A Proof of Thom's Theorem[1])

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§ 0. Introduction

The paper is designed to give a simple proof of a theorem of THOM (Théorème II. 10 of [11]), which states that the cohomology of the stable THOM object **MO** is a free module over the STEENROD algebra A over Z_2.

The proof is divided into three parts: we first recall that the stable cohomology is a coalgebra M over Z_2, and show that the graded dual M^* is a polynomial algebra; we then prove that M^* is an algebra over A^* (the graded dual of A); lastly we show that M^* is isomorphic to a free comodule over A^*. As a corollary of the proof of the main theorem, we give a short proof of the structure theorem for the unoriented cobordism ring $\mathfrak{N}_*$.

It seems possible to prove the theorems of WALL [12] on **MSO** in a similar way.

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§ 1. Cohomology of the Thom Spectrum

Let $0(n)$ be the n-dimensional real orthogonal group, $B_{0(n)}$ the classifying space for $0(n)$, γ_n the classifying n-plane bundle over $B_{0(n)}$. Let $\eta_n : E \to B_{0(n)}$ be the n-disk bundle associated with γ_n, $\dot{\eta}_n : \partial E \to B_{0(n)}$ the $(n-1)$-sphere bundle associated with η_n. Let $MO(n)$ be the space obtained from E by collapsing ∂E to a point. $MO(n)$ is called the THOM space of $0(n)$ ([11], [7], [3]).

The inclusion $0(n) \times 1 \subset 0(n+1)$ induces a map

$$MO(n) \divideontimes S^1 \to MO(n+1) \tag{1.1}$$

which yields isomorphisms of cohomology and homotopy in dimensions

$$n + k, \; k < n .$$

Thus a spectrum **MO** is obtained:

$$\mathbf{MO} = (\text{point}, \; MO(1), MO(2), \ldots, MO(k), MO(k+1), \ldots) . \tag{1.2}$$

The cohomology groups of MO are defined as follows (we will only consider coefficients Z_2):

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$$H^k(MO; Z_2) = H^{n+k}(MO(n); Z_2) \quad k < n\,. \tag{1.3}$$

We will write M for $\sum_k H^k(MO; Z_2)$. The Steenrod algebra operates on M. The A-module structure of M is given by Thom's theorem:

Theorem 1. (Thom). The A-module M is a free A-module, with free generators $u(\omega)$ in one-to-one correspondence with partitions ω of integers into integers, none of which have the form $2^t - 1$ for $t > 0$.

The theorem was first proved in [11]. A new proof will be given in § 3.

The additive structure of M is easily determined. Let $s: B_{0(n)} \to E$ be the zero cross section of η_n, above. We still denote by s the map induced by s into $MO(n) = E/\partial E$. It is well known [7] that s^* is a monomorphism, and that Image $s^* = w_n H^*(B_{0(n)}; Z_2)$, where w_n is the top Stiefel-Whitney class.

Since $H^*(B_{0(n)}; Z_2) = Z_2[w_1, \ldots, w_n]$, we have the result that

$$M \cong Z_2[w_1, \ldots\ w_k, \ldots]\,, \tag{1.4}$$

as graded vector spaces, where grade $(w_k) = k$.

It has been noted [9] that, although M does not have a natural algebra structure, it does have a natural coalgebra [8] structure. Consider the usual inclusion

$$0(m) \times 0(n) \subset 0(m+n)\,; \tag{1.5}$$

it induces a map

$$\varrho_{m,n}: MO(m) \divideontimes MO(n) \to MO(m+n)\,. \tag{1.6}$$

The maps $\varrho_{m,n}$ induce

$$\varrho^*: M \to M \otimes M\,, \tag{1.7}$$

which make M into a coalgebra over Z_2 (the symbol $\otimes$ of course stands for $\otimes_{Z_2}$), and the coproduct ϱ^* is consistent with the operation of A on M, that is, the following diagram is commutative:

$$\begin{tikzcd} A \otimes M \arrow[r, "\psi \otimes \varrho^*"] \arrow[d, "\pi"'] & A \otimes A \otimes M \otimes M \arrow[r, "1 \otimes T \otimes 1"] & A \otimes M \otimes A \otimes M \arrow[d, "\pi \otimes \pi"] \\ M \arrow[rr, "\varrho^*"] & & M \otimes M\,, \end{tikzcd} \tag{1.8}$$

where $\pi: A \otimes M \to M$ is the action of A on M, $\psi: A \to A \otimes A$ is the coproduct [6] in A, and T is the twist map which interchanges factors.

We can describe the map ϱ^* very easily, because the following diagram is commutative:

$$\begin{tikzcd} MO(m) \divideontimes MO(n) \arrow[rr, "\varrho"] & & MO(m+n) \\ M(0(m) \times 0(n)) \arrow[u, "\approx"'] & & \\ B_{0(m)\times 0(n)} \arrow[u, "s"'] \arrow[rr, "\sigma"] & & B_{0(m+n)}, \arrow[uu, "s"'] \end{tikzcd} \tag{1.9}$$

where σ is the WHITNEY direct sum map, induced from (1.5).

Under the isomorphism (1.4) ϱ^* corresponds to σ^*, but σ^* is well known (as the WHITNEY direct sum theorem [5]):

$$\sigma^*(w_k) = \sum_{i+j=k} w_i \otimes w_j . \tag{1.10}$$

§ 2. Comodules over A^*

Let A^* be the graded dual of the STEENROD algebra A over Z_2. Let φ be the product and ψ the coproduct of A; we will denote by φ^* the coproduct and ψ^* the product of A^*. If we let $\varepsilon : A \to Z_2$ be the augmentation of A and $\eta : Z_2 \to A$ the unit of A, then the dual maps ε^* and η^* are the unit and augmentation of A^*. According to [6], A^* is the algebra of polynomials $Z_2[\xi_1, \ldots, \xi_n, \ldots]$, grade $\xi_n = 2^n - 1$, with the coproduct given by

$$\varphi^*(\xi_n) = \sum_{i=O}^{n} \xi_{n-i}^{2^i} \otimes \xi_i . \tag{2.1}$$

The notion of a comodule L over A^* is just the obvious dualization of the notion of a module over A:

Definition. A Z_2-module L is called a comodule over A^* if there exists a map

$$\mu : L \to A^* \otimes L , \tag{2.2}$$

called the coaction of A^*, such that the following two diagrams are commutative:

$$\begin{tikzcd} L \arrow[r, "\mu"] \arrow[d, "\mu"'] & A^* \otimes L \arrow[d, "1 \otimes \mu"] \\ A^* \otimes L \arrow[r, "\varphi^* \otimes 1"] & A^* \otimes A^* \otimes L , \end{tikzcd} \tag{2.3}$$

$$\begin{tikzcd} L \arrow[r, "\mu"] \arrow[d, "1"'] & A^* \otimes L \arrow[d, "\eta^* \otimes 1"] \\ L \arrow[r, "\cong"] & Z_2 \otimes L . \end{tikzcd} \tag{2.4}$$

We immediately cite examples of A^*-comodules.

1. A^* itself is a comodule over A^* under φ^* as coaction.

2. If N is a graded module over A (suppose that N is finite dimensional in each grading) with action

$$\lambda : A \otimes N \to N\,, \tag{2.5}$$

then the graded dual N^* is a comodule over A^* with coaction the dual of λ:

$$\lambda^* : N^* \to A^* \otimes N^*\,. \tag{2.6}$$

3. If V is a vector space over Z_2, we can construct a free comodule $F = A^* \otimes V$ by letting

$$\mu : F \to A^* \otimes F \tag{2.7}$$

be just $\varphi^* \otimes 1$.

Free comodules have the expected properties: we just quote two, which we will use in the proof of Theorem 1.

Proposition 1. Let V be a Z_2-module and $F = A^* \otimes V$ a free A^*-comodule on V. Suppose we are given a comodule L over A^* and a Z_2-map

$$f : L \to V\,. \tag{2.8}$$

Then there exists a unique A^*-comodule map

$$g : L \to F \tag{2.9}$$

which makes the following diagram commutative:

$$\begin{array}{ccc} L & \xrightarrow{\quad g \quad} & F \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle 1} \\ A^* \otimes L & \xrightarrow{1 \otimes f} & A^* \otimes V\,. \end{array} \tag{2.10}$$

The map g is said to be induced by f.

Proof. Define $g = (1 \otimes f)\,\mu$. The following commutative diagram proves that g is a map of A^*-comodules:

$$\begin{array}{ccccc} L & \xrightarrow{\quad \mu \quad} & A^* \otimes L & \xrightarrow{\qquad 1 \otimes f \qquad} & A^* \otimes V \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \varphi^* \otimes 1} & & \downarrow{\scriptstyle \varphi^* \otimes 1} \\ A^* \otimes L & \xrightarrow{1 \otimes \mu} & A^* \otimes A^* \otimes L & \xrightarrow{1 \otimes 1 \otimes f} & A^* \otimes A^* \otimes V\,. \end{array} \tag{2.11}$$

Definition. We say that the Z_2-vector space L is an algebra over A^* if 1) it is an A^*-comodule with coaction μ (2.2), and 2) it is a Z_2-algebra with multiplication

$$h : L \otimes L \to L \tag{2.12}$$

such that the following diagram is commutative:

$$\begin{tikzcd}
L \otimes L \arrow[rr, "h"] \arrow[d, "\mu \otimes \mu"'] & & L \arrow[dd, "\mu"] \\
A^* \otimes L \otimes A^* \otimes L \arrow[d, "1 \otimes T \otimes 1"'] & & \\
A^* \otimes A^* \otimes L \otimes L \arrow[rr, "\psi^* \otimes h"] & & A^* \otimes L .
\end{tikzcd} \tag{2.13}$$

Proposition 2. Let V be a Z_2-algebra, $F = A^* \otimes V$ the free A^*-comodule on V. Then

i) F is an A^*-algebra under $(\psi^* \otimes h')(1 \otimes T \otimes 1)$, where $h' : V \otimes V \to V$ is the product in V,

ii) If L is an A^*-algebra, and

$$f : L \to V \tag{2.14}$$

is a map of Z_2-algebras, then the comodule map induced by f

$$g : L \to F \tag{2.15}$$

is a map of A^*-algebras.

Proof. Part i) is an immediate consequence of the fact that A^* is a Hopf algebra under ψ^*, φ^*. The reader is invited to draw the appropriate commutative diagram.

We prove that g is a map of algebras by referring to the commutative diagram (2.16):

$$\begin{tikzcd}
L \otimes L \arrow[rr, "h"] \arrow[d, "\mu \otimes \mu"] & & L \arrow[d, "\mu"] \\
A^* \otimes L \otimes A^* \otimes L \arrow[r, "1 \otimes T \otimes 1"] \arrow[d, "1 \otimes f \otimes 1 \otimes f"] & A^* \otimes A^* \otimes L \otimes L \arrow[r, "\psi^* \otimes h"] \arrow[d, "1 \otimes 1 \otimes f \otimes f"] & A^* \otimes L \arrow[d, "1 \otimes f"] \\
A^* \otimes V \otimes A^* \otimes V \arrow[r, "1 \otimes T \otimes 1"] & A^* \otimes A^* \otimes V \otimes V \arrow[r, "\psi^* \otimes h'"] & A^* \otimes V .
\end{tikzcd} \tag{2.16}$$

§ 3. Proof of Thom's Theorem

Let n be a fixed positive integer,

$$R^{(n)} = Z_2[w_1, \ldots, w_n] \tag{3.1}$$

a graded polynomial algebra on n indeterminates $w_i, i = 1, \ldots, n$, with grade $(w_k) = k$. We make $R^{(n)}$ into a Hopf algebra by setting

$$\sigma^*(w_k) = \sum_{i+j=k} w_i \otimes w_j \,. \tag{3.2}$$

Let

$$S^{(n)} = Z_2[y_1, \ldots, y_n] \,, \tag{3.3}$$

where grade $(y_i) = 1, i = 1, \ldots, n$.

Suppose ω is a partition of a non-negative integer k:

$$\omega = (i_1, \ldots, i_q), \ \omega \in \Pi(k) \,. \tag{3.4}$$

If all of $i_1, \ldots, i_q$ are positive, we write

$$\| \omega \| = q \,, \tag{3.5}$$

if $k = 0$, we set $\| \omega \| = 0$.

If $\| \omega \| \leq n$, we will denote by $s(\omega)$ the smallest symmetric polynomial in $S^{(n)}$ containing the monomial $y_1^{i_1} \ldots y_q^{i_q}$ (see [7], for example).

Let us make $S^{(n)}$ into a Hopf algebra by setting

$$\sigma^*(y_i) = y_i \otimes 1 + 1 \otimes y_i \,; \tag{3.6}$$

then

$$\sigma^*(s(\omega)) = \sum_{(\omega_1, \omega_2) = \omega} s(\omega_1) \otimes s(\omega_2) \tag{3.7}$$

(compare [5]). We may thus consider $R^{(n)}$ as a Hopf subalgebra of $S^{(n)}$, by identifying w_i with $s((1, \ldots, 1)), (1, \ldots, 1) \in \Pi(i)$. Under this identification, a Z_2-basis of $R^{(n)}$ is furnished by the set of elements

$$\{s(\omega) \mid \omega \in \Pi(k), k \geq 0, \| \omega \| \leq n\} \,. \tag{3.8}$$

If we consider the normal inclusions $R^{(n)} \subset R^{(n+1)}$, $S^{(n)} \subset S^{(n+1)}$, we see that we can define Hopf algebra retractions $f^{(n+1)} : R^{(n+1)} \to R^{(n)}$, $g^{(n+1)} : S^{(n+1)} \to S^{(n)}$ which make the following diagram commutative:

$$\begin{array}{ccc} R^{(n+1)} & \xrightarrow{f^{(n+1)}} & R^{(n)} \\ \cap\big\downarrow & & \big\downarrow\cap \\ S^{(n+1)} & \xrightarrow{g^{(n+1)}} & S^{(n)} . \end{array} \tag{3.9}$$

The maps are defined by:

$$\begin{aligned} f^{(n+1)}(w_j) &= w_j \quad \text{if } j \leq n \\ &= 0 \quad \text{if } j = n+1 \\ g^{(n+1)}(y_j) &= y_j \quad \text{if } j \leq n \\ &= 0 \quad \text{if } j = n+1 \,. \end{aligned} \tag{3.10}$$

We remark that $f^{(n+1)}$ is an isomorphism in gradings $< n+1$. If we now consider the Hopf algebra

$$R = Z_2[w_1, \ldots, w_k, \ldots], \tag{3.11}$$

where we set

$$\sigma^*(w_k) = \sum_{i+j=k} w_i \otimes w_j, \tag{3.12}$$

we can define Hopf algebra epimorphisms

$$\begin{aligned} &h^{(n)} : R \to R^{(n)} \\ &h^{(n)}(w_j) = w_j \quad j \leq n, \\ &h^{(n)}(w_j) = 0 \quad j > n. \end{aligned} \tag{3.13}$$

Given $\omega \in \Pi(k)$, we define

$$\tilde{s}(\omega) = h^{(n)-1}(s(\omega)), \quad n > k. \tag{3.14}$$

The definition makes sense, for $h^{(n)}$ is an isomorphism in gradings $< n + 1$, and $s(\omega)$ is independent of the choice of $n > k$, according to (3.9).

From (3.8) we see that the set of elements

$$\{\tilde{s}(\omega) \mid \omega \in \Pi(k), \ k \geq 0\} \tag{3.15}$$

forms a Z_2-basis of R.

Let R^* be the graded dual of R. Let $\tilde{s}(\omega)^*$ be the dual basis to (3.15). The elements $\tilde{s}(\omega)^*$ are characterized by:

$$\langle \tilde{s}(\omega)^*, \tilde{s}(\omega')\rangle = \begin{cases} 1 & \omega' = \omega, \\ 0 & \omega' \neq \omega. \end{cases} \tag{3.16}$$

Let

$$x_k = \tilde{s}((k))^*. \tag{3.17}$$

Proposition 3. As an algebra,

$$R^* = Z_2[x_1, \ldots, x_k, \ldots]. \tag{3.18}$$

Proof. Let $T = Z_2[\tilde{x}_1, \ldots, \tilde{x}_k, \ldots]$, grade $(\tilde{x}_k) = k$. Since R has commutative, associative coproduct, R^* is a commutative, associative algebra, therefore the assignment $f(\tilde{x}_k) = x_k$ defines an algebra map

$$f : T \to R^*. \tag{3.19}$$

We claim that f is an epimorphism. To prove this, it is sufficient to show that for each $\omega \in \Pi(k)$, $k \geq 0$ the element $\tilde{s}(\omega)^*$ is in the image of f. This follows from the

Lemma. If $\omega = 1^{\lambda_1} \ldots q^{\lambda_q} \ldots k^{\lambda_k}$ (where λ_q is the number of times q occurs in ω), then

$$\tilde{s}(\omega)^* = x_1^{\lambda_1} \ldots x_q^{\lambda_q} \ldots x_k^{\lambda_k}.$$

Proof of Lemma. The result follows from the equation

$$\langle x_1^{\lambda_1}\dots x_k^{\lambda_k}, \tilde{s}(\omega')\rangle = \langle \underbrace{x_1\otimes\dots\otimes x_1}_{\lambda_1}\otimes\dots\otimes \underbrace{x_k\otimes\dots\otimes x_k}_{\lambda_k}, \sigma^{(m)}\tilde{s}(\omega')\rangle\,, \tag{3.20}$$

where $m = \sum_i \lambda_i$, and $\sigma^{(m)}$ denotes the coproduct σ^* iterated $m-1$ times.

The proof of Proposition 3 is now immediate: since f preserves grading, and T with R have the same dimension in each grading, we know that since f is an epimorphism, it is also a monomorphism.

Corollary: As an algebra,

$$M^* = Z_2[x_1, \dots, x_k, \dots]\,, \tag{3.21}$$

where $$x_k = \tilde{s}((k))^*, \text{ grade } (x_k) = k\,.$$

Proof. Proposition 3 and (1.4), (1.10).

For the next proposition, we hark back to the isomorphism

$$s^* : M_t = w_n H^t(B_{0(n)}; Z_2)$$

of A-modules for $t < n$ (1.3). For what follows, we always suppose that n was picked large. The elements $\tilde{s}(\omega)$ (3.14) satisfy

$$s^*(\tilde{s}(\omega)) = w_n s(\omega)\,. \tag{3.22}$$

Proposition 4. Let $k = 2^t - 1$, $\vartheta \in A$, $\omega \in \Pi(q)$, grade $\vartheta = k - q$. Then

$$\begin{aligned} \langle x_k, \vartheta\,\tilde{s}(\omega)\rangle &= 0 \text{ if } \omega \neq (q), q = 2^s - 1\,,\\ \langle x_k, \vartheta\,\tilde{s}((q))\rangle &= \langle \xi_{t-s}^{2^s}, \vartheta\rangle \text{ if } q = 2^s - 1\,. \end{aligned} \tag{3.23}$$

Proof. Consider the A-map $h: A \to M$ defined by $h(1) = \tilde{s}((0))$. This is the well-known CARTAN-SERRE representation of A ([4], [10]), for

$$s^* h(\vartheta) = s^*(\vartheta\,\tilde{s}((0))) = \vartheta\, s^*\,\tilde{s}((0)) = \vartheta\, w_n\,. \tag{3.24}$$

If we identify w_n with $s(1^n) = y_1\dots y_n$, we get ([2], p. 43)

$$\vartheta\, w_n = \vartheta(y_1\dots y_n) = \sum_{(i_1,\dots,i_n)} \langle \xi_{i_1}\dots\xi_{i_n}, \vartheta\rangle\, y_1^{2^{i_1}}\dots y_n^{2^{i_n}}\,. \tag{3.25}$$

To find $\vartheta\,\tilde{s}(\omega)$, where $\omega = 1^{\lambda_1}\dots k^{\lambda_k}$, it is sufficient to take

$$\vartheta(y_1^{\lambda_1+1}\dots y_k^{\lambda_k+1} y_{k+1}\dots y_n)$$

and symmetrize the result. In particular, if $\omega = (2^s - 1$, we see that

$$\vartheta(y_1^{2^s} y_2\dots y_n) = \sum \langle \xi_{i_1}^{2^s}\xi_{i_2}\dots\xi_{i_n}, \vartheta\rangle\, y_1^{2^{i_1+s}} y_2^{2^{i_2}}\dots y_n^{2^{i_n}}\,, \tag{3.26}$$

which proves part of Proposition 4. Let us call a partition

$$\omega \in \Pi(k),\ \omega = 1^{\lambda_1}\dots k^{\lambda_k}$$

honest, if for at least one λ_j we have $0 < \lambda_j < k$. It is then an immediate consequence of (3.25) that if ω is an honest partition, $\vartheta \in A$ and $\vartheta \tilde{s}(\omega) = \Sigma c_{\omega'} \tilde{s}(\omega')$, $c_{\omega'} \in Z_2$, then $c_{\omega'} \neq 0$ implies ω' is an honest partition. For partitions $\omega = (q)$, $q \neq 2^s - 1$, we prove again using (3.25) that $\vartheta \tilde{s}(\omega)$ is in the subspace spanned by elements $\tilde{s}(\omega')$, where ω' is an honest partition.

Proposition 5. Let $\mu^* : M^* \to A^* \otimes M^*$ be the coaction of A^* on M^*. Then

$$\mu^*(x_{2^t-1}) = \sum_{s=0}^{t} \xi_{t-s}^{2^s} \otimes x_{2^s-1}, \tag{3.27}$$

where we set $x_0 = 1$.

Proof. Let $\mu^*(x_k) = \Sigma \alpha_\omega \otimes \tilde{s}(\omega)^*$. The term $\alpha_\omega \otimes \tilde{s}(\omega)^*$ occurs in $\mu^*(x_k)$ with a non-zero coefficient if and only if for $\vartheta \in A$, grade $\vartheta =$ grade α_ω we have

$$\langle x_k, \vartheta \tilde{s}(\omega)\rangle = \langle \alpha_\omega, \vartheta \rangle. \tag{3.28}$$

Proposition 4 completes the proof.

Corollary. Let $q : A^* \to M^*$ be a map of Z_2-algebras, defined by

$$q(\xi_k) = x_{2^k-1}.$$

Then q is a monomorphism of A^*-algebras.

Proof. (2.1) and (3.27).

Let $H^* = Z_2[u_2, \ldots, u_k, \ldots]$, $k \neq 2^t - 1$, any $t > 0$, grade $(u_k) = k$. Let

$$f : M^* \to H^* \tag{3.29}$$

be an epimorphism of algebras, defined by

$$\begin{aligned} f(x_k) &= u_k \text{ if } k \neq 2^t - 1 \text{ for any } t > 0, \\ &= 0 \quad \text{if } k = 2^t - 1, t > 0. \end{aligned} \tag{3.30}$$

Consider the free A^*-comodule $F = A^* \otimes H^*$. According to Proposition 2, F is an A^*-algebra. Furthermore, Proposition 1 shows that there exists a comodule map g induced by f; Proposition 2 asserts that g is a map of algebras.

Let $H^{*(m)}$ be the subalgebra of H^* generated by $1, f(x_1), \ldots, f(x_m)$.

Lemma.

$$g(x_{2^t-1}) = \xi_t \otimes 1, \tag{3.31}$$

$$g(x_k) \equiv 1 \otimes u_k \bmod \bar{A}^* \otimes H^{*(k-1)} \quad \text{if } k \neq 2^t - 1, t > 0. \tag{3.32}$$

Proof. Formula (3.31) follows from (3.27). The assertion (3.26) follows from the remark that $\mu^*(x_k) \equiv 1 \otimes x_k \bmod \bar{A}^* \otimes M^{*(k-1)}$, where $M^{*(k-1)}$ is the subalgebra generated by $1, x_1, \ldots, x_{k-1}$.

Proposition 6. The map

$$g : M^* \to A^* \otimes H^* \tag{3.33}$$

induced by f (3.30) yields an isomorphism of algebras over A^*.

Proof. Since M^* and $A^* \otimes H^*$ are graded, have the same (finite) dimension in each grading as Z_2-modules, and g is grading preserving, it is sufficient to prove that g is an epimorphism. Let us prove this by showing that the image of g contains $A^* \otimes H^{*(m)}$. This is true for $m = 1$, for $H^{*(1)} = \{1\}$, and $\xi_t \otimes 1 \in \operatorname{Image} g$, according to (3.31). Suppose $\operatorname{Im}(g) \supset A^* \otimes H^{*(m-1)}$. If $m = 2^t - 1$ for some $t > 0$, then $H^{*(m)} = H^{*(m-1)}$, and we are done; suppose, therefore, that $m \neq 2^t - 1$ for any $t > 0$. According to (3.32) and the induction hypothesis, there is an element $z_m \in \bar{A}^* \otimes M^{*(m-1)}$ such that

$$g(x_m + z_m) = 1 \otimes u_m .$$

Since g is a map of algebras, this proves that $A^* \otimes M^{*(m)} \subset \operatorname{Im} g$. Induction completes the proof.

Proof of Theorem 1.

Consider the dual map to g:

$$g^* : A \otimes H \to M . \tag{3.34}$$

Since g^* is an isomorphism of A^*-algebras, g is an isomorphism of A-coalgebras. A Z_2-basis of H is given by the dual basis of the basis of H^* consisting of monomials in the u_k, $k \neq 2^t - 1$, $t > 0$.

This completes the proof of THOM's Theorem. We cannot, however, restrain ourselves from taking the argument one step further. Let $\mathfrak{N}_*$ be the unoriented cobordism ring [11]. According to a fundamental theorem of THOM (Théorème IV. 8 [11]), there is a naturally defined isomorphism

$$T : \Pi_{n+k}(MO(n)) \to \mathfrak{N}_{*k} \quad k < n . \tag{3.35}$$

Furthermore, the product in $\mathfrak{N}^*$ corresponds under this isomorphism to the map induced by (1.6) [9].

We can use the ADAMS spectral sequence [1] as in [7] to compute the homotopy of **MO**. It is sufficient to look at the ADAMS spectral sequence for $p = 2$. The E_2-term is given by

$$E_2^{s,t} = \operatorname{Ext}_A^{s,t}(M, Z_2) . \tag{3.36}$$

Since M is a coalgebra over A with coproduct ϱ^*, $\mathrm{Ext}_A^{s,t}(M, Z_2)$ is an algebra; furthermore, the multiplication in the E_∞ terms corresponds to the multiplication in homotopy induced by ϱ^{**}_{*}. However, since M is $A \otimes H$ as an A-coalgebra, we have

$$\mathrm{Ext}_A^{*,*}(M, Z_2) = \mathrm{Ext}_A^{0,*}(M, Z_2) \cong H^* \tag{3.37}$$

as an algebra. Thus $E_2^{s,t} = 0$ unless $s = 0$, hence the ADAMS spectral sequence collapses in the nicest way imaginable—and we have the following theorem, also first proved by THOM:

Theorem 2. The ring $\mathfrak{N}_*$ is a polynomial ring over Z_2 in generators u_k, where $k = 2, \ldots, k \neq 2^t - 1$ for any $t > 0$.

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