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Asymptotic FitzGerald inequalities

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Dedicated to Professor Albert Pfluger on his seventieth birthday

Abstract. In this article we use the asymptotic behavior of the positive semi-definite FitzGerald matrix to get by elementary means Hayman's Regularity Theorem and a sharpening of an approximation theorem of Lebedev. Moreover we show that there is an absolute constant n_0 such that for any $f = z + a_2z^2 + \cdots \in S$ with $|a_2| < 1.78$ we have $|a_n| < n$ for all $n > n_0$.

1. Introduction

Let S denote the class of all normalized univalent functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ defined in the unit disk $U = \{z; |z| < 1\}$. The Bieberbach Conjecture states that for functions in S one has $b_n = |a_n| \leq n$ for all $n \in \mathbf{N}$. It is known to be true for $n \leq 6$. The best known estimate for all coefficients is $b_n \leq (1.069)n$ (Horowitz [1976]). On the other hand Hayman's Regularity Theorem (Hayman [1955]) states that $\lim_{n\to\infty} b_n/n \leq 1$ for each $f \in S$, and that equality holds only for the Koebe-functions $k(z) = z/(1-\eta z)^2$, $|\eta| = 1$, for which $b_n = n$. This implies that $b_n \leq n$ for $n \geq n_0(f)$.

The first author (Bshouty [1976a, 1976b]) has shown, that (a) if $b_2 < 1.61$, then $b_n < n$ for all $n \in \mathbf{N}$, (b) if $b_2 < 1.75$, then there is an absolute constant n_0 (independent of f) such that $a_n < n$ for all $n > n_0$.

The proofs of these two results were based on the FitzGerald inequalities and uses lengthy calculations. In this paper we investigate the asymptotic behavior of the FitzGerald inequalities to get by elementary means Hayman's Regularity Theorem, an improvement of the above result (b) and a sharpening of an approximation theorem in S.

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2. Asymptotic behavior of the elements of the FitzGerald matrix

The following theorem, due to FitzGerald [1972], is given in another formulation (Pommerenke [1975]):

THEOREM A (FitzGerald Inequalities) *Let* $f(z)=z+\sum_{k=2}^{\infty} a_k(f)z^k$ *be in* S *and define*

$$q_{mn}(f)=q_{nm}(f)=\left(\sum_{j=1}^{n+m-1}\beta_j(m,n)b_j^2(f)\right)-b_m^2(f)b_n^2(f) \tag{1}$$

where $b_j(f)=|a_j(f)|$; $\beta_j(m,n)=\beta_j(n,m)$, $j\in\mathbf{N}$, *and for* $m<n$:

$$\beta_j(m,n)=\begin{cases} m-|j-n| & \text{for } |j-n|<m \\ 0 & \text{if otherwise.}\end{cases}$$

Then the FitzGerald matrix

$$Q(f)=(q_{m,n}(f))_{m,n\in\mathbf{N}} \tag{2}$$

is positive semi-definite.

The elements q_{mn} are complicated in themselves and it is not easy to handle them, but asymptotically they behave nicely. The following lemma approximates q_{mn} for large n.

LEMMA 1. *Let* $\{f_k\}$, $k\in\mathbf{N}$, *be a sequence in* S *that converges locally uniformly to a function* $f\in S$. *Denote by*

$$\delta_n=\sup_k b_n(f_k)/n \quad \text{and} \quad \delta=\limsup_{n\to\infty}\delta_n \tag{3}$$

Then for any β *with*

$$\liminf_{n\to\infty} b_n(f_n)/n\leqslant\beta\leqslant\limsup_{n\to\infty} b_n(f_n)/n \tag{4}$$

there is a subsequence $\{f_{n_k}\}$; $k\in\mathbf{N}$, $k\to\infty$, *such that for fixed* m

$$\lim_{k\to\infty} q_{mn_k}(f_{n_k})/n_k^2=[m^2-b_m^2(f)]\cdot\beta^2 \tag{5}$$

and

$$\lim_{k\to\infty} q_{n_k n_k}(f_{n_k})/n_k^4 \leq (7/6)\delta^2 - \beta^4 \tag{6}$$

Proof. Let us first note that for $m < n$

$$\sum_{j=1}^{n+m-1} \beta_j(m, n) = \sum_{j=n-m+1}^{n+m-1} \beta_j(m, n) = m^2 \tag{7}$$

and

$$\sum_{j=1}^{2n-1} j^2 \cdot \beta_j(n, n) = n^2(7n^2 - 1)/6 \leq 7n^4/6 \tag{8}$$

In addition, we have for $k = \pm 1, \pm 2, \ldots, \pm m$

$$|b_n^2 - b_{n-k}^2| = 0(n^{3/2}) \quad \text{for} \quad n \to \infty \tag{9}$$

that follows directly from the FitzGerald inequalities (see e.g. Pommerenke [1975] and Horowitz [1972]). Note that we could replace in (9) $0(n^{3/2})$ by $0(n)$ using a result of Milin that $|b_n - b_{n-1}| \leq 4.18$ for $f \in S$.

The relations (1) and (7) and the definition of the $\beta_j(m, n)$ give for fixed $m < n$:

$$\begin{aligned} q_{mn}(f_n) - (m^2 - b_m^2(f_n)) \cdot b_n^2(f_n) &= \sum_{j=1}^{n+m-1} \beta_j(m, n)[b_j^2(f_n) - b_n^2(f_n)] \\ &= \sum_{j=n-m+1}^{n+m-1} \beta_j(m, n)[b_j^2(f_n) - b_n^2(f_n)] = 0(n^{3/2}) \quad \text{as} \quad n \to \infty. \end{aligned}$$

Now choose a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\lim_{k\to\infty} b_{n_k}(f_{n_k})/n_k = \beta$ and (5) is established.

For (6) we use this argument. Given $\gamma > \delta$, then there is an $n_0(\gamma)$ such that $b_n(f_k) < \gamma \cdot n$ for all $k \in \mathbf{N}$ and for all $n > n_0(\gamma)$. Choose $\{f_{nk}\}$ as before; thus for $n_k > n_0(\gamma)$ we have by (1), (8) and using the fact that $\beta_j(n_k, n_k) = j$ for $1 \leq j \leq n_0(\gamma) < n_k$:

$$\begin{aligned} q_{n_k n_k}(f_{n_k}) + b_{n_k}^4(f_{n_k}) &= \left(\sum_{j=1}^{n_0(\gamma)} + \sum_{j=n_0(\gamma)+1}^{2n_k-1}\right) \beta_j(n_k, n_k) b_j^2(f_{n_k}) \\ &\leq (1.069) \cdot \sum_{j=1}^{n_0(\gamma)} \beta_j(n_k, n_k) \cdot j^2 + \sum_{n_0(\gamma)+1}^{2n_k-1} \gamma^2 \cdot j^2 \cdot \beta_j(n_k, n_k) \\ &\leq \text{const}\,(\gamma) + 7\gamma^2 n_k^4/6. \end{aligned}$$

Divide both sides by n_k^4 and pass to the limit. The result (6) follows, if we let γ tend to δ by choosing simultaneously an appropriate subsequence of $\{f_{n_k}\}$.

Let now $\{g_n \in S\}$, $n \in \mathbf{N}$, satisfy

$$\sup\{b_n(f); f \in S\} = b_n(g_n). \tag{10}$$

The asymptotic Bieberbach limit α is given by

$$\alpha = \limsup_{n\to\infty} b_n(g_n)/n. \tag{11}$$

Evidently $\alpha \geqslant 1$.

COROLLARY. *There exists a subsequence* $\{g_{n_k}\}$, $k \in \mathbf{N}$ *of* $\{g_n\}$, $n \in \mathbf{N}$, *such that*

(a) g_{n_k} *converges locally uniformly to a function* $g \in S$

(b) $\lim_{k\to\infty} q_{mn_k}(g_{n_k})/n_k^2 = (m^2 - b_m^2(g)) \cdot \alpha^2$

(c) $\limsup_{k\to\infty} q_{n_k n_k}(g_{n_k})/n_k^4 \leqslant (7\alpha^2/6) - \alpha^4$.

Proof. Take first a subsequence $\{g_{n_k}\}$, $k \in \mathbf{N}$ of $\{g_n\}$, $n \in \mathbf{N}$, such that $\lim_{k\to\infty} b_{n_k}(q_{n_k})/n_k = \alpha$ and then a subsequence of it such that (a) holds.

3. Hayman's regularity theorem:

Before we study the asymptotic FitzGerald inequalities in the general form, we want to show the procedure at the simple case $f_n = f \in S$ for all $n \in \mathbf{N}$, by proving in a elementary way Hayman's Regularity Theorem. The idea of the method is significant for all what follows. Another lengthier proof was done by Horowitz [1972].

We consider for $j_1 < j_2 < \ldots < j_p$ the principal minor

$$Q(j_1, j_2, \ldots, j_p)(f) = (q_{j_s j_t}(f))_{1 \leqslant s \leqslant p, 1 \leqslant t \leqslant p} \tag{12}$$

of the positive semi-definite matrix $Q(f)$ in (2) corresponding to the FitzGerald inequalities for a given function $f \in S$. The matrix (12) is again positive semi-definite. Denote by

$$\alpha = \limsup_{n\to\infty} b_n(f)/n \quad \text{and} \quad \tau = \liminf_{n\to\infty} b_n(f)/n.$$

We apply Lemma 1 with $f_n = f$, $\delta = \sigma$ and $\beta_p \in [\tau, \sigma]$ to the last row and last column of $Q(j_1, \ldots, j_p)(f)$, after dividing both of them by j_p^2. This means that we let $j_1, \ldots, j_{p-1}$ to be fixed and pick then according to Lemma 1 a subsequence $\{n_k\}$ from j_p such that $\lim_{k\to\infty} b_{n_k}(f)/n_k = \beta_p \in [\tau, \sigma]$. Since the elementwise limit of a finite positive semi-definite matrix stays positive semi-definite and since the addition of a positive term to a diagonal element does not effect the positive semi-definiteness, we conclude that the matrix $Q(j_1, j_2, \ldots, j_{p-1}, \beta_p)(f)$ with the elements:

$$\begin{cases} q_{j_s j_t}(f) & \text{for} \quad 1 \leq s \leq p-1, 1 \leq t \leq p-1 \\ \beta_p^2(j_s^2 - b_{j_s}^2(f)) & \text{for } 1 \leq s \leq p-1, t = p \\ \beta_p^2(j_t^2 - b_{j_t}^2(f)) & \text{for } s = p, 1 \leq t \leq p-1 \\ 7\sigma^2/6 - \beta_p^4 & \text{for } s = p, t = p \end{cases}$$

is positive semi-definite. Note that $\beta_p < 1.07$. Therefore there is no confusion about j_p and β_p in the above notation. This same procedure we apply to the $(p-1)^{\text{th}}$ column and $(p-1)^{\text{th}}$ row in letting $j_1, j_2, \ldots, j_{p-2}$ fixed. After dividing both of them by j_{p-1}^2 we choose again a subsequence $\{n_k\}$ from j_{p-1}, such that $\lim_{k\to\infty} b_{n_k}(f)/n_k = \beta_{p-1} \in [\tau, \sigma]$. Then the matrix $Q(j_1, j_2, \ldots, j_{p-2}, \beta_{p-1}, \beta_p)(f)$ with the elements:

$$\begin{cases} q_{j_s j_t}(f) & \text{for} \quad 1 \leq s \leq p-2, 1 \leq t \leq p-2 \\ \beta_t^2(j_s^2 - b_{j_s}^2(f)) & \text{for} \quad 1 \leq s \leq p-2, p-1 \leq t \leq p \\ \beta_s^2(j_t^2 - b_{j_t}^2(f)) & \text{for} \quad p-1 \leq s \leq p, 1 \leq t \leq p-2 \\ 7\sigma^2/6 - \beta_t^4 & \text{for} \quad t = s, p-1 \leq t \leq p \\ \beta_p^2(1 - \beta_{p-1}^2) & \text{for} \quad s = p-1, t = p \quad \text{and} \quad s = p, t = p-1 \end{cases} \tag{13}$$

is again positive semi-definite.

We continue by the same way for the $(p-2)^{\text{th}}$ column and $(p-2)^{\text{th}}$ row and so on. We finally get that the $p \times p$ matrix $Q(\beta_1, \ldots, \beta_p)$ with the elements

$$\begin{cases} \beta_s^2(1 - \beta_t^2) & \text{for} \quad s > t \\ \beta_t^2(1 - \beta_s^2) & \text{for} \quad t > s \\ 7\sigma^2/6 - \beta_t^4 & \text{for} \quad t = s, 1 \leq t \leq p \end{cases} \tag{14}$$

is again positive semi-definite.

Now we want to show that $\sigma=\tau\leqslant 1$ and that equality holds only for the Koebe-functions.

First we put in (14) $\beta_1=\beta_2=\ldots=\beta_p=\sigma$. The matrix $Q(\sigma,\sigma,\ldots,\sigma)$ we denote (for later purposes) by $M_p(\sigma)$ and their elements are

$$m_{st}(\sigma)=\begin{cases}7\sigma^2/6-\sigma^4 & \text{for} \quad s=t\\ \sigma^2(1-\sigma^2) & \text{for} \quad s\neq t\end{cases}. \tag{15}$$

The determinant of $M_p(\sigma)$ is $\sigma^{2p}(6p-6\sigma^2p+1)/6^p$ and has to be non-negative for all $p\in\mathbf{N}$, that implies $\sigma\leqslant 1$.

Next put $j_1=2$, $\beta_2=\beta_3=\ldots=\beta_{p+1}=\sigma$. The determinant of $Q(2,\sigma,\sigma,\ldots,\sigma)(f)$ is then

$$q_{22}\cdot\sigma^{2p}(6p-6\sigma^2p+1)/6^p-p\cdot(4-b_2^2)^2\sigma^{2p-1}/6^{p-1}\geqslant 0$$

for all $p\in\mathbf{N}$. We assume now that $\sigma=1$, then

$$q_{22}\geqslant(4-b_2^2)^2 6p \quad \text{for all} \quad p\in\mathbf{N}$$

that implies $b_2=2$. By the area theorem it follows that f is a Koebe-function for which $b_n=n$ and therefore $\sigma=\tau=1$.

It remains to show that $\sigma=\tau$ for $0<\sigma<1$. Choose $\beta_1=\beta_2=\ldots=\beta_n=\tau$ and $\beta_{n+1}=\beta_{n+2}=\ldots=\beta_{2n}=\sigma$. The determinant of $Q(\tau,\tau,\ldots,\tau,\sigma,\sigma,\ldots.,\sigma)$ is $\sigma^{2n}6^{-2(n-1)}(7\sigma^2-6\tau^2)^{n-1}\cdot\{n^2[\tau^2(1-\sigma^2)(1-\tau^2)-\sigma^2(1-\tau^2)^2]+0(n)\}\geqslant 0$ for all $n\in\mathbf{N}$. Since $\tau<1$ and $\sigma>0$ we have $\tau^2(1-\sigma^2)\geqslant\sigma^2(1-\tau^2)$ that implies $\tau\geqslant\sigma$; i.e. $\tau=\sigma$.

4. Asymptotic FitzGerald inequalities

Let $\{f_n\}$, $n\in\mathbf{N}$, be a sequence of univalent functions in S that converges locally uniformly to a function $f\in S$. With $c(f_n)$ we denote the limit

$$c(f_n)=\lim_{k\to\infty} b_k(f_n)/k \tag{16}$$

which by Hayman's Regularity Theorem exists. We may assume that the limit

$$d=\lim_{n\to\infty} c(f_n) \tag{17}$$

exists. Otherwise we pick a subsequence of $\{f_n\}$, $n\in\mathbf{N}$.

For each fixed $f_n, n>q-1$ we consider the principal minor $Q(j_1, j_2, \ldots, j_{q-1}, n, j_{q+1}, \ldots . j_p)(f_n)$ of $Q(f_n)$ in (2) where $j_1<j_2<\ldots<j_{q-1}<n<j_{q+1}<\ldots<j_p$. We apply now the same procedure as before in section 3 to the columns and rows $j_p, j_{p-1}, \ldots, j_{q+1}$. The obtained matrix $Q(j_1, j_2, \ldots, j_{q-1}, n, c(f_n), c(f_n), \ldots, c(f_n))(f_n)$ is positive semi-definite.

Next, we go with n to infinity and keep $j_1, \ldots, j_{q-1}$ fixed. Choose β with

$$\liminf_{n\to\infty} b_n(f_n)/n \leq \beta \leq \limsup_{n\to\infty} b_n(f_n)/n.$$

By Lemma 1 we have a subsequence $\{f_{n_k}\}, k\in\mathbf{N}$, of $\{f_n\}$, $n\in\mathbf{N}$, such that $\lim_{k\to\infty} b_{n_k}(f_{n_k})/n_k=\beta$ and such that the relations (5) and (6) in Lemma 1 hold. After dividing the $j_q{}^{\text{th}}$ column and row of $Q(j_1, \ldots, j_{q-1}, n_k, c(f_{n_k}), \ldots, c(f_{n_k})(f_{n_k})$ by n_k^2 we let n_k go to infinity. The elements $q_{j_sj_t}(f_{n_k})$, $1\leq s, t\leq q-1$ contain only the first $2j_{q-1}$ coefficients of f_{n_k}; therefore $\lim_{k\to\infty} q_{j_sj_t}(f_{n_k})=q_{j_sj_t}(f)$ for $1\leq s, t\leq q-1$. We denote the so obtained positive semi-definite matrix with $Q(j_1, \ldots, j_{q-1}, \beta, d, d, \ldots, d)(f)$.

Once more we apply the procedure described in section 3 to $j_{q-1}, j_{q-2}, \ldots, j_r$ with respect to the function f. The result is

THEOREM 1 (Asymptotic FitzGerald inequalities) *Let $\{f_n\}$, $n\in\mathbf{N}$, be a sequence of functions in S, such that*

a) *f_n converges locally uniformly to a function $f\in S$*

b) $\liminf_{n\to\infty} b_n(f_n)/n \leq \beta \leq \limsup_{n\to\infty} b_n(f_n)$

c) $c(f)=\lim_{n\to\infty} b_n(f)/n$

d) $d=\lim_{n\to\infty} c(f_n)$

Then $A=Q(j_1, j_2, \ldots, j_{r-1}, c(f), \ldots, c(f), \beta, d, \ldots, d)(f)$ is a positive semi-definite matrix.

Denote by E_{mn} the $m\times n$ matrix whose elements are all equal to be one. Moreover let $H_{mn}(f)$ be the $m\times n$ matrix defined by its elements $h_{st}(f)=j_t^2-b_{j_t}^2(f)$. With the notation of $Q(j_1, \ldots, j_{r-1})(f)$ in (12), $M_p(x)$ in (15), and δ in (3) the matrix A has the following form:

$$\begin{pmatrix} Q(j_1,\dots,j_{r-1}(f), & c(f)H_{r-1,q-r}(f), & \beta^2 H_{r-1,1}(f), & d^2 H_{r-1,p-q}(f) \\ c(f)H^T_{r-1,q-r}(f), & M_{q-r}(c(f)), & \beta^2(1-c^2(f))E_{q-r,1}, & d^2(1-c^2(f))E_{q-r,p-q} \\ \beta^2 H^T_{r-1,1}(f), & \beta^2(1-c^2(f))E_{1,q-r}, & (7\delta^2/6-\beta^4)E_{1,1}, & d^2(1-\beta^2)E_{1,p-q} \\ d^2 H^T_{r-1,p-q}(f), & d^2(1-c^2(f))E_{p-q,q-r}, & d^2(1-\beta^2)E_{p-q,1}, & M_{p-q}(d) \end{pmatrix}$$

where H^T is the transposed matrix of H.

5. Applications

For the first two applications we need

LEMMA 2. *Let* $\{f_n\}$, $n\in\mathbf{N}$ *be a sequence of univalent functions in* S, *that converges locally uniformly to a function* f *in* S *and suppose that* $c(f)>0$. *Then* $7\delta^2c^2(f)\geq 6\beta^4$, *where* β *is choosen as in Theorem* 1 *and* δ *is defined in* (3).

Proof. Consider the $(n+1)\times(n+1)$ principal minor $Q(c(f),\dots,c(f),\beta)$ of the matrix A in Theorem 1. Then its determinant is with $c=c(f)$:

$$(c^2/6)^n(1-c^2)[n(7\delta^2-6\beta^4/c^2)+\beta^4/c^4]+c^{2n}6^{-(n+1)}(7\delta^2-6\beta^4/c^2)\geq 0,$$

THEOREM 2. *For* $n\in\mathbf{N}$, *let* g_n *be in* S *such that* $b_n(g_n)=\sup(b_n(f);\ f\in S)$ (*see* (10)). *Then we have for any limit function* g *of* $\{g_n\}$, $n\in\mathbf{N}: c(g)\geq 0.92$.

Proof. Let $\{g_{n_k}\}$, $k\in\mathbf{N}$ be a subsequence of $\{g_n\}$, $n\in\mathbf{N}$, that converges locally uniformly to g. We take $\beta=\delta$. Note that δ of this subsequence is at least one.

First we show that $c(g)>0$. In fact the determinant of the 2×2 submatrix $Q(c(g),\delta)$ of A is

$$(7c^2(g)-6c^4(g))(7\delta^2-6\delta^4)/36-\delta^4(1-c^2(g))^2\geq 0$$

that excludes $c(g)=0$.

Now apply Lemma 2 to $f_k=g_{n_k}$ and we get $c^2(g)\geq 6\delta^2/7\geq 6/7$ or $c(g)\geq \sqrt{6/7}>0.92$.

THEOREM 3. *Let* $f(z)=z+\sum_{k=2}^{\infty}a_kz^k$ *be in* S. *If* $|a_2|<1.78$, *then there is an absolute constant* n_0 (*independent of* f), *such that* $|a_n|<n$ *for all* $n>n_0$.

Proof. Suppose, to the contrary, that there exists a sequence $\{h_k\}$, $k \in \mathbf{N}$, of univalent functions in S such that

a) h_k converges locally uniformly to a function $h_0 \in S$,

b) $b_2(h_k) < 1.78$,

c) $b_{n_k}(h_k) \geqslant n_k$ for a sequence n_k going to infinity.

We pick now for each n_k one of the functions of $\{h_j\}$, $j = 0, 1, \ldots$, that maximizes b_{n_k} and denote it by f_k. We choose a subsequence $\{f_{k_j}\}, j \in \mathbf{N}$, of $\{f_k\}$, $k \in \mathbf{N}$, that converges locally uniformly to a function $f \in S$ and we take $\beta = \delta \geqslant 1$. Evidently $b_2(f) \leqslant 1.78$.

As before $c(f) > 0$, since Det $Q(c(f), \delta) \geqslant 0$ implies for $c(f) = 0$ that $\delta = 0$. By Lemma 2 we have

$$7c^2(f)\delta^2 - 6\delta^4 \geqslant 0 \quad \text{or} \quad c^2(f) \geqslant 6\delta^2/7 \geqslant 6/7.$$

This implies by a theorem of Jenkin and Hayman (see Hayman [1958]), that $b_2(f) > 1.78$ that contradicts the assumptions.

Let $S_x = \{f \in S; c(f) = x\}$. Given $0 \leqslant x_1 < x_2 \leqslant 1$, Lebedev [1941] proved that each function in S_{x_2} can be approximated locally uniformly by univalent functions in S_{x_1}. We show that the converse is not true for any function in S_{x_1}.

THEOREM 4. *Let $0 \leqslant x_1 < x_2 \leqslant 1$. Then no function in S_{x_1} can be approximated locally uniformly by functions in S_{x_2}.*

Proof. Let $\{f_n\}$, $n \in \mathbf{N}$, be a sequence in S_{x_2} that converges locally uniformly to a function $f \in S$. We consider the $2n \times 2n$ principal minor $Q(c(f), \ldots, c(f), d, \ldots, d)$ with n elements $d = x_2$ of the matrix A in Theorem 1. Its determinant is

$$c^{2n}(f)d^{2n}6^{2-2n}\{n^2(1-c^2)(1-d^2/c^2)+0(n)\} \quad \text{as} \quad n \to \infty.$$

For $c(f) = 0$ we get from the determinant of $Q(c(f), d)$ that $d = 0$, what is a contradiction of the assumption. Let $c(f) > 0$. Then we have, for $n \to \infty$, that $c^2(f) \geqslant d^2$, i.e. $f \in S_x$, $x \geqslant x_2$.

COROLLARY. *The functional $c(f)$ is upper semi-continuous on S.*

Proof. Let $\{f_n\}$, $n \in \mathbf{N}$, be sequence in S. We pick a subsequence $\{f_{n_k}\}$, $k \in \mathbf{N}$ of $\{f_n\}$, $n \in \mathbf{N}$, such that

a) f_{n_k} converges locally uniformly to a function $f \in S$

b) $\lim\limits_{k \to \infty} c(f_{n_k}) = \limsup\limits_{n \to \infty} c(f_n) = d.$

Applying the proof of Theorem 4 the Corollary follows.

Remarks. Using the same method as above for Theorem 2, 3 and 4 we get the following results:

1) The principal minor $Q(k, c(f), \ldots, c(f))$ of the matrix A gives rise to

$$q_{kk}(1-c^2(f)) \geq c^2(f)(k^2 - b_k^2(f))^2 \quad \text{for all} \quad k \in \mathbf{N}$$

This gives a bound of $c(f)$ in terms of $b_k(f)$. Take for example $k=2$. Then

$$(1+2b_2^2+b_3^2-b_2^4)(1-c^2(f)) \geq c^2(f)(4-b_2^2)^2.$$

Note that the problem of estimating $c(f)$ as a function of $b_2(f)$ was completely solved by Jenkins [1954] even for a larger class of functions.

2) Let $\{g_n\}$, $n \in \mathbf{N}$ be a sequence in S that satisfies (10) and denote by α the asymptotic Bieberbach limit in (11). Take a subsequence $\{g_{n_k}\}$, $k \in \mathbf{N}$ of $\{g_n\}$, $n \in \mathbf{N}$, such that

a) g_{n_k} converges locally uniformly to a function g in S,

b) $\lim\limits_{k \to \infty} b_{n_k}(g_{n_k})/n_k = \alpha$,

The $(2n+1) \times (2n+1)$ principal minor $Q(c(g), \ldots, c(g), \alpha, d, \ldots, d)$ gives rise to

$$(7c^2(g) - 6\alpha^2)(6\alpha^2 - 5d^2) \geq d^2 c^2(g).$$

This inequality does not say much unless some lower bound of d is known. If one can show the continuity of the functional $c(\cdot)$ for a specific subsequence $\{g_{n_k}\}$, $k \in \mathbf{N}$, of above, then one can conclude the Asymptotic Bieberbach Conjecture $\alpha = c(g) = d = 1$.

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