x_2; wx_1^e, w_1, w_2, \ldots \rangle,$$

where $w, w_1, w_2, \ldots,$ are words with zero exponent sum on x_1 and x_2, and where $e \geq 1$. Then the G_{ab}-module G'/G'' is isomorphic to $\mathbf{Z}G_{ab}$ modulo the ideal generated by $\pi_*(\bar{D}_1(wx_1^e))$, $\pi_*(\bar{D}_2(wx_1^e))$, $\pi_*(\bar{D}_2 w_i/(1-\xi_1))$ $(i = 1, 2, \ldots)$.

5.5 Now we use the G_{ab}-module presentation of G'/G'' obtained in Proposition 5.1 in order to decide whether $G'/G'' \in \mathcal{k}_t$. If $e = 1$ then $\pi_*(\xi_1) = 1$ and $C = G_{ab}$ is infinite cyclic generated by $t = x_2 G'$. It then follows from (5.2) that $\pi_*(\bar{D}_2 r) = 0$, so that $G'/G'' = \mathbf{Z}C/(\pi_*(\bar{D}_1 r))$ and Lemma 5.4 below applies. In the general case, G'/G'' requires two defining relations, and it seems difficult to extract the information needed to decide whether $G'/G'' \in \mathcal{k}_t$ by a direct approach. This is so even for the simple examples to be discussed in Section 6.2.

For our indirect approach we consider in $\mathbf{Z}G_{ab}$ the two ideals

$$I' = \mathbf{Z}G_{ab} \cdot \pi_*(1-\xi_1) + \mathbf{Z}G_{ab} \cdot \pi_*(\bar{D}_1 r)$$

$$I'' = \mathbf{Z}G_{ab} \cdot \pi_*(1 + \xi_1 + \cdots + \xi_1^{e-1}) + \mathbf{Z}G_{ab} \cdot \pi_*\left(\frac{\bar{D}_2 r}{1-\xi_1}\right).$$

Let Λ denote the ring $\Lambda = \mathbf{Z}[\xi_1]/(1+\xi_1+\cdots+\xi_1^{e-1})$. One has the obvious ring homomorphisms $\rho: \mathbf{Z}G_{ab} \twoheadrightarrow \mathbf{Z}[\xi_2, \xi_2^{-1}]$ and $\sigma: \mathbf{Z}G_{ab} \twoheadrightarrow \Lambda[\xi_2, \xi_2^{-1}]$. These induce isomorphisms

$$\rho_*: \mathbf{Z}G_{ab}/I' \xrightarrow{\cong} \mathbf{Z}[\xi_2, \xi_2^{-1}]/(\rho\pi_*(\bar{D}_1 r)),$$

$$\sigma_*: \mathbf{Z}G_{ab}/I'' \xrightarrow{\cong} \Lambda[\xi_2, \xi_2^{-1}]/\left(\sigma\pi_*\left(\frac{\bar{D}_2 r}{1-\xi_1}\right)\right).$$

which show that both $\mathbf{Z}G_{ab}/I'$ and $\mathbf{Z}G_{ab}/I''$ have 1-relator presentations over suitable rings.

LEMMA 5.3. *Let C be the infinite cyclic group generated by $t = x_2G' \in G_{ab}$. Then the C-module G'/G'' is in the class $\mathfrak{k}_t$ if and only if both $\mathbf{Z}G_{ab}/I'$ and $\mathbf{Z}G_{ab}/I''$ are in $\mathfrak{k}_t$.*

Proof. By Proposition 5.1, G'/G'' is isomorphic to $\mathbf{Z}G_{ab}/I$, where I is the ideal generated by $\pi_*(\bar{D}_1 r)$ and $\pi_*(\bar{D}_2 r)$. As $\bar{D}_2 r$ is a multiple of $1-\xi_1$ we have $I \subseteq I'$ and because of formula (5.5) $I \subseteq I''$. Thus, if the C-module $\mathbf{Z}G_{ab}/I$ is in $\mathfrak{k}_t$, so are its homomorphic images $\mathbf{Z}G_{ab}/I'$ and $\mathbf{Z}G_{ab}/I''$ by Lemma 3.2. For the converse let $\mu: \mathbf{Z}G_{ab} \to \mathbf{Z}G_{ab}$ denote multiplication by $\pi_*(\bar{D}_2 r/(1-\xi_1))$. Then $\mu(I') \subseteq I$ and one has the exact sequence

$$\mathbf{Z}G_{ab}/I' \xrightarrow{\mu_*} \mathbf{Z}G_{ab}/I \to \mathbf{Z}G_{ab}/I'' \to 0,$$

where μ_* is induced by μ and the second arrow stands for the canonical projection. Now, if $\mathbf{Z}G_{ab}/I'$ and $\mathbf{Z}G_{ab}/I''$ are in $\mathfrak{k}_t$, so is $\mathbf{Z}G_{ab}/I$, being an extension of a homomorphic image of $\mathbf{Z}G_{ab}/I'$ by $\mathbf{Z}G_{ab}/I''$ (Lemma 3.2).

5.6 The next two Lemmata, finally, allow to decide when $\mathbf{Z}G_{ab}/I'$ and $\mathbf{Z}G_{ab}/I''$ belong to $\mathfrak{k}_t$.

LEMMA 5.4. *Let R be a commutative ring with unit whose underlying additive group is finitely generated, C an infinite cyclic group, and $f(t) \in RC$. Then the RC-module $M = RC/(f(t))$ is in $\mathfrak{k}_t$ if and only if there is a $g(t) \in RC$ such that $f(t)g(t)$ is a polynomial in $R[t]$ with leading coefficient* 1.

Proof. Let $M \in \mathfrak{k}_t$ and let B be a finitely generated base group of M containing $m = 1 + (f(t))$. Then the Abelian group generated by m, tm, $t^2m, \ldots,$ is contained in

B and hence is finitely generated. Therefore there is a natural number d such that $t^{d+1}m \in gp(m, tm, t^2m, \ldots, t^dm)$. In other words, there is a polynomial

$$h(t) = r_0 + r_1 t + r_2 t^2 + \cdots + r_d t^d + t^{d+1} \in R[t] \tag{5.6}$$

with $h(t)m = 0$, i.e., $h(t) = f(t)g(t)$. Conversely if $h(t) = f(t)g(t)$ with $h(t)$ as in (5.6) then, as R is a finitely generated $\mathbf{Z}$-module, $gp_R(m, tm, t^2m, \ldots, t^dm)$ is a finitely generated base group of M.

It follows from Lemma 5.4 that $RC/(f(t)) \in \mathcal{k}_t$ if the leading coefficient of $p(t)$ is a unit of R; and in the absence of zero divisors, in particular for $R = \mathbf{Z}$, this condition is both necessary and sufficient for $RC/(f(t)) \in \mathcal{k}_t$. The same is true for the ring $R = \Lambda = \mathbf{Z}[x]/(1 + x + \cdots + x^{e-1})$. For, although Λ is not a domain (when e is not prime), one has

LEMMA 5.5. *Let*

$$\Lambda = \mathbf{Z}[x]/(1 + x + x^2 + \cdots + x^{e-1})$$

and let $f(t)$ and $g(t)$ be two polynomials in $\Lambda[t]$. If the product $f(t)g(t)$ has leading coefficient 1 *then the leading coefficients of $f(t)$ and $g(t)$ are units in Λ.*

5.7 In order to prove Lemma 5.5 we need a special result about roots of unity and cyclotomic polynomials. The cyclotomic polynomials $\Phi_m(x) \in \mathbf{Z}[x]$, $m \geq 1$, can be defined inductively by $\Phi_1(x) = x - 1$ and

$$\prod_{d \mid m} \Phi_d(x) = x^m - 1, \qquad m \geq 2, \tag{5.7}$$

where $d \mid m$ stands for "d is a divisor of m." Moreover, there is an explicit description of $\Phi_m(x)$ given by the formula

$$\Phi_m(x) = \prod_{d \mid m} (x^{m/d} - 1)^{\mu(d)}, \qquad m \geq 1, \tag{5.8}$$

where, as is the custom, $\mu : \mathbf{N} \to \mathbf{Z}$ denotes the Möbius function

$$\mu(d) = \begin{cases} 0 & \text{if there is a prime } p \text{ with } p^2 \mid d \\ (-1)^r & \text{if } d \text{ is a product of } r \text{ different primes} \\ 1 & \text{if } d = 1 \end{cases}$$

(c.f. [13], p. 181).

LEMMA 5.6. *Let p be a prime, $r>1$ an integer prime to p, and $\zeta_r \in \mathbf{C}$ a complex primitive r-th root of unity. Then $\Phi_p(\zeta_r) \in \mathbf{C}$ is a unit in the ring of algebraic integers.*

Proof. Since r is prime to p, ζ_r^p is again a primitive r-th root of unity. Therefore there is an integer q, such that $\zeta_r^{pq} = \zeta_r$. As $\Phi_p(x) = 1 + x + \cdots + x^{p-1}$, one has

$$\begin{aligned}\Phi_p(\zeta_r)(1+\zeta_r^p+\zeta_r^{2p}+\cdots+\zeta_r^{(q-1)p}) &= 1+\zeta_r+\zeta_r^2+\cdots+\zeta_r^{pq-1}\\ &= (1-\zeta_r^{pq})/(1-\zeta_r) = 1,\end{aligned}$$

whence $\Phi_p(\zeta_r)$ is a unit in the ring of algebraic integers.

LEMMA 5.7. *Let p be a prime, m a natural number and $\zeta_m \in \mathbf{C}$ a complex primitive m-th root of unity. Then $\Phi_{mp}(\zeta_m) \in \mathbf{C}$ is not a unit in the ring of algebraic integers.*

Proof. Let $m = np^\alpha$ with n prime to p. Using (5.7) or (5.8) one can show that

$$\Phi_{mp}(x) = \Phi_{np}(x^{p^\alpha});$$

but $\zeta_m^{p^\alpha}$ is a primitive n-th root of unity and so we are reduced to the case where m is prime to p. In that case we have $\mu(d)+\mu(dp)=0$ for every natural number d dividing m, and (5.8) yields

$$\begin{aligned}\Phi_{mp}(x) &= \prod_{d|m}\left(\frac{x^{(m/d)p}-1}{x^{(m/d)}-1}\right)^{\mu(d)}\\ &= \prod_{d|m}(x^{(m/d)(p-1)}+x^{(m/d)(p-2)}+\cdots+x^{(m/d)}+1)^{\mu(d)}\\ &= \prod_{d|m}\Phi_p(x^{(m/d)})^{\mu(d)}.\end{aligned}$$

Evaluating at $x=\zeta_m$ yields a product decomposition of $\Phi_{mp}(\zeta_m)$. By Lemma 5.6 all factors $\Phi_p(\zeta_m^{m/d})^{\pm 1}$ with $d>1$ are units in the ring of algebraic integers, whereas the factor corresponding to $d=1$ is $\Phi_p(1)=p$. Therefore the inverse of $\Phi_{mp}(\zeta_m)$cannot be an algebraic integer.

5.8 We are now in a position to prove Lemma 5.4. Let

$$\Lambda = \mathbf{Z}[x]/(1+x+x^2+\cdots+x^{e-1}).$$

As the statement of Lemma 5.4 is trivial for $\Lambda=0$ assume $e\geq 2$. Consider two polynomials in $\mathbf{Z}[x, t]$,

$$f(x,t)=f_0(x)+f_1(x)t+f_2(x)t^2+\cdots+f_k(x)t^k,$$
$$g(x,t)=g_0(x)+g_1(x)t+g_2(x)t^2+\cdots+g_l(x)t^l,$$

with $f_i(x)$, $g_i(x)\in\mathbf{Z}[x]$, and assume that

$f_k(x)\not\equiv 0\not\equiv g_l(x) \bmod 1+x+\cdots+x^{e-1}$. Assume furthermore that the product $f(x,t)g(x,t)$ represents a polynomial in $\Lambda[t]$ with leading coefficient 1. We have to show that both $f(x,t)$ and $g(x,t)$ represent polynomials whose leading coefficients are units in Λ, and clearly this amounts to prove that

$$f_k(x)\cdot g_l(x)\not\equiv 0 \bmod 1+x+\cdots+x^{e-1}.$$

By hypothesis there exists an integer $j\geq 0$ such that the following congruences mod $1+x+\cdots+x^{e-1}$ hold:

$$\begin{aligned}
&f_k(x)\cdot g_l(x) &&\equiv 0\\
&f_k(x)\cdot g_{l-1}(x)+f_{k-1}(x)\cdot g_l(x) &&\equiv 0\\
&\quad\vdots\\
&f_k(x)\cdot g_{l-j+1}(x)+\cdots+f_{k-j+1}(x)\cdot g_l(x) &&\equiv 0\\
&f_k(x)\cdot g_{l-j}(x)+\cdots+f_{k-j}(x)\cdot g_l(x) &&\equiv 1.
\end{aligned}$$

As

$$f_k(x)\not\equiv 0 \bmod 1+x+\cdots+x^{e-1},$$

not every e-th root of unity $\zeta\neq 1$ is a zero of $f_k(x)$. Let ζ_m be an m-th primitive root of unity, where m is an integer $\neq 1$ dividing e, such that $f_k(\zeta_m)\neq 0$. If we evaluate the above congruences at $x=\zeta_m$ we obtain successively that

$$0=g_l(\zeta_m)=g_{l-1}(\zeta_m)=\cdots=g_{l-j+1}(\zeta_m),$$

and hence

$$f_k(\zeta_m)\cdot g_{l-j}(\zeta_m)=1.$$

This shows that $f_k(\zeta_m) \in \mathbf{C}$ is a unit in the ring $\bar{\mathbf{Z}}$ if all algebraic integers. It follows by Lemma 5.7 that, in $\bar{\mathbf{Z}}$, $f_k(\zeta_m)$ cannot be a multiple of $\Phi_{mp}(\zeta_m)$ for any prime p. Hence the cyclotomic polynomial $\Phi_{mp}(x)$, which is irreducible over $\mathbf{Q}$, does not occur in the prime factorization of $f_k(x)$. Therefore we have $f_k(\zeta_{mp}) \neq 0$ for every prime p and all primitive mp-th roots of unity ζ_{mp}. Iterating this argument eventually yields that $f_k(\zeta_e) \neq 0$ for every primitive e-th root of unity ζ_e. Now the situation is symmetric in f and g, hence the same holds for g_l. This shows that

$$f_k(x)g_l(x) \not\equiv 0 \bmod 1+x+\cdots+x^{e-1}, \text{ as asserted.}$$

6. Two generator one relator groups in $\mathfrak{A}^2$: the decision procedure

6.1 By the results of the preceding Section 5 we are now in a position to decide whether a given two generator one relator group in the variety $\mathfrak{A}^2$ is finitely presented (in $\mathfrak{D}$). We summarize this decision procedure.

Let G be a one relator group of the form

$$G = \langle x_1, x_2;\ w(x_1, x_2)x_1^e \rangle, \tag{6.1}$$

where $w = w(x_1, x_2)$ is a commutator word and e an integer ≥ 1. Let F be the free group on $\{x_1, x_2\}$, $\xi_i = x_i[F, F] \in F_{ab}$, $i = 1, 2$, and let $\bar{D}_1 : F \to \mathbf{Z}F_{ab}$ denote the abelianized Fox-derivation with respect to x_1. As $w \in F'$, $\bar{D}_1 w \in \mathbf{Z}F_{ab}$ is a multiple of $1-\xi_2$. Let

$$L(\xi_1, \xi_2) \in \mathbf{Z}[\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1}]$$

be the Laurent polynomial

$$L(\xi_1, \xi_2) = \frac{\bar{D}_1 w}{1-\xi_2},$$

and determine coefficients $z_{\min}$, $z_{\max}$, $\lambda_{\min}$, $\lambda_{\max}$ as follows: Firstly, take the Laurent polynomial

$$(1-\xi_2)L(1, \xi_2) + e = \sum_{i=-\infty}^{+\infty} z_i \xi_2^i \in \mathbf{Z}[\xi_2, \xi_2^{-1}],$$

and write $z_{\min}$ and $z_{\max}$, respectively, for the first and last non-vanishing coefficient $z_i \in \mathbf{Z}$ (note that the coefficients of this L-polynomial sum up to e).

Secondly, consider the ring

$$\Lambda = \mathbf{Z}[\xi_1]/(1+\xi_1+\xi_1^2+\cdots+\xi_1^{e-1})$$

(notice that $\Lambda = 0$ when $e=1$) and the obvious map

$$\vartheta : \mathbf{Z}F_{ab} \twoheadrightarrow \Lambda[\xi_2, \xi_2^{-1}],$$

and take the Laurent polynomial

$$\vartheta(L(\xi_1, \xi_2)) = \sum_{i=-\infty}^{+\infty} \lambda_i \xi_2^i \in \Lambda[\xi_1, \xi_2^{-1}].$$

Write $\lambda_{\min}$ and $\lambda_{\max}$, respectively, for the first and last non-vanishing coefficient $\lambda_i \in \Lambda$; if $\lambda_i = 0$ for all i set $\lambda_{\min} = 0 = \lambda_{\max}$. Then one has

THEOREM D. *The metabelian group G/G'' is (almost) finitely presented if and only if either of the following conditions holds:*

(i) $z_{\min} = \pm 1$ and $\lambda_{\min}$ is a unit in Λ,

(ii) $z_{\max} = \pm 1$ and $\lambda_{\max}$ is a unit in Λ.

Proof. Using the notation of Section 5.5 one has

$$(1-\xi_2)L(1, \xi_2) + e = \rho\pi_*(\bar{D}_1 r)$$

and

$$\vartheta(L(\xi_1, \xi_2)) = -\sigma\pi_*(\bar{D}_2 r/(1-\xi_1)).$$

Hence, by Lemmata 5.4 and 5.5, condition (i) is equivalent with requiring that both C-modules $\mathbf{Z}G_{ab}/I'$ and $\mathbf{Z}G_{ab}/I''$ be in $\mathcal{k}_t$ (if $e=1$, then Λ is trivial and 0 is a unit), and this in turn is equivalent with $G'/G'' \in \mathcal{k}_t$ by Lemma 5.3. Similarly, (ii) is equivalent with $G'/G'' \in \mathcal{k}_{t^{-1}}$. The claim thus follows from Theorem C. □

Remarks. 1) The assertion of Theorem D holds more generally for every quotient $G/\mathfrak{V}(G)$, where $\mathfrak{V}$ is a variety with $\mathfrak{A}^2 \subseteq \mathfrak{V} \subseteq \mathfrak{N}_c\mathfrak{A}$ (see Corollary C1).

2) Recall that *any* two-generator one-relator group G whose Abelianization G_{ab} is of torsion-free rank 1 has a presentation of the form (6.1) (cf. [14], Theorem 3.5).

3) The question whether a given element $\lambda \in \Lambda$ is a unit can be decided, e.g., by regarding λ as an endomorphism of the (finitely generated free-Abelian) additive group of Λ and computing its determinant.

6.2 Illustration. The one relator groups

$$G=\langle x_1, x_2; x_2^r x_1^m x_2^{-r}=x_1^n\rangle=\langle x_1, x_2; [x_2^r, x_1^m]x_1^{m-n}\rangle \qquad mnr\neq 0,$$

have been studied in [5]. If $e=m-n=0$ then either $|r|=|m|=1$ and G is free-Abelian of rank two, or G/G'' maps onto a wreath product of the form $\mathbf{Z}_p \sim \mathbf{Z}$ and hence cannot be finitely related (e.g. by Theorem A).

Assume now $e=m-n>0$ (the case $e<0$ can easily be reduced to this). The Fox-derivative of $w=x_2^r x_1^m x_2^{-r} x_1^{-m}$ with respect to x_1 is

$$D_1 w = x_2^r(1-x_1^m x_2^{-r} x_1^{-m})D_1(x_1^m),$$

hence

$$\bar{D}_1 w = -(1-\xi_2^r)\cdot\frac{1-\xi_1^m}{1-\xi_1},$$

and we obtain the L-polynomial

$$L(\xi_1, \xi_2) = -\left(\frac{1-\xi_1^m}{1-\xi_1}\right)\cdot\left(\frac{1-\xi_2^r}{1-\xi_2}\right).$$

Next we have to compute

$$(1-\xi_2)\cdot L(1, \xi_2)+e=-m(1-\xi_2^r)+e=m\xi_2^r+n,$$

from which we deduce, by Theorem D, that a necessary condition for G/G'' to be finitely presented is $|m|=1$ or $|n|=1$.

On the other hand one has

$$L(\xi_1, \xi_2)=u(1+\xi_1+\cdots+\xi_1^{|m|-1})(1+\xi_2+\cdots+\xi_2^{|r|-1}),$$

with $u=\pm\xi_1^\alpha\xi_2^\beta$, hence the second condition in Theorem D requires that $1+\xi_1+\cdots+\xi_1^{|m|-1}$ be a unit in

$$\Lambda=\mathbf{Z}[\xi_1]/(1+\xi_1+\cdots+\xi_1^{e-1}).$$

But as $e=m-n$, this is equivalent with $1+\xi_1+\cdots+\xi_1^{|n|-1}$ being a unit in Λ, and hence the second condition of Theorem D is automatically satisfied if $|m|=1$ or $|n|=1$.

We summarize: *if*

$$G=\langle x, y; x^r y^m x^{-r}=y^n\rangle, \qquad mnr\neq 0, \qquad m\neq n,$$

then G/G'' is almost finitely presented (over some ring K) if and only if $|m|=1$ *or* $|n|=1$. (cf. [5]; in particular, this establishes the addendum to the theorem on p. 48, though not with the method sketched on p. 51.)

REFERENCES

[1] BAUMSLAG, G.: *Finitely presented metabelian groups.* Proc. Second Internat. Conf. Theory of Groups, Canberra 1973, 65–74; Lecture Notes in Math. No 372, Springer Berlin-Heidelberg-New York 1974.
[2] BAUMSLAG, G.: *Subgroups of finitely presented metabelian groups.* J. Australian Math. Soc. *14* (1973), 98–110.
[3] BAUMSLAG, G.: *Multiplicators and metabelian groups.* J. Australian Math. Soc. *22* (1976), 305–312.
[4] BAUMSLAG, G. and BIERI, R.: *Constructable solvable groups,* Math. Z. *151* (1976), 249–257.
[5] BAUMSLAG, G. and STREBEL, R.: *Some finitely generated, infinitely related metabelian groups with trivial multiplicator.* J. Algebra *40* (1976), 46–62.
[6] BIERI, R.: *Homological dimension of discrete groups.* Queen Mary College Math. Notes, Queen Mary College, London 1976.
[7] BOLER, J.: *Subgroups of finitely presented metabelian groups of finite rank.* J. Austral. Math. Soc., *22*(A) (1976), 501–508.
[8] CROWELL, R. H. and FOX, R. H.: *Introduction to knot theory.* Ginn and Co., 1963.
[9] FOX, R. H.: *Free differential calculus, I. Derivation in the free group ring.* Ann. of Math. *57* (1953), 547–560.
[10] HALL, P.: *Finiteness conditions for soluble groups.* Proc. London Math. Soc. (3) *4* (1954), 419–436.
[11] HALL, P.: *The Frattini subgroups of finitely generated groups,* Proc. London Math. Soc. (3) *11* (1961), 327–352.
[12] KARRASS, A. and SOLITAR, D.: *Subgroups of HNN-groups and groups with one defining relation.* Canad. J. of Math. *23* (1971), 627–643.
[13] HASSE, H.: *Vorlesungen über Zahlentheorie. Die Grundlehren der Mathematischen Wissenschaften* 59, Springer Verlag, Berlin-Heidelberg-New York 1964.
[14] MAGNUS, W., KARRASS, A. and SOLITAR, D.: *Combinatorial group-theory.* Interscience 1966.
[15] ROBINSON, D. J. S.: *Finiteness conditions and generalized soluble groups.* Ergebnisse der Math. und ihrer Grenzgebiete, Springer Verlag, Berlin-Heidelberg-New York 1972.
[16] ŠMEL'KIN, A. L.: *On soluble products of groups.* Sibirsk. Mat. Ž. *6* (1965), 212–220; MR 32 #2464.
[17] STREBEL, R.: *Homological methods applied to the derived series of groups.* Comment. Math. Helv. *49* (1974), 302–332.
[18] THOMSON, M.: *Finitely presented soluble groups.* Trans. Amer. Math. Soc., to appear.
[19] TROTTER, H. F.: *Torsion-free metabelian groups with infinite cyclic quotient groups.* Proc. Second Intern. Conf. Theory of Groups, Canberra 1973, 655–666; Lecture Notes in Math. No 372, Springer 1974.

Mathematisches Institut
der Universität Freiburg
Albertstr. 23b
7800 Freiburg i. Br.
Germany

Mathematisches Institut
der Universität Heidelberg
Im Neuenheimer Feld 288
6900 Heidelberg
Germany

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