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# Bilinear Forms on $k$-Vectorspaces of Denumerable Dimension in the Case of char $(k) = 2$

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**Introduction.** The classification, up to metric isomorphism, of finite dimensional $k$-vector spaces $E$, supplied with a symmetric bilinear form $\Phi : E \times E \to k$, is a rather difficult problem; it has been solved for particular fields $k$, such as the field of rationals, reals, $p$-adic numbers or function fields in one variable over a finite constant field. KAPLANSKY has shown that for $k$-vector spaces $(E, \Phi)$ of a denumerable (algebraic) dimension, these problems vanish in a large number of cases, $E$ admitting an orthonormal basis for an extensive class of underlying fields ([4]; for an investigation of such fields see [3]). In the denumerable case, an exceptional role is once more played by the fields of characteristic 2. For perfect fields of characteristic 2 KAPLANSKY has proved the following

**Theorem.** For every $\aleph_0$-dimensional $k$-space $(E, \Phi)$, $\Phi$ a non degenerate bilinear form, precisely one of the following four possibilities holds: (1) $E$ possesses an orthonormal basis, (2) $E$ possesses a symplectic basis, (3) $E$ is an orthogonal sum $E = E_0 \oplus L$ where $E_0$ is spanned by a symplectic basis and $L$ is one-dimensional, (4) $E$ is an orthogonal sum $E_0 \oplus L$, where $E_0$ has a symplectic basis and $L$ is two-dimensional, spanned by an orthogonal basis ([4] p. 15). KAPLANSKY has asked what becomes of this theorem if the assumption that every element in the coefficient field be a square, is dropped.

In the following, we investigate the case of an arbitrary field of characteristic 2. Complete results as regards the classification problem are obtained for all fields $k$ of finite dimension over their subfields $k^2$ (Theorem 2). As a side-result we obtain an invariant characterization of the $k$-spaces $(E, \Phi)$ of denumerable dimension which admit of orthogonal bases, $k$ an arbitrary field of characteristic 2 (Theorem 3).

## I. Notations and Results

Let $k$ be a commutative field. A $k$-vector space $(E, \Phi)$ is a $k$-vector space $E$ supplied with a symmetric bilinear form $\Phi : E \times E \to k$. $(E, \Phi)$ is called semisimple if $E \cap E^{\perp} = (0)$. In the following, an isomorphism $(E, \Phi) \cong (G, \psi)$ is a vector space isomorphism $\vartheta : E \to G$ such that $\psi(\vartheta x, \vartheta y) = \Phi(x, y)$

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for all $x, y \in E$. If there is no risk of confusion, we simply talk about $E$ instead of $(E, \Phi)$ and, we write $(x, y)$ and $||x||$ respectively for $\Phi(x, y)$ and the "length" $\Phi(x, x)$ of $x \in E$. A subspace $H$ of $(E, \Phi)$ is always considered as being supplied with the restriction $\Phi/_H$ of $\Phi$ to $H$. The radical of $H$ (rad $H$) is defined as $H \cap H^\perp$. A subspace $H \subset E$ is said to be closed if $H^{\perp\perp} = H$. If $H$ is a closed subspace of $(E, \Phi)$ and $F$ a finite dimensional subspace of $(E, \Phi)$ then $H + F$ is closed.

2. The following lemma, proved by KAPLANSKY in [4], will be used in the proof of Lemma 4 below. Lemma: Let $(E, \Phi)$ be a semi-simple $k$-vector space of infinite dimension over an arbitrary field $k$. Let furthermore $F$ be a finite dimensional subspace of $E$, spanned by the basis $f_1, \ldots, f_n$, $V$ a subspace of $E$ with $V^\perp = (0)$. Then there exists a vector $x \in E$ with $x \in V$, $x \notin V \cap F$ and $\Phi(x, f_i) = \beta_i$ for arbitrarily prescribed $\beta_i \in k$.

3. Bases being the central object below, the following notations prove convenient. If $\alpha_1, \ldots, \alpha_n \in k$ then $\langle \alpha_1, \ldots, \alpha_n \rangle$ is an $n$-dimensional $k$-space $(E, \Phi)$ possessing an orthogonal basis $e_1, e_2, \ldots, e_n$ with $||e_i|| = \alpha_i$. "$P$" invariably denotes a hyperbolic plane, i.e., a two-dimensional space $(E, \Phi)$ having a basis $e_1, e_2$ with $||e_1|| = ||e_2|| = 0$ and $(e_1, e_2) = 1$. $\Sigma P$ is an orthogonal sum of hyperbolic planes (i.e., a space spanned by a symplectic basis). $\Sigma\langle\alpha\rangle$ is a space $(E, \Phi)$ spanned by an orthogonal basis (finite or infinite), each basis vector of length $\alpha, \alpha \neq 0$. If $\Sigma\langle\alpha\rangle$ is of denumerable dimension, we denote it by $E_{(\alpha)}$.

4. In the following investigations, $k$ will always be a field of characteristic 2 unless stated otherwise. Every such field is a vector space over its subfield $k^2$ of squares.

5. If $(E, \Phi)$ is a semi-simple $k$-vector space with $\dim E \leq \aleph_0$ then $E$ is an orthogonal sum $\Sigma P \oplus E_0$, where $E_0$ is spanned by an orthogonal basis.

6. Let $(E, \Phi)$ be a $k$-vector-space. We have $||x + y|| = ||x|| + ||y||$ for all $x, y \in E$ as $\operatorname{char} k = 2$. Thus, if $H$ is a subspace of $E$, then the range of the restriction $||H||$ is a subspace of the $k^2$-vector space $k$. This range will be denoted by "$||H||$" throughout. In particular, the set of all isotropic vectors $x$ in $E$ ($||x|| = 0$) is a vector space. This subspace of $E$ is invariably denoted by $E_*$. (The subspace of vectors satisfying condition $(T)$ in [1] p. 66.) The subspaces $E_*$, $E_*^\perp$, $E_*^{\perp\perp}$, rad $E_*$ etc. will play an important role since they are invariant subspaces under orthogonal transformations. We notice that $\operatorname{rad}(E_*^\perp) \subset E_*$ by the definition of $E_*$, hence $\operatorname{rad}(E_*^\perp) \subset \operatorname{rad}(E_*^\perp) \cap E_* = \operatorname{rad} E_*$. Therefore $\operatorname{rad} E_*^\perp = \operatorname{rad} E_*$, the converse inclusion being trivial. This means in particular that $\operatorname{rad} E_* (= \operatorname{rad}(E_*^\perp) = (E_* + E_*^\perp)^\perp)$ is a closed space.

## II. Bases

Let us mention a few words about the fields. When describing $k$-spaces $(E, \Phi)$ in terms of orthogonal bases, it is clear that the non-square elements of $k$ play an important role. Let $g_k$ be the multiplicative group of non-zero elements in $k$ modulo square factors. If $g_k$ is finite, then its order is a power of 2 since every element of $g_k$ is of order 2. If char $k \neq 2$ then one can find, for every natural $n$, fields with $g_k$ of order $2^n$ (even among the denumerable fields, [3]). On the other hand, if char $k = 2$ then $k^2$ is a subfield of $k$ and the elements of $g_k$ are precisely the straight lines through the origin of the $k^2$-vector space $k$. In other words, the order of $g_k$ is either 1 or equal to card $(k)$. In particular, since $g_k$ is of order 1 for finite fields, $g_k$ is either of order 1 or infinite. In the following discussion of isomorphisms between $\aleph_0$-dimensional $k$-spaces the fields with finite dimension $[k : k^2]$ over their subfields $k^2$ are seen to play a special role. Since a simple characterization of all non isomorphic spaces over such fields can be given (Theorem 2), let us mention a few elementary facts about these fields.

Clearly, if $[k : k^2]$ is finite, then $[k : k^2]$ is a power of 2. Furthermore, if $\bar{k}$ is a finite algebraic extension of $k$, $[k : k^2]$ finite, then $[\bar{k}: \bar{k}^2] = [k: k^2]$ ($[\bar{k}: k^2] = [\bar{k}: \bar{k}^2]\,[\bar{k}^2: k^2] = [\bar{k} : k]\,[k : k^2]$ and $[\bar{k}^2 : k^2] = [\bar{k} : k]$). From this follows that $[\bar{k}: \bar{k}^2] \leq [k: k^2]$ for an arbitrary algebraic extension $\bar{k}$ of $k$. ($<$ is witnessed by the transition to the algebraic closure.) On the other hand, if $\bar{k} = k(\xi_1, \ldots, \xi_n)$, where $\xi_1, \ldots, \xi_n$ are independent transcendentals over $k$, we have $[\bar{k}: \bar{k}^2] = [k: k^2] \cdot 2^n$ (a basis for $\bar{k}$ over $\bar{k}^2$ is given by the elements $\alpha_i \xi_1^{\varepsilon_1} \xi_2^{\varepsilon_2} \ldots \xi_n^{\varepsilon_n}$, $\varepsilon_j = 0, 1$ and $\alpha_i$ running through a $k^2$ basis of $k$). In particular:

*If $k$ is a field of characteristic 2 with finite $[k : k^2]$, then $[\bar{k} : \bar{k}^2]$ is finite for an arbitrary over field $\bar{k}$ of $k$, provided its transcendence degree over $k$ is finite.* The fields $k$ with finite $[k : k^2]$ form thus a considerable class.

Let again $k$ be an arbitrary field of characteristic 2. It is well known that WITT's Cancellation Theorem does not hold for bilinear forms in the case of char $k = 2$. Instead, we have the following orthogonal isomorphisms:

**Lemma 1.** $\langle\alpha\rangle \oplus \langle\alpha, \alpha\rangle \cong \langle\alpha\rangle \oplus P$ *($0 \neq \alpha \in k$, $P$ a hyperbolic plane and all the sums orthogonal).*

**Lemma 2.** $\langle\alpha, \alpha\rangle \oplus \bigoplus_{i \in I} \langle\beta_i\rangle \cong \langle\bar{\alpha}, \bar{\alpha}\rangle \oplus \bigoplus_{i \in I} \langle\beta_i\rangle$ *provided that the elements $\{\alpha, \beta_i\}_{i \in I}$ are independent over $k^2$ and span the same subspace of $k$ (over $k^2$) as the elements $\{\bar{\alpha}, \beta_i\}_{i \in I}$ (card $I$ is finite or infinite; all sums are orthogonal).*

**Proofs.** 1. Let $e_1, e_2, e_3$ be an orthogonal basis of $\langle\alpha\rangle \oplus \langle\alpha, \alpha\rangle$ with $\|e_i\| = \alpha$. Introduce a new basis $\bar{e}_1, \bar{e}_2, \bar{e}_3$ by $\bar{e}_1 = e_1 + e_2 + e_3$, $\bar{e}_2 = e_1 + e_2$, $\bar{e}_3 = \alpha^{-1}(e_2 + e_3)$.

2. Let $e_{00}, e_0, e_i\ (i \in I)$ be an orthogonal basis of $\langle\alpha, \alpha\rangle \oplus \bigoplus_{i\in I} \langle\beta_i\rangle$ with $\|e_{00}\| = \|e_0\| = \alpha$, $\|e_i\| = \beta_i$. Since $\{\alpha, \beta_i\}_{i\in I}$ and $\{\bar{\alpha}, \beta_i\}_{i\in I}$ span the same subspace of $k$ we have $\bar{\alpha} = \lambda_0^2 \alpha + \sum_1^n \lambda_i^2 \beta_i$ for suitable $\lambda_0, \lambda_1, \ldots, \lambda_n$. Since the elements $\{\bar{\alpha}, \beta_i\}_{i\in I}$ are independent over $k^2$ we have $\lambda_0 \neq 0$. For a fixed choice of $\lambda_0, \lambda_1, \ldots, \lambda_n$ introduce the following basis

$$\begin{aligned}
\bar{e}_{00} &= \frac{\bar{\alpha}}{\lambda_0 \alpha} e_{00} + \left(\lambda_0 + \frac{\bar{\alpha}}{\lambda_0 \alpha}\right) e_0 + \sum_2^n \lambda_i e_i \\
\bar{e}_0 &= \lambda_0 e_0 + \sum_2^n \lambda_i e_i \\
2 \leq i \leq n : \bar{e}_i &= \frac{\lambda_i \beta_i}{\lambda_0 \alpha} (e_{00} + e_0) + e_i \\
n < i : \bar{e}_i &= e_i .
\end{aligned}$$

We shall list a few consequences some of which will be of importance later.

**Corollary 1.** (i) $\bigoplus_{i\in I} E_{(\alpha_i)} \oplus \Sigma P = \bigoplus E_{(\alpha_i)}$ *(all sums orthogonal).*

(ii) $\langle\alpha_1 \alpha_1 \alpha_2 \alpha_2 \ldots \alpha_m \alpha_m\rangle \cong \langle\bar{\alpha}_1 \bar{\alpha}_1 \bar{\alpha}_2 \bar{\alpha}_2 \ldots \bar{\alpha}_m \bar{\alpha}_m\rangle$ *provided the elements* $\alpha_1, \ldots, \alpha_m$ *are independent over* $k^2$ *and span the same subspace of* $k$ *(over* $k^2$*) as the elements* $\bar{\alpha}_1, \ldots, \bar{\alpha}_m$.

(iii) $\bigoplus_{j=1}^m \langle\alpha_j\, \alpha_j\rangle \oplus \bigoplus_{i\in I} \langle\beta_i\rangle \cong \bigoplus_{j=1}^m \langle\bar{\alpha}_j\, \bar{\alpha}_j\rangle \oplus \bigoplus_{i\in I} \langle\beta_i\rangle$ *provided the elements* $\{\alpha_1, \ldots, \alpha_m, \beta_i\}_{i\in I}$ *are independent over* $k^2$ *and span the same subspace of* $k$ *as the elements* $\{\bar{\alpha}_1, \ldots, \bar{\alpha}_m, \beta_i\}_{i\in I}$ *(card* $I$ *is finite or infinite,* $m$ *is a natural number, all sums are orthogonal).*

We remark that the transformation of Lemma 2 does not lend itself to a generalization of (ii) and (iii) to the case of infinite $m$. (We have not succeded in proving or disproving the infinite analogue of (ii) by any other means; cf. Proposition 3.)

Another lemma which we shall use is the following:

**Lemma 3.** *Let* $(E, \Phi)$ *be a k-vector space of denumerable dimension, semi-simple with respect to the bilinear form* $\Phi : E \times E \to k$ *and* $k$ *a field of arbitrary characteristic. Let furthermore* $R$ *be a closed, totally isotropic subspace of* $E$ ($R^{\perp\perp} = R$

*and* $R \subset R^{\perp}$). *There exists a basis* $(r_i)_{i \in I}$ *of* $R$ *and a subspace* $R'$ *of* $E$ *admitting an orthogonal basis* $(r_i')_{i \in I}$ *such that* $R \oplus R'$ *decomposes into an orthogonal sum of semi-simple planes* $K_i = k(r_i, r_i')$,

$$R \oplus R' = \bigoplus_{i \in I} K_i \quad \text{card}\, I = \dim R = \dim R'$$

*and, furthermore, such that* $R \oplus R'$ *admits of an orthogonal supplement in* $E$: $E = (R \oplus R') \oplus H, H \perp R \oplus R'$.

In the case of char $k \neq 2$, the planes $K_i$ are hyperbolic and $R \oplus R'$ thus possesses a sympletic basis (cf. Bourbaki, Formes Sesquilinéaires p. 78).

**Proof.** Let $S$ and $T$ be finite dimensional semi-simple subspaces with the following properties:

$$S \perp T, T \subset R^{\perp}, S = \bigoplus_{i=1}^{n} K_i, K_i = k(r_i, r_i') \text{ and } r_i \in R \tag{1}$$

$$(T \oplus S) \cap R = k(r_i)_{1 \leq i \leq n} \,. \tag{2}$$

Let $(e_m)_{m \geq 1}$ be some fixed basis of the space $E$ and let $e_m$ be the first basis vector not contained in $S \oplus T$. We construct finite dimensional spaces $K$ and $L$ in $(S \oplus T)^{\perp}$ such that $S' = S \oplus K$ and $T' = T \oplus L$ satisfy the properties (1) and (2) with $S'$ and $T'$ in lieu of $S$ and $T$ and such that $e_m \in S' \oplus T'$. In this fashion we obtain a decomposition of $E$ of the required form:

$$E = \cup S \oplus T = (\cup S) \oplus (\cup T)\,, \quad H = \cup T \text{ and } R \oplus R' = \cup S\,.$$

Since $S \oplus T$ is semi-simple and finite dimensional, we may decompose $e_m$: $e_m = e_m' + e_m''$ with $e_m' \in S \oplus T$ and $e_m'' \perp S \oplus T$. Thus we may without loss of generality assume that $e_m \perp S \oplus T$.

**First case.** $e_m \in R$. Therefore $\|e_m\| = 0$ and, since $(S \oplus T)^{\perp}$ is semi-simple, there exists $r'$ with $(e_m, r') \neq 0$. The space $k(e_m, r')$ is semi-simple and we put $S' = S + k(e_m, r')$ and $T' = T$. We have to determine $(T' + S') \cap R$. Let $r \in (T' \oplus S') \cap R$, $r = t + s + \lambda e_m + \mu r'$ with $t \in T$, $s \in S$ and $r \in R$. Since $T \subset R^{\perp}$ we obtain $0 = (v, R) = (t, R)$ hence $t = 0$ as $T$ is semi-simple. Therefore, (since $R \subset R^{\perp}$) we obtain $0 = (r, e_m) = \mu(e_m, r')$. Thus $\mu = 0$ and $v = s + \lambda e_m$. Since $e_m \in R$ in our case therefore $s \in R$ i.e., $s \in S \cap R = k(r_i)_{i \leq n}$ by (2). Thus $(T' \oplus S') \cap R = k(r_1, \ldots, r_m, e_m)$ which, upon relabeling $e_m$ as $r_{n+1}$ (and $r'$ as $r_{n+1}'$), is (2). The remaining conditions are trivially satisfied.

**Case 2.** $e_m \notin R$ and $e_m \in R^\perp$. We first convince ourselves that $e_m \notin R + (S \oplus T)$; assume that $e_m = r + s + t$ with $r \in R$, $s \in S$ and $t \in T$. Since $e_m \perp S + T$ and $T \subset R^\perp$, we have in particular $0 = (e_m, T) = (t, T)$; hence $t = 0$ as $T$ is semi-simple. Since $e_m \in R^\perp$ in the present case, and $R \subset R^\perp$, we obtain furthermore $0 = (e_m, R \cap S) = (s, R \cap S)$ i. e., $S \perp S \cap R$. From the explicit form of $S = \oplus k(r_i, r_i')$ we see that necessarily $s \in R \cap S$. Thus $e_m = r + s \in R$, a contradiction. Since $(R + S + T)^{\perp\perp} = R + S + T$, we conclude from $e_m \notin R + S + T$ that $(R + S + T)^\perp \not\subset e_m{}^\perp$. Hence there exists a vector $t \in (R + S + T)^\perp = R^\perp \cap (S + T)^\perp$ with $(e_m, t) \neq 0$. Thus, if $||e_m|| = 0$ then $k(e_m, t)$ is a semi-simple space and we put $S' = S$, $T' = T + k(e_m, t)$. If, on the other hand, $||e_m|| \neq 0$, we simply put and $S' = S$ and $T' = T + k(e_m)$. We have to determine $(T' \oplus S') \cap R$. Let, in the first case, $r \in T' \oplus S'$ i.e., $r = s + t + \lambda e_m + \mu t$ with $s \in S, t \in T$ and $r \in R$. Since $e_m \in R^\perp$ and $||e_m|| = 0$ we find $0 = (r, e_m) = \mu(t, e_m)$, therefore $\mu = 0$. Since $t \in R^\perp \cap (S \oplus T)^\perp$ we then find $0 = (r, t) = \lambda(e_m, t)$. Hence $\lambda = 0$. This shows that $(T' \oplus S') \cap R = (T \oplus S) \cap R$. In the other case, $||e_m|| \neq 0$, it is even simpler to verify that $(T' \oplus S') \cap R = (T \oplus S) \cap R$. The remaining conditions (1) are trivially satisfied for $S'$ and $T'$.

**Case 3.** $e_m \notin R^\perp$. As in the second case one verifies that $e_m \notin R^\perp + S + T$. Since $(R^\perp + S + T)^{\perp\perp} = R^\perp + S + T$, we conclude from $e_m \notin R^\perp + S + T$ that $(R^\perp + S + T)^\perp \not\subset e_m^\perp$. In other words there exists a vector $r \in (R^\perp + S + T)^\perp = R^{\perp\perp} \cap (S \oplus T)^\perp = R \cap (S \oplus T)^\perp$ with $(e_m, r) \neq 0$. Since $r \in R$ we have $||r|| = 0$ and the space $k(r, e_m)$ is semi-simple. We put $S' = S \oplus k(r, e_m)$ and $T' = T$. Upon relabeling $r$ as $r_{n+1}$ (and $e_m$ as $r'_{n+1}$) the conditions (1) and (2) are verified as in case 1. Q.E.D.

Lemma 3 often finds application in the following situation. Suppose that $G$ is a subspace of $E$ such that the radical $R = G \cap G^\perp$ of $G$ happens to be a closed subspace of $E$. We then have a decomposition $E = (R \oplus R') \oplus H$, $H \perp (R \oplus R')$. Furthermore, one can always find an algebraic complement $L$ of $R$ in $G$ such that $L \subset H$. For, if $L_0$ is some algebraic complement of $R$ in $G$ then $L_0 \perp R$. Every vector $l_0 \in L_0$ has a decomposition $l_0 = r + r' + h$. Since $l_0 \perp R$ necessarily $r' = 0$. In other words, $L_0 \subset R \oplus H$ which shows that there is a complement $L$ of $R$ in $G$ with $L \subset H$.

We are interested in decompositions of $E$ of the following sort: $E$ is an orthogonal sum $E = \oplus E_i$ such that the ranges $||E_i||$ of the summands are either 0 or 1-dimensional subspaces of the $k^2$-vector space $||E||$ and such that the elements spanning the non trivial $||E_i||$ are linearly independent over $k^2$. In other words,

$$E = \Sigma P \oplus \Sigma \langle \alpha_1 \rangle \oplus \Sigma \langle \alpha_2 \rangle \oplus \dots$$

where the $P_s$ are hyperbolic planes and where the field elements $\alpha_1, \alpha_2, \dots$ are linearly independent over $k^2$. In view of Lemma 1 we may assume that the summands $\Sigma \langle \alpha_i \rangle$ are either of infinite dimension or of dimension $\leq 2$. Thus, collecting 1-, 2- and $\aleph_0$-dimensional summands we may rewrite the above decomposition as follows:

$$E = \Sigma P \oplus \bigoplus_{i \in I_1} E_{(\beta_i)} \oplus \bigoplus_{i \in I_2} \langle \gamma_i \gamma_i \rangle \oplus \bigoplus_{i \in I_3} \langle \delta_i \rangle \tag{1}$$

where all the field elements $\beta_i, \gamma_j, \delta_l$ together are independent over $k^2$.

We shall determine those $k$-space $(E, \Phi)$ which admit of a decomposition of type (1). We first have

**Proposition 1.** *If $E$ admits of a decomposition* (1) *then*

$$E_*^{\perp} \oplus E_*^{\perp\perp} = (\mathrm{rad}\, E_*)^{\perp}. \tag{2}$$

**Proof.** Let for every $i \in I_1$ the space $E_{(\beta i)}$ be spanned by the vectors $(e_{i\iota})_{\iota \geq 1}$. $(E_{(\beta_i)})_*$ is spanned by the vectors $(e_{i1} + e_{i\iota})_{\iota \geq 1}$ and, the orthogonal complement of $(E_{(\beta_i)})_*$ in $E_{(\beta_i)}$ is $(0)$. Let furthermore, for every $i \in I_2$, $\langle \gamma_j \gamma_j \rangle$ be spanned by the vectors $f_j, f_j'$. Since all the elements $\beta_i, \gamma_j, \delta_e$ together are independent over $k^2$ (by assumption), we obtain for $E_*$ from (1)

$$E_* = \Sigma P \oplus \bigoplus E_{(\beta_i)*} \oplus \bigoplus_{i \in I_2} k(f_i + f_i') \oplus (0)\,.$$

Furthermore

$$E_*^{\perp} = (0) \oplus \bigoplus k(f_i + f_i') \oplus \bigoplus \langle \delta_i \rangle \quad \text{and} \quad E_*^{\perp\perp} = \Sigma P \bigoplus_{I_1} E_{(\beta_i)*} \oplus \bigoplus_{I_2} k(f_i + f_i').$$

From this we readily read off that (2) holds.

Condition (2) is not always satisfied. The simplest kind of counter-example is the following. Let $E$ be spanned by the basis vectors $\{e_i\}_{i \geq 1} \cup \{f_i\}_{i \geq 1} \cup \{g_0\}$ and let $\Phi$ be defined on the basis as follows: $\| e_i \| = \alpha$ and $(e_i, e_j) = 0\ (i \neq j, i, j \geq 1)$, $\| f_i \| = \beta_i$ and $(f_i, f_j) = 0\ (i \neq j, i, j \geq 1)$, $\| g_0 \| = \gamma$ and $(e_i, f_j) = 0$, $(e_i, g_0) = \alpha$, $(f_i, g_0) = \beta_i$, $(i, j \geq 1)$ for $\alpha, \gamma, \beta_1, \beta_2, \dots$ independent over $k^2$ (a field with $[k\colon k^2] \geq \aleph_0$ is required). Here $\mathrm{rad}\, E_* = 0$ and $(\mathrm{rad}\, E_*)^{\perp} = E$, but $E_*^{\perp} + E_*^{\perp\perp}$ falls short of $E$ by one dimension. We remark that (2) is equivalent to $E_*^{\perp} \oplus E_*^{\perp\perp}$ being closed.

We shall prove that the converse of Proposition 1 is true. This is accomplished by reducing the general case to the cases of spaces $E$ with $E_*^{\perp} = (0)$ or $E_*^{\perp} = E_*$. We start out with these special cases.

**Lemma 4.** *Let $(E, \Phi)$ be a semi-simple space of denumerable dimension with $E_*^\perp = (0)$. Then for every $\alpha \in ||E||$ and every orthogonal decomposition $E = H \oplus H^\perp$ with finite dimensional $H$ we have $\alpha \in ||H^\perp||$.*

**Proof.** Let $E = H \oplus H^\perp$ be any decomposition with finite dimensional $H$, furthermore $\alpha$ some arbitrarily fixed element in $||E||$. We apply Lemma 1.2 with $E_*$ and $H$ in the roles of $V$ and $F$ respectively. Since $\alpha \in ||E||$, there exists some vector $x_0 \in E$ with $||x_0|| = \alpha$. Hence there exists a vector $x \in E_*$ with $(x, f_i) = -(x_0, f_i)$, $f_1, \ldots, f_n$ a fixed basis of $H$. Therefore $(x_0 + x, f_i) = 0$ i.e., $x_0 + x \perp H$. Since $x \in E_*$ we have $||x_0 + x|| = ||x_0|| = \alpha$.

**Proposition 2.** *Let $(E, \Phi)$ be a semi-simple space of denumerable dimension with $||E|| \neq 0$. We have an orthogonal decomposition*

$$E = \bigoplus_{i \in I} E_{(\pi_i)}$$

*where $\{\pi_i\}_{i \in I}$ is a $k^2$-basis for $||E||$ if and only if $E_*^\perp = (0)$.*

**Proof.** If $E$ admits such a decomposition it is readily verified that $E_*^\perp = (0)$. Let us then assume that $E_*^\perp = (0)$. We construct a decomposition of $E$ of the required type step by step. Let $F = \Sigma P \oplus \Sigma \langle\pi_1\rangle \oplus \ldots \oplus \Sigma \langle\pi_n\rangle$ be a finite dimensional subspace of $E$, the $P_s$ hyperbolic planes and the field elements $\pi_1, \ldots, \pi_n$ linearly independent over $k^2$. Let furthermore $(e_i)_{i \geq 1}$ be some fixed basis for the space $E$ and assume that $e_m$ is the first basis vector not contained in $F$. We shall construct a finite dimensional subspace $H$ in $F^\perp$ such that $e_m \in F \oplus H$ and $F' = F \oplus H$ is of the form $\Sigma P \oplus \Sigma\langle\pi_1\rangle \oplus \ldots \oplus \Sigma\langle\pi_r\rangle$ with $\pi_1, \ldots, \pi_r$ linearly independent over $k^2$.

Since $F$ is finite dimensional and semi-simple, we may decompose $e_m$: $e_m = e_m' + e_m''$ with $e_m' \in F$ and $e_m'' \perp F$. Three cases are possible: $||e_m''|| = 0$ and $e_m''$ is contained in some hyperbolic plane $P' \subset F^\perp$ or $||e_m''|| \neq 0$ or $||e_m''|| = 0$ and $e_m'' \in \langle\delta, \delta\rangle \subset F^\perp$ for some $0 \neq \delta \in k$. In the first case we may choose $P'$ for $H$ and we put $F' = F \oplus P'$. In the second case we put $F' = F \oplus k(e_m'')$ provided that $e_m'' \notin ||F||$. If, on the other hand, we should have $e_m'' = \sum_1^n \lambda_i^2 \pi_i$ with, say $\lambda_1 \neq 0$, then we apply Lemma 4 a finite number of times and find a sequence of mutually orthogonal vectors $h_1, h_2, \ldots, h_n$ in $(F + k(e_m''))^\perp$ with $||h_1|| = ||e_m''||$, $||h_i|| = \pi_i$, $2 \leq i \leq n$. By Lemma 2 the space $H$ spanned by $e_m''$, $h_1, h_2, \ldots, h_n$ is isomorphic to $\langle\pi_1 \pi_1 \pi_2 \pi_3, \ldots, \pi_n\rangle$ and we put $F' = F \oplus H$. The third case is treated in the same way, the first two vectors for the construction of $H$ already at hand. Thus, in all three cases we find $F' = F \oplus H$, $e_m \in F'$ where $F'$ again is of the form $\Sigma P \oplus \Sigma\langle\pi_1\rangle \oplus \ldots \oplus \Sigma\langle\pi_r\rangle$, the $\pi_i s$ linearly independent over $k^2$. In this fashion we find an orthogonal decomposition of $E$ as follows, $E = \cup F = \Sigma P \oplus \Sigma\langle\pi_1\rangle \oplus \Sigma\langle\pi_2\rangle \oplus \ldots$. In view of the independence of the $\pi_i s$ we have $E_* = \Sigma P \oplus (\Sigma\langle\pi_1\rangle)_* \oplus \ldots$. Not all of the summands $\Sigma\langle\pi_i\rangle$ can be (0) since $||E|| \neq 0$. Thus, if one of the summands should be finite dimensional we would have $E_*^\perp \neq (0)$, contrary to assumption. Hence all the summands $\Sigma\langle\pi_i\rangle$ are infinite dimensional. Application of Corollary 1 finally yields $E \cong E_{(\pi_1)} \oplus E_{(\pi_2)} \oplus \ldots$.

**Corollary 2.** *If $(E, \Phi)$ is a space with $E_*^\perp = (0)$ whose range $||E|| \neq 0$ is spanned by the elements $\pi_1, \ldots, \pi_m$ (not necessarily independent over $k^2$) then E is isomorphic to $E_{(\pi_1)} \oplus \ldots \oplus E_{(\pi_m)}$.*

**Proof.** By Proposition 2 $E \cong E_{(\sigma_n)} \oplus \ldots \oplus E_{(\sigma_1)}$ where $\sigma_1 \ldots \sigma_n$ is a $k^2$-basis for $||E||$. Let then $\pi_1, \ldots, \pi_n$ $(n \leq m)$ be a subset of elements independent over $k^2$. By Corollary 1 (ii) we have

$$\langle\pi_1\pi_1\rangle \oplus \ldots \oplus \langle\pi_n\pi_n\rangle \cong \langle\sigma_1\sigma_1\rangle \oplus \ldots \oplus \langle\sigma_n\sigma_n\rangle.$$

Hence trivially $E_{(\sigma_1)} \oplus \ldots \oplus E_{(\sigma_n)} \cong E_{(\pi_1)} \oplus \ldots \oplus E_{(\pi_n)}$. Let $\pi_{n+1} = \sum_{i=1}^{r} \lambda_i^2 \pi_i$. After renumbering $\pi_1 \ldots \pi_n$ we may assume that $\lambda_i \neq 0$, $1 \leq r \leq i$. Hence by Corollary 1 (ii) $\langle\pi_{n+1}\pi_{n+1}\pi_2 \ldots \pi_r\rangle \cong \langle\pi_1\pi_1\pi_2 \ldots \pi_n\rangle$. Thus $E_{(\pi_{n+1})} \oplus E_{(\pi_2)} \oplus \ldots \oplus E_{(\pi_r)} \cong E_{(\pi_1)} \oplus \ldots \oplus E_{(\pi_r)}$ can be arranged in a trivial fashion. In this manner we obtain $E_{(\pi_1)} \oplus \ldots \oplus E_{(\pi_m)} \cong E$.

**Proposition 3.** *Let $(E, \Phi)$ be a semi-simple space of at most denumerable dimension. We have an orthogonal decomposition*

$$E = \bigoplus_{i \in I} \langle\pi_i\pi_i\rangle$$

*where the $\pi_i$ form some $k^2$-basis for $||E||$ if and only if $E_*^\perp = E_*$.*

**Proof.** If $E$ admits such a decomposition we trivially have $E_*^\perp = E_*$. Conversely, let us assume that $E_*^\perp = E_*$. We first remark that $E$ cannot contain a triple of mutually orthogonal vectors of the same length $\neq 0$. For, assume that $z_1, z_2, z_3$ were such vectors, $\|z_1\| = \|z_2\| = \|z_3\| \neq 0$. We decompose according to the decomposition $E = E_* \oplus L$: $z_1 = e_1 + l_1$, $z_2 = e_2 + l_2$, $z_3 = e_3 + l_3$. Thus $\|l_1\| = \|l_2\| = \|l_3\|$. Since $L$ contains no isotropic vectors we must necessarily have $l_1 = l_2 = l_3$. Since $E_*$ is totally isotropic in our case, the three orthogonality conditions reduce to $0 = (e_1 + e_2, l_1) + \|l_1\|$, $0 = (e_1 + e_3, l_1) + \|l_1\|$, $0 = (e_2 + e_3, l_1) + \|l_1\|$. Adding the first two of these equations we obtain $(e_2 + e_3, l_1) = 0$ which contradicts the third one as $\|l_1\| \neq 0$. We now construct a decomposition of $E$ step by step as in the proof of Proposition 2. Let $F = \langle \pi_1 \pi_1 \rangle \oplus \langle \pi_2 \pi_2 \rangle \oplus \oplus \ldots \oplus \langle \pi_n \pi_n \rangle$ be a finite dimensional subspace of $E$, $\pi_1, \pi_2, \ldots, \pi_n$ linearly independent over $k^2$. Furthermore, let $e_m$ again be the first basis vector of some fixed basis for $E$ not contained in $F$. Without loss of generality we may proceed assuming that $e_m \perp F$. We consider first the case that $\|e_m\| \neq 0$. We try to find a vector $l \in F^\perp \cap E_*$ with $(l, e_m) \neq 0$. Suppose that there is no such vector $l$, in other words $F^\perp \cap E_* \subset e_m^\perp$. Since $E_*$ is closed in our case, we find $(F + E_*^\perp)^\perp = F^\perp \cap E_*^{\perp\perp} = F^\perp \cap E_* \subset e_m^\perp$ therefore $e_m \in (F + E_*^\perp)^{\perp\perp} = F + E_*^\perp$ i.e., $e_m \in F + E_*^\perp = F + E_*$. Thus $e_m = f + f_0$ with $\|e_m\| = \|f\| \neq 0$.

Since $f \in F$ we should therefore have three mutually orthogonal vectors of the same length $\|e_m\| \neq 0$, a contradiction (if $F$ contains one vector of some length $\alpha \neq 0$, then it contains, by virtue of its form, two orthogonal vectors of that length). Thus we must have $F^\perp \cap E_* \not\subset e_m^\perp$ and there exists a vector $l \in F^\perp \cap E_*$ with $(e_m, l) \neq 0$. Hence $e_m$ and $e_m' = e_m + \dfrac{\|e_m\|}{(l, e_m)} l$ are mutually orthogonal vectors of $F^\perp$ with $\|e_m\| = \|e_m'\|$. We put $F' = F \oplus k(e_m, e_m')$. There remains the possibility that $\|e_m\| = 0$. Since $E_*$ is totally isotropic, $e_m$ cannot be contained in a hyperbolic plane, therefore $e_m \in \langle \delta, \delta \rangle \subset F^\perp$ for some $0 \neq \delta \in k$ ($F^\perp$ is semi-simple). Since there cannot be more than two orthogonal vectors of the same length $\neq 0$ we must have $\delta \notin \|F\|$ and we put $F' = F \oplus \langle \delta \delta \rangle$ similar to the former case. In this fashion we obtain a decomposition of $E$ of the required form, $E = \cup F = \langle \pi_1 \pi_1 \rangle \oplus \langle \pi_2 \pi_2 \rangle \oplus \ldots$ where all the $\pi_i s$ are linearly independent over $k^2$.

We now prove the converse of Proposition 1.

**Theorem 1.** *Let* char $k = 2$ *and* $(E, \Phi)$ *a semi-simple k-space of denumerable dimension and let* $E_*$ *be the subspace of vectors of length zero. If*

$$E_*^\perp + E_*^{\perp\perp} = (\mathrm{rad}\, E_*)^\perp$$

*then E admits of an orthogonal decomposition*

$$E = \bigoplus_{i \in I_1} E_{(\gamma_i)} \oplus \bigoplus_{i \in I_2} \langle \beta_i, \beta_i \rangle \oplus \bigoplus_{i \in I_3} \langle \alpha_i \rangle \tag{I}$$

*or*

$$E = \bigoplus_{i \in I_1} P_i \oplus \bigoplus_{i \in I_2} \langle \beta_i, \beta_i \rangle \oplus \bigoplus_{i \in I_3} \langle \alpha_i \rangle \tag{II}$$

*where, in the first case, the elements of the union* $\{\gamma_i\}_{i \in I_1} \cup \{\beta_i\}_{i \in I_2} \cup \{\alpha_i\}_{i \in I_3}$ *are a $k^2$-basis of the range* $\|E\|$ *over* $k^2$, *in the second case the same for the elements of the union* $\{\beta_i\}_{i \in I_2} \cup \{\alpha_i\}_{i \in I_3}$ *(the* $P_i s$ *are hyperbolic planes).*

**Proof.** Let $R = \operatorname{rad}(E_*^{\perp\perp}) = (E_* + E_*^{\perp})^{\perp}$. Since $R$ is totally isotropic and closed, we can apply Lemma 3 and obtain a decomposition

$$E = (R \oplus R') \oplus H, \quad H \perp (R \oplus R')$$

$$R \oplus R' = \bigoplus_{i \in I_2} k(r_i, r_i'), \quad R = \bigoplus k(r_i)_{i \in I_2}. \tag{1}$$

Since $R \perp E_*^{\perp\perp}$, we can find an algebraic complement $S$ of $R$ in $E_*^{\perp\perp}$ with $S \perp R'$ (see the remark following the proof of Lemma 3). Hence $S \perp R \oplus R'$:

$$E_*^{\perp\perp} = R \oplus S, \quad S \subset H. \tag{2}$$

Furthermore $S$ is semi-simple. If $T$ is the orthogonal of $S$ in $H$, we obtain from (2) $E_*^{\perp} = E_*^{\perp\perp\perp} = R \oplus T$. On the other hand, by the assumption of the theorem $R \oplus H = R^{\perp} = E_*^{\perp} + E_*^{\perp\perp} = R \oplus (S \oplus T)$. Since $S + T \subset H$ therefore $S + T = H$. Furthermore, since $S$ is semi-simple, the sum $S + T$ is direct. Thus $E$ is decomposed into three orthogonal summands:

$$E = (R \oplus R') \oplus S \oplus T \tag{3}$$

and it remains to discuss the spaces $R \oplus R'$, $S$ and $T$. With regard to $S$ we first remark that

$$E_* = R \oplus S_*. \tag{4}$$

For $R \oplus S_* \subset E_*$ is trivial. Conversely, if $x \in E_* \subset E_*^{\perp\perp} = R \oplus S$ we have $x = r + s$ with $r \in R$ and $s \in S$. Therefore $0 = \|x\| = \|r\| + \|s\| = \|s\|$ and $s \in S_*$. This shows $E_* \subset R + S_*$. Let then $S_*^{\perp_s}$ be the orthogonal of $S_*$ in $S$. Since $S_*^{\perp_s} \subset S$ and $S \perp R$ we have $S_*^{\perp_s} \subset E_*^{\perp}$ by (4). Also $S_*^{\perp_s} \subset S \subset E_*^{\perp\perp}$, hence $S_*^{\perp_s} \subset E_*^{\perp} \cap E_*^{\perp\perp} = R$. Therefore $S_*^{\perp_s} = (0)$ as

$S_*^{\perp s} \subset S$ and $S \cap R = (0)$. Thus, $S$ is semi-simple and $S_*^{\perp s} = (0)$. Two cases are possible for $S$: Either $S = S_*$ in which case $S$ is a sum of hyperbolic planes or else $S \neq S_*$ in which case the range $\|S\|$ is different from 0 and Proposition 2 can be quoted: Thus

$$\text{either} \quad S = \bigoplus_{i \in I_1} P_i \quad \text{or} \quad S = \bigoplus_{i \in I_1} E_{(\gamma_i)} . \tag{5}$$

From (4) we learn that $R' \cap E_* = (0)$. Therefore, taking orthogonals in $R + R'$, we obtain $(R + R')_* = R = R^\perp = (R + R')_*^\perp$ and we may cite Proposition 3:

$$R \oplus R' = \bigoplus_{i \in I_2} \langle \beta_i, \beta_i \rangle . \tag{6}$$

Finally $E_* \cap T = (0)$ by (4), i.e., $T$ contains no isotropic vectors. Hence $T$ possesses an orthogonal basis, $T = \bigoplus_{i \in I_3} \langle \alpha_i \rangle$ where all the $\alpha_i s$ are independent over $k^2$. Summarizing the facts about the decomposition (3) we see that $E$ admits of an orthogonal decomposition of the form

$$E = \bigoplus_{i \in I_1} E_{(\gamma_i)} \oplus \bigoplus_{i \in I_2} \langle \beta_i \beta_i \rangle \oplus \bigoplus_{i \in I_3} \langle \alpha_i \rangle \quad \text{or} \quad E = \bigoplus_{i \in I_1} P_i \oplus \bigoplus_{i \in I_2} \langle \beta_i \beta_i \rangle \oplus \bigoplus_{i \in I_3} \langle \alpha_i \rangle .$$

A dependence $0 = \Sigma \nu_i^2 \gamma_i + \Sigma \mu_i^2 \beta_i + \Sigma \varkappa_i^2 \alpha_i$ defines an isotropic vector $x = \Sigma \nu_i c_i + \Sigma \mu_i b_i + \Sigma \varkappa_i a_i$, $\Sigma \nu_i c_i \in S$, $\Sigma \mu_i b_i \in R + R'$ and $\Sigma \varkappa_i a_i \in T$. By (4) $x \in E_* = R + S_*$ and thus $\varkappa_i = 0$, $\|\Sigma \nu_i c_i\| = \Sigma \nu_i^2 \gamma_i = 0$ and $\|\Sigma \mu_i b_i\| = \Sigma \mu_i^2 \beta_i = 0$. However, the $\gamma_i s$ are linearly independent over $k^2$ by Proposition 2. Therefore $\nu_i = 0$. Proposition 3 guarantees the independence of the $\beta_i s$ and therefore $\mu_i = 0$. This proves that the elements $\gamma_i, \beta_j, \alpha_e$ together are independent over $k^2$ and the proof of Theorem 1 is complete.

Theorem 1 can be used to discuss the problem of isomorphism between $\aleph_0$-dimensional $k$-spaces $(E, \Phi)$ in a large number of cases. We shall give here a complete discussion of the cases where the underlying field $k$ is of finite dimension over its subfield $k^2$. Thus, let $k$ be a field with $[k : k^2]$ finite. For a space $(E, \Phi)$ we have codim $E_* \leq [k : k^2]$ or else an algebraic complement of $E_*$ in $E$ should contain an isotropic vector which is impossible. Since dim $E_*^\perp \leq$ codim $E_*$, the space $E_*^\perp$ is finite dimensional and $E_*^{\perp\perp} + E_*^\perp$ is therefore closed. Hence every space of denumerable dimension over such a field admits of a basis as described by Theorem 1. (The following discussion also includes that of spaces $(E, \Phi)$ with $\|E\|$ finite dimensional over $k^2$, $k$ an arbitrary field.)

**Theorem 2.** *Let $k$ be a field of characteristic 2 of finite dimension $n$ over its subfield $k^2$ ($n = [k : k^2]$), $(E, \Phi)$ an $\aleph_0$-dimensional semi-simple space over $k$. Then* (i) *$E$ is of the form:*

$$E = E_{(\gamma_1)} \oplus \ldots \oplus E_{(\gamma_r)} \oplus \langle \beta_1 \beta_1 \beta_2 \beta_2 \ldots \beta_s \beta_s \rangle \oplus \langle \alpha_1 \alpha_2 \ldots \alpha_t \rangle \quad r \geq 1 \quad \text{(I)}$$

*or*

$$E = \sum^{\infty} P \oplus \langle \beta_1 \beta_1 \beta_2 \beta_2 \ldots \beta_p \beta_p \rangle \oplus \langle \alpha_1 \alpha_2 \ldots \alpha_q \rangle, \quad \text{(II)}$$

*where all the sums are orthogonal and, in the first case, the elements $\gamma_1, \ldots, \gamma_r, \beta_1, \ldots, \beta_s, \alpha_1, \ldots, \alpha_t$ are independent over $k^2$ and the same for $\beta_1, \ldots, \beta_p, \alpha_1, \ldots, \alpha_q$ in the second case (thus $r + s + t \leq n$, $p + q \leq n$).*

(ii) *$E$ is uniquely determined, up to orthogonal isomorphism, by its range $\| E \|$, the range $\| E_*^{\perp\perp} \|$ and by the space $E_*^{\perp}$. (In particular, the numbers $r$, $s$ and $t$, respectively $p$ and $q$ are orthogonal invariants of the space $E$.)*

(iii) *In terms of the above bases: If $\| E_*^{\perp\perp} \| \neq 0$ (i.e., $E_*$ not closed) then $E$ is of type* (I), *if $\| E_*^{\perp\perp} \| = 0$ (i.e., $E_*$ closed) then $E$ is of type* (II). (*Thus* (I) *and* (II) *represent non isomorphic spaces.*) *A space of type* (I) *is uniquely determined, up to orthogonal isomorphism, by $\| E \|$, the subspace of $k$ (over $k^2$) spanned by the elements $\gamma_1, \ldots, \gamma_r$ and by the space $\langle \alpha_1, \ldots, \alpha_t \rangle$. A space of type* (II) *is uniquely determined, up to isomorphism, by $\| E \|$ and by the space $\langle \alpha_1, \ldots, \alpha_q \rangle$.*

**Proof.** It only remains to discuss the question of isomorphisms. For a space of type (I) let $E_{(\gamma_i)}$ be spanned by a basis $\{e_{ij}\}_{j \geq 1}$. $E_{(\gamma_i)*}$ is then spanned by the vectors $e_{i1} + e_{ij}$ $(j \geq 1)$ and the orthogonal of $E_{(\gamma_i)*}$ in $E_{(\gamma_i)}$ is $0$. Let $\langle \beta_1 \beta_1, \ldots, \beta_s \beta_s \rangle$ be spanned by a basis $\{e_i, e_i'\}_{1 \leq i \leq s}$ and let $R$ be the totally isotropic space $k(e_i + e_i')_{1 \leq i \leq s}$. We then have, by virtue of the independence of the lements $\gamma_1, \ldots, \beta_1, \ldots, \alpha_1, \ldots$

$$E_* = E_{(\gamma_1)*} \oplus \ldots \oplus E_{(\gamma_r)*} \oplus R, \quad E_*^{\perp} = R \oplus \langle \alpha_1, \ldots, \alpha_t \rangle,$$

$$E_*^{\perp\perp} = E_{(\gamma_1)} \oplus \ldots \oplus E_{(\gamma_r)} \oplus R.$$

Let $\bar{E}$ be another space falling into category (I), $\bar{E} = E_{(\bar{\gamma}_1)} \oplus \ldots \oplus E_{(\bar{\gamma}_{\bar{r}})} \oplus \oplus \langle \bar{\beta}_1 \bar{\beta}_1, \ldots, \bar{\beta}_{\bar{s}} \bar{\beta}_{\bar{s}} \rangle \oplus \langle \bar{\alpha}_1, \ldots, \bar{\alpha}_{\bar{t}} \rangle$ such that $\| E \| = \| \bar{E} \|$, $\| E_*^{\perp\perp} \| = \| \bar{E}_*^{\perp\perp} \|$ and $E_*^{\perp} \cong \bar{E}_*^{\perp}$. We have to prove that $E \cong \bar{E}$. Since $\gamma_1, \ldots, \gamma_r$ and $\bar{\gamma}_1, \ldots, \bar{\gamma}_{\bar{r}}$ are independent over $k^2$ we first have $r = \bar{r}$ (since $\| E_*^{\perp\perp} \| = \| \bar{E}_*^{\perp\perp} \|$). By Corollary 2 we see that $E_*^{\perp\perp} \cong \bar{E}_*^{\perp\perp}$. Hence we may introduce a new basis in $\bar{E}_*^{\perp\perp}$ such that $\bar{\gamma}_i, = \gamma_i$, $1 \leq i \leq r$. From the isomorphism $R \oplus \langle\alpha_1, \ldots, \alpha_t\rangle \cong \bar{R} \oplus \langle\bar{\alpha}_1, \ldots, \bar{\alpha}_{\bar{t}}\rangle$ we conclude that $\langle\alpha_1, \ldots, \alpha_t\rangle \cong$ $\cong \langle\bar{\alpha}_1, \ldots, \bar{\alpha}_{\bar{t}}\rangle$ since $R$ and $\bar{R}$ are totally isotropic orthogonal summands and since both $\langle\alpha_1, \ldots, \alpha_t\rangle$ and $\langle\bar{\alpha}_1, \ldots, \bar{\alpha}_{\bar{t}}\rangle$ are semi-simple (even non-isotropic by the independence of the $\alpha s$). Thus $t = \bar{t}$ and we may introduce a new basis in $\langle\bar{\alpha}_1, \ldots, \bar{\alpha}_t\rangle$ such that $\bar{\alpha}_i = \alpha_i$, $1 \leq i \leq t$. Finally, since $||E|| = ||\bar{E}||$ and since $\gamma_1, \ldots, \beta_1, \ldots, \alpha_1, \ldots$ and $\bar{\gamma}_1, \ldots, \bar{\beta}_1, \ldots, \bar{\alpha}_1, \ldots$ are independent over $k^2$ we have $r + s + t = \bar{r} + \bar{s} + \bar{t}$; therefore $s = \bar{s}$ as $r = \bar{r}$ and $t = \bar{t}$. Furthermore, having introduced the new bases in $\bar{E}_*^{\perp\perp}$ and $\langle\bar{\alpha}_1, \ldots, \bar{\alpha}_t\rangle$ we may cite Corollary 1 (ii), $\langle\gamma_1, \ldots, \gamma_r\rangle \oplus \langle\beta_1\beta_1, \ldots, \beta_s\beta_s\rangle \oplus \langle\alpha_1, \ldots, \alpha_t\rangle \cong$ $\cong \langle\bar{\gamma}_1, \ldots, \bar{\gamma}_r\rangle \oplus \langle\bar{\beta}_1\bar{\beta}_1, \ldots, \bar{\beta}_s\bar{\beta}_s\rangle \oplus \langle\bar{\alpha}_1, \ldots, \bar{\alpha}_t\rangle$. A fortiori $E_{(\gamma_1)} \oplus \ldots \oplus E_{(\gamma_r)} \oplus$ $\oplus \langle\beta_1\beta_1, \ldots, \beta_s\beta_s\rangle \oplus \langle\alpha_1, \ldots, \alpha_t\rangle \cong \bar{E}_{(\bar{\gamma}_1)} \oplus \ldots \oplus \bar{E}_{(\bar{\gamma}_r)} \oplus \langle\bar{\beta}_1\bar{\beta}_1, \ldots, \bar{\beta}_s\bar{\beta}_s\rangle \oplus$ $\oplus \langle\bar{\alpha}_1, \ldots, \bar{\alpha}_t\rangle$ and thus $E \cong \bar{E}$. The simpler case of spaces falling into category (II) is treated in the same way. This proves Theorem 2.

Theorem 2 may also be expressed in the following way: If $[k : k^2]$ is finite and $(E, \Phi)$ an $\aleph_0$-dimensional, semi-simple $k$-space with $E_*$ not closed, then there exist three finite dimensional $k$-spaces $F, G$ and $H$ such that $F \oplus G \oplus H$ contains no isotropic vectors and $E$ is isomorphic to the (external) orthogonal sum $(\overset{\infty}{\Sigma} F) \oplus G \oplus G \oplus H$. $E$ is uniquely determined by the ranges $||F + G + H||$, $||F||$ and by the space $H$; on the other hand, if $E_*$ is closed, then there exist two finite dimensional $k$-spaces $G$ and $H$ such that $G \oplus H$ contains no isotropic vector and $E$ is isomorphic to the (external) orthogonal sum $(\overset{\infty}{\Sigma} P) \oplus G \oplus G \oplus H$. In this case $E$ is uniquely determined by the ranges $||G + H||$ and by the space $H$.

We should like to mention that Theorem 2 alone can be obtained more directly by proving Theorem 1 only for spaces $E$ with $||E||$ of finite dimension over $k^2$. This is done by an induction on $\dim_{k^2}||E||$. For $\dim_{k^2}||E|| = 0$ we have $E = \Sigma P$. After induction assumption two cases arise which have to be treated differently: First case, there exists some decomposition $E = H \oplus H^\perp$ with finite dimensional $H$ such that $\dim_{k^2}||H^\perp|| < \dim_{k^2}||E||$. Hence there is a basis of the required sort for $H^\perp$ by the induction assumption. The required basis for $E$ is then found easily by applications of Corollary 1. Second case, there is no such decomposition of $E$. In that case, one proves directly that $E = E_{(\pi_1)} \oplus \ldots \oplus E_{(\pi_n)}$ where $\pi_1, \ldots, \pi_n$ span $||E||$. This is accomplished along the line of the proof of Proposition 2, where now the assumption of our case replaces the function of Lemma 4.

Thus, for fields $k$ with finite $[k : k^2]$ a complete list of non isomorphic $k$-spaces $(E, \Phi)$ of denumerable dimension can easily be given on the basis of Theorem 2, provided one knows the *finite* dimensional, non-isotropic $k$-spaces $(\langle\alpha_1, \ldots, \alpha_t\rangle!)$. It is advantageous to first subdivide the spaces according to the dimensions of $E/E_*$, $E_*^{\perp}$ and rad $(E_*)$. In the notations of Theorem 2: $p + q$, $r + s + t = \dim(E/E_*)$; $p + q$, $s + t = \dim(E_*^{\perp})$; $p, s = \dim(\text{rad } E_*)$ $p + q$, $r + s + t \leq [k : k^2]$. We may use uniformly the notations $r, s, t$ by interpreting a triple $(r, s, t)$ with $r = 0$ as belonging to a space of type (II). There are $\frac{(n+1)(n+2)(n+3)}{6}$ ordered triples $(r, s, t)$ with $0 \leq r + s + t \leq n$; they yield a subdivision of all semi-simple $\aleph_0$-dimensional $k$-spaces $(E, \Phi)$ according to their dimensions of $E/E_*$, $E_*^{\perp}$ and rad $E_*$ into $\frac{(n+1)(n+2)(n+3)}{6}$ classes $(n = [k : k^2])$. The particular choices for $\gamma_1, \ldots, \gamma_r$, $\beta_1, \ldots, \beta_s$, $\alpha_1, \ldots, \alpha_t$ are then taken. For the sake of illustration, we give a complete list for an underlying field $k$ with $[k: k^2] = 2$:

| $\dim E/E_*$ $r+s+t$ | $\dim E_*^{\perp}$ $s+t$ | dim (rad $E_*$) $s$ | |
|---|---|---|---|
| 0 | 0 | 0 | $\overset{\infty}{\Sigma} P$ |
| 1 | 0 | 0 | $E_{(\nu)}$ |
| 1 | 1 | 0 | $\overset{\infty}{\Sigma} P \oplus \langle\nu\rangle$ |
| 1 | 1 | 1 | $\overset{\infty}{\Sigma} P \oplus \langle\nu, \nu\rangle$ |
| 2 | 0 | 0 | $E_{(\alpha)} \oplus E_{(\beta)}$ |
| 2 | 1 | 0 | $E_{(\nu)} \oplus \langle\mu\rangle$ $\nu \neq \mu$ |
| 2 | 1 | 1 | $E_{(\alpha)} \oplus \langle\beta, \beta\rangle$, $E_{(\nu)} \oplus \langle\alpha, \alpha\rangle$ $\nu \neq \alpha$ |
| 2 | 2 | 0 | $\overset{\infty}{\Sigma} P \oplus \langle\alpha, \nu\rangle$ $\nu \neq \alpha$ |
| 2 | 2 | 1 | $\overset{\infty}{\Sigma} P \oplus \langle\beta, \beta\rangle \oplus \langle\alpha\rangle$, $\overset{\infty}{\Sigma} P \oplus \langle\alpha, \alpha\rangle \oplus \langle\nu\rangle$ $\nu \neq \alpha$ |
| 2 | 2 | 2 | $\overset{\infty}{\Sigma} P \oplus \langle\alpha, \alpha\rangle \oplus \langle\beta, \beta\rangle$ |

All the sums are orthogonal, $\{\alpha, \beta\}$ is some fixed basis of $k$ over $k^2$; $\nu$ and $\mu$ run independently through a fixed set of representatives of $g_k$ (the multiplicative group of $k$ modulo square factors), subject only to conditions listed in the table. All the spaces thus obtained are mutually non isomorphic and they are, up to orthogonal isomorphisms, all semi-simple $k$-spaces $(E, \Phi)$ of denumerable dimension.

## III. Orthogonal bases

Let $k$ be an arbitrary field of characteristic 2. If the semi-simple $k$-space $(E, \Phi)$ is finite dimensional, then either $E = \Sigma P$ or $E$ possesses an orthogonal basis (Lemma 1). Let $(E, \Phi)$ be a space of denumerable dimension. $E$ is an orthogonal sum $\Sigma P \oplus E_0$ where $E_0$ possesses an orthogonal basis. If $\dim_k (E/E_*)$ is infinite (i.e., $\dim_{k^2} ||E||$ is infinite), then $\dim E_0$ is infinite and $E$ has an orthogonal basis by virtue of Lemma 1. Thus, if $E$ does not admit of an orthogonal basis, then $E/E_*$ is of finite dimension and there exists a decomposition of $E$ as described in Theorem 2 (necessarily of type (II)): $E = \Sigma P \oplus E_0$, where $E_0$ is finite dimensional and spanned by an orthogonal basis. Conversely, a space of this form does not admit of an orthogonal basis for, $\Sigma P \oplus E_0 \subset \bigoplus^{\infty} k(e_i)$ gives $E_0 \subset \bigoplus_1^N k(e_i)$ for a suitable $N$ and thus, for the respective orthogonals, we obtain $\bigoplus_{N+1}^{\infty} k(e_i) \subset \Sigma P$. This is a contradiction as $||e_i|| \neq 0$ for an orthogonal basis of a semi-simple space. Thus, a space $(E, \Phi)$ of denumerable dimension admits of no orthogonal basis if and only if $E_*$ is closed and $E/E_*$ finite dimensional. These conditions may be formulated in various ways. Here is a selection:

**Theorem 3.** *Let $k$ be an arbitrary field of characteristic 2, $(E, \Phi)$ a semi-simple $k$-space of denumerable dimension. The following statements are equivalent:*

(j) *$E$ possesses no orthogonal basis;*

(jj) *$E/E_*$ is finite dimensional and $E_*$ is closed;*

(jjj) *$E_*^{\perp}$ is finite dimensional and* $\dim E/E_* = \dim E_*^{\perp}$;

(jv) *$E/E_*$ is finite dimensional and* $\dim (\operatorname{rad} E_*) = \dim E/(E_* + E_*^{\perp})$.

## IV. Automorphisms

We shall add here a few remarks about the group $\mathfrak{O}(E, \Phi)$ of all metric automorphisms of a space $(E, \Phi)$, i.e., the group of all vector space automorphisms $T: E \to E$ which satisfy $\Phi(Tx, Ty) = \Phi(x, y)$ for all $x, y \in E$. The underlying field $k$ is of characteristic 2 and $\dim E = \aleph_0$. The structure of the group $\mathfrak{O}(E, \Phi)$ is unknown in the general case. If $(E, \Phi)$ satisfies the conditions

$$E_*^{\perp} + E_*^{\perp\perp} \text{ is closed, } \dim(\operatorname{rad} E_*) < \aleph_0 \tag{1}$$ [1]

– which always takes place when the underlying field is of finite dimension $[k : k^2]$ over $k^2$ – then the study of $\mathfrak{O}(E, \Phi)$ can be reduced to the study of simpler groups. They are the (sympletic) group $\mathfrak{O}(E, \Phi)$, where the $\aleph_0$-dimensional space $(E, \Phi)$ is an orthogonal sum of hyperbolic planes, and the group $\mathfrak{O}(E, \Phi)$, where $(E, \Phi)$ is an orthogonal sum $E_{(\alpha_1)} \oplus E_{(\alpha_2)} \oplus \dots$ and the elements $\alpha_1, \alpha_2, \dots$ independent over $k^2$ (cf. 1.3 for notations). This reduction, possible for the spaces subject to (1), shall be carried out here.

For a space satisfying (1) there is decomposition (Theorem 1):

$$E = E_0 \oplus (R + R') \oplus E_1, \tag{2}$$

where $E_0$, $R \oplus R'$ and $E_1$ are orthogonal summands such that

$$R = \operatorname{rad} E_*, \; E_* = E_{0*} \oplus R, \; E_*^{\perp} = R \oplus E_1, \; E_*^{\perp\perp} = E_0 \oplus R \tag{3}$$

and, furthermore, $R \oplus R'$ is an orthogonal sum of planes $k(r_i, r_i')$, $i \in I$ for $\{r_i\}_{i \in I}$ and $\{r_i'\}_{i \in I}$ a basis of $R$ and $R'$ respectively. For every $T \in \mathfrak{O}(E, \Phi)$ we have $T(E_*) = E_*$, $T(R) = R$, $T(E_*^{\perp}) = E_*^{\perp}$ and $T(E_*^{\perp\perp}) = E_*^{\perp\perp}$. When $x \in R' \oplus E_1$ we write $Tx = x + Lx$. Hence $\|Lx\| = 0$ and $Lx \in E_* \subset E_*^{\perp\perp}$,

$$Lx \in E_0 \oplus R \text{ for } x \in R' \oplus E_1. \tag{4}$$

In particular, if $x \in R$ and $y \in R'$ then $(x, y) = (Tx, Ty) = (Tx, y + Ly) = (Tx, y)$ since $Tx \in R \perp E_0 \oplus R$. Therefore $(x - Tx, y) = 0$ for all $y \in R'$ or $x - Tx \in R'^{\perp}$, $R'^{\perp} \cap R = 0$; hence $x - Tx = 0$ since $x - Tx$ also belongs to $R$. Thus the restriction $T/R$ of $T$ to $R$ leaves the vectors of $R$ fixed,

$$T|_R = 1_R. \tag{5}$$

[1] We recall an earlier example where the second condition is satisfied but not the first. See the remark at the end of this section.

Let then $x \in E_1$ and $y \in R'$. Since $E_1 \subset E_*^\perp$ and $T(E_*^\perp) = E_*^\perp$ we have $Lx \in R$; hence $(x, y) = (Tx, Ty) = (x + Lx, y + Ly) = (x, y) + (Lx, y)$. Thus $(Lx, y) = 0$ for all $y \in R'$ i.e., $Lx \in R'^\perp$, $R'^\perp \cap R = 0$ and therefore $Lx = 0$ as $Lx \in R$. In other words,

$$T|_{E_1} = 1_{E_1} . \tag{6}$$

Thus, every automorphism of $E$ leaves $E_*^\perp$ pointwise fixed. Therefore we have for every $x \in R'$ and $y \in E_*^\perp$ that $(x, y) = (Tx, Ty) = (Tx, y)$ hence $x - Tx \in E_*^{\perp\perp} = E_0 + R$ for every $x \in R'$. Therefore, and in view of (5) and (6) we can decompose the image $Tx$ for every $x \in (R \oplus R') + E_1$ as follows, $Tx = x + L_0 x + L_1 x$ with $L_0 x \in E_0$ and $L_1 x \in R$. Computing $\| Tx \|$ shows furthermore that even $L_0 x \in E_{0*}$. We therefore have $(x \in R \oplus R' \oplus E_1)$

$$Tx = x + L_0 x + L_1 x \tag{7}$$

where the projections $L_0$ and $L_1$ are linear maps

$$L_0 : R \oplus R' \oplus E_1 \to E_{0*}, \ L_0(R \oplus E_1) = (0);$$

$$L_1 : R \oplus R' \oplus E_1 \to R, \ L_1(R \oplus E_1) = (0).$$

On the other hand, for $x \in E_0 \subset E_*^{\perp\perp} = E_0 \oplus R$ we have

$$(x \in E_0) \quad Tx = L_2 x + L_3 x \quad L_2 x \in E_0, \ L_3 x \in R . \tag{8}$$

Since $R$ is totally isotropic and orthogonal to $E_0$, $L_2 : E_0 \to E_0$ is a metric automorphism of $E_0$; $L_3$ is some linear map $E_0 \to R$. If we express $Tx$ for an arbitrary $x \in E$ by using (7) and (8), then the condition that $(x, y) = (Tx, Ty)$ for all $x, y \in E$, $T \in \mathfrak{O}(E, \Phi)$ is equivalent with the conditions

$$(x, L_3 y) + (L_0 x, L_2 y) = 0 \quad \text{for all} \quad x \in R', y \in E_0 \tag{9}$$

$$(x, L_1 y) + (L_1 x, y) + (L_0 x, L_0 y) = 0 \quad \text{for all} \quad x, y \in R' \tag{10}$$

(9) and (10) permits a discussion of $\mathfrak{O}(E, \Phi)$ as in the finite dimensional case

([2]). First, the system (9) and (10) admits of solutions $L_0$ and $L_1$ for arbitrarily prescribed $L_2$ and $L_3$, $L_2$ an automorphism of $E_0$ and $L_3: E_0 \to R$ a linear map. Indeed. For given $L_2$ and $L_3$ (9) defines a linear map $L_0: R' \to E_{0*}$ in a unique manner. We then extend it to $L_0: R \oplus R' \oplus E_1 \to E_{0*}$ by defining $L_0(R \oplus E_1) = (0)$. Appealing to the basis of $R \oplus R' = \bigoplus_I k(r_i, r'_i)$ we put $L_1 r'_i = \Sigma \alpha_{ij} r_j$. Condition (10) is satisfied with the previously found $L_0$ provided that $\alpha_{ij} + \alpha_{ji} = (L_0 r'_i, L_0 r'_j)$. Since $(L_0 r'_i, L_0 r'_i) = \| L_0 r'_i \| = 0$ as $L_0 r'_i \in E_{0*}$, there are always solutions for the unknowns $\alpha_{ij}$; (this is the only place where use is made of the assumption (1) that $\dim R < \aleph_0$). This proves our assertion. Thus, if $T$ runs through $\mathfrak{O}(E, \Phi)$ then the restriction $T|_{E_0 \oplus R}$ (it leaves $E_0 \oplus R = E_*^{\perp\perp}$ invariant!) runs through the group $\mathfrak{G}$ of all automorphisms of the space $E_0 \oplus R$ that leave $R$ pointwise fixed (as we have just proved, every element of $\mathfrak{G}$ can be extended to an automorphism of $E$). $T \to T|_{E_0 \oplus R}$ defines an epimorphism

$$\varphi: \mathfrak{O}(E, \Phi) \to \mathfrak{G} . \tag{11}$$

The kernel $\mathfrak{C} = \ker \varphi$ can easily be described. $T \in \mathfrak{C}$ means that $T|_{E_0 \oplus R}$ is the identical transformation of $E_0 \oplus R$. For such a $T$ and every $x \in E_0 \oplus R \oplus E_1$, $y \in R'$ we obtain from $(x, y) = (Tx, Ty) = (x, Ty)$ that $y - Ty \in (E_0 + R + E_1)^{\perp} = R$. Thus

$$Tx = x + L_4 x, \quad L_4 x \in R, \quad x \in E, \quad L_4(E_0 + R + E_1) = (0) \tag{12}$$

$(x, y) = (Tx, Ty)$ yields

$$(y, L_4 x) + (L_4 y, x) = (0) . \tag{13}$$

Conversely, every linear map $L_4: R' \to R$ meeting (13) defines an element $T \in \mathfrak{C}$ by means of (12). $\mathfrak{C}$ is thus seen to be isomorphic to the additive group of linear maps $L: R \to R'$ satisfying (13). Thus, as $s = \dim R$ is finite, $\mathfrak{C} \cong k^{\frac{s(s+1)}{2}}$. Let us turn to the group $\mathfrak{G}$. It contains the subgroup $\mathfrak{G}_0$ of automorphisms $T': E_0 \oplus R \to E_0 \oplus R$ of the form $T': x \to x + L_5 x$ where $L_5$ is an arbitrary linear map $L_5: E_0 \oplus R \to R$ with $L_5(R) = (0)$. $\mathfrak{G}_0$ is an invariant subgroup of $\mathfrak{G}$ and $\mathfrak{G}/\mathfrak{G}_0 \cong \mathfrak{O}(E_0, \Phi|_{E_0})$. $\mathfrak{G}_0$ is isomorphic to the additive group of all linear maps $L: E_0 \to R$, and $\mathfrak{G}_0 \cong k^\omega$ or $\mathfrak{G}_0 \cong (1)$.

Thus, if we put $\mathfrak{C}_0 = \varphi^{-1}\mathfrak{G}_0$, we have the series of invariant subgroups

$$\mathfrak{C} \subset \mathfrak{C}_0 \subset \mathfrak{O}(E, \Phi)$$

with $\mathfrak{C} \cong k^{\frac{s(s+1)}{2}}$, $\mathfrak{C}_0/\mathfrak{C} \cong \mathfrak{G}_0$, $\mathfrak{O}(E, \Phi)/\mathfrak{C}_0 \cong \mathfrak{O}(E_0, \Phi|_{E_0})$, $s = \dim(\mathrm{rad}\, E_*)$. $E_0$ is an algebraic complement of $\mathrm{rad}\, E_*$ in $E_*^{\perp\perp}$; it is either an orthogonal sum of hyperbolic planes or an orthogonal sum $E_{(\alpha_1)} \oplus \ldots \oplus E_{(\alpha_n)}$, the elements $\alpha_1, \alpha_2, \ldots, \alpha_n$ independent over $k^2$.

**Remark** (added in proof). The condition in (1) that $\dim R = \dim(\mathrm{rad}\, E_*) < \aleph_0$ is quite unnecessary for the discussion that followed. Setting $L_1 r_i' = \Sigma \alpha_{ij} r_j$ the matrix equation $\alpha_{ij} + \alpha_{ji} = (L_0 r_i', L_0 r_j')$ admits row-finite solutions (which actually define a map $L_1$); for example $\alpha_{ij} = 0$ $(j \geq i)$, $\alpha_{ij} = (L_0 r_i', L_0 r_j')$ for $j < i$. For the normal series of groups obtained we have in the case $\dim R = \aleph_0$: $G_0 \cong k^\omega$ and $C \cong k^\omega$.

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