Homotopy Groups of Maps and Exact Sequences.

Homotopy Groups of Maps and Exact Sequences

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1. Introduction

In [3] we described two exact sequences arising in homotopy theory, dual to each other, which contain as special cases many of the familiar sequences of algebraic topology (e.g., homotopy sequence, cohomology sequence, cohomotopy sequence, coefficient sequences). Certain other sequences (e.g. the homotopy and cohomology sequences of a triple and the homotopy sequence of a triad) may be deduced as special cases of sequences involving objects and maps in the category of pairs of pairs. It has seemed worthwile to make a *systematic study of the two exact sequences in the category of pairs* corresponding to the two sequences mentioned above in the category of based spaces. We should point out that the latter sequences are more accurately described as functors of the product category $\mathfrak{T} \times \mathfrak{P}(\mathfrak{T})$, where $\mathfrak{T}$ is the category of based spaces and $\mathfrak{P}(\mathfrak{T})$ is the category of pairs from $\mathfrak{T}$. Thus the two sequences we introduce in this paper (section 4) are functors of the product category $\mathfrak{P}(\mathfrak{T}) \times \mathfrak{P}^2(\mathfrak{T})$ and this explains the introduction of the *category of pairs of pairs* $\mathfrak{P}^2(\mathfrak{T})$ mentioned above.

The *relative groups* $\Pi_n(A, \beta)$, $\Pi_n(\alpha, B)$ of [3] are essentially mixed constructions and cannot, without suitable conventions, be meaningfully regarded as sets of homotopy classes of maps of $\Sigma^n A$ into β or α into $\Omega^n B$. Because of their hybrid nature we prefer in this paper to use a new symbol $P_n(A, \beta)$, $P_n(\alpha, B)$ for these groups, thus pointing the contrast with the groups $\Pi_n(A, B)$, $\Pi_n(\alpha, \beta)$, which are sets of classes of maps $\Sigma^n A \to B$ or $\Sigma^n \alpha \to \beta$. It thus appears in our formulation that the groups $P_n(A, \beta)$, $P_n(\alpha, B)$ only represent as it were a halfway stage in the process of relativization, in that only one of the variables is taken from the category $\mathfrak{P}(\mathfrak{T})$, the other variable remaining an object of $\mathfrak{T}$; and that full relativization of the basic construct $\Pi_n(A, B)$ leads to the groups $\Pi_n(\alpha, \beta)$. In specializing such relative groups one is led to define cohomology groups and homotopy groups (of pairs) whose "coefficients" lie in a *cohomology or homotopy operation.*

If our object were just to derive the classical homotopy sequence in the category of pairs we could have based ourselves on the KAN theory of categories with homotopy [7]. However not only have we wished to discuss far

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[1]) This research was partly supported by the U.S. Department of Army through its European Research Office.

more general sequences but we have also wanted to bring out the additional structure present in the categories $\mathfrak{P}(\mathfrak{T})$ and $\mathfrak{P}^2(\mathfrak{T})$. In $\mathfrak{P}(\mathfrak{T})$ the objects have an obvious *groupoid structure;* that is, objects may sometimes be multiplied. Moreover the groupoid structure is associative and admits left and right identities. In $\mathfrak{P}^2(\mathfrak{T})$ we have two such groupoid structures: for an object of $\mathfrak{P}^2(\mathfrak{T})$ is a map Ψ,

$$\begin{tikzcd} \cdot \arrow[r, "\beta_1"] \arrow[d, "\sigma"'] & \cdot \arrow[d, "\sigma'"] \\ \cdot \arrow[r, "\beta_2"'] & \cdot \end{tikzcd} \qquad \Downarrow \Psi$$

and so we have horizontal and vertical composition. In addition there is a *transposition operation* in $\mathfrak{P}^2(\mathfrak{T})$, converting Ψ into the map Ψ^T,

$$\begin{tikzcd} \cdot \arrow[r, "\sigma"] \arrow[d, "\beta_1"'] & \cdot \arrow[d, "\beta_2"] \\ \cdot \arrow[r, "\sigma'"'] & \cdot \end{tikzcd} \qquad \Downarrow \Psi^T$$

and there are obvious relations connecting transposition with the two composition operations. Our proof of the exactness of the *sequences of a triple* (section 7) is designed to exploit this additional structure.

Section 2 consists of a review of those classical exact sequences described in [3], together with a small amount of generalization of the sequences and the groups which enter into them. Section 3, which is preparatory for the two subsequent sections, describes certain *canonical homotopy constructions* whose naturality enables them to do service both in $\mathfrak{T}$ and in $\mathfrak{P}(\mathfrak{T})$; in particular we use them in section 4 to prove the exactness of the two basic sequences in $\mathfrak{P}(\mathfrak{T})$. In section 5 we show how, just as for the familiar sequences in $\mathfrak{T}$, the presence of a *fibration* or *cofibration* leads to the replacement of the hybrid terms in the appropriate sequence by a pure term. It also turns out that the mapping track functor and mapping cylinder functor lead, as one would hope, to the replacement of arbitrary maps in $\mathfrak{P}(\mathfrak{T})$ by fibrations or cofibrations. The mixed sequences of section 6 are specializations of the sequences of section 4 and are, more precisely, functors of the product category $\mathfrak{T} \times \mathfrak{P}^2(\mathfrak{T})$. The two homotopy sequences of a *triad* are special cases of one of the sequences of this section, the invariance of the groups $P_n(A, \Psi)$ under transposition corresponding to the fact of the presence of the same triad homotopy group in both triad sequences. In section 7 we study the transposition operation more closely and then proceed to obtain the two exact sequences of a triple, generalizing the classical homotopy and cohomology sequences respectively. The sequences of section 7 are further generalized in section 8 to sequences more genuinely based on the category $\mathfrak{P}(\mathfrak{T})$ (i.e. not involving objects of $\mathfrak{T}$ and functors from $\mathfrak{T}$ to $\mathfrak{P}(\mathfrak{T})$). It turns out, perhaps a little surprisingly, that the "missing" group in these sequences is not the likely looking candidate $\Pi_n(\alpha, \beta)$ but a different group $\tilde{\Pi}_n(\alpha, \beta)$. We call the latter a *twisted homotopy group* because there are basic functions $\iota_1 : \mathfrak{P} \to \mathfrak{P}^2$, $\varrho_1 : \mathfrak{P} \to \mathfrak{P}^2$ such that $\Pi_n(\alpha, \beta)$ may be identified with $\Pi_{n-1}(\iota_1\alpha, \varrho_1\beta)$, while $\tilde{\Pi}_n(\alpha, \beta)$ is, by definition, $\Pi_{n-1}((\iota_1 \alpha)^T, \varrho_1\beta) = \Pi_{n-1}(\iota_1\alpha, (\varrho_1\beta)^T)$. Of course, the various exact sequences discussed fit into a pattern of exact sequences; this pattern often takes the precise form of an exact triangle of exact sequences and we have usually displayed such connections between the sequences we define.

The last section is an appendix giving a *combinatorial* treatment of *homology groups* and their exact sequences intended to show a parallel with the exact sequences involving cohomology groups which arose by specialization from the general theory.

Throughout we have kept the *dual aspects* of homotopy theory in the foreground. Thus every general result has its dual counterpart and all our notations are designed to bring out the duality relations. A result and its dual are given the same numerical index, one index appearing plain and the other with a superscript star. In particular, the duality does not, of course, permit us to regard a *pair* as an inclusion nor a triple as a system of two inclusions; *thus a pair is*, as in [3], *just a map* and a triple is a system $[\lambda, \mu]$ of two maps

$$P \xrightarrow{\lambda} Q \xrightarrow{\mu} R .$$

Similarly, as indicated above, a pair of pairs is essentially just a commutative square of maps, though it is also imbued with a *sense* in that one passes from top to bottom (or left to right) across the square.

The homotopy and homology sequences of a triple were applied in [5] to establish the homotopy and homology decompositions of a map.

2. Review of exact sequences in the category of spaces

We recall that $\mathfrak{T}$ denotes the category of spaces with base point and based maps and $\mathfrak{P}(\mathfrak{T})$ – or just $\mathfrak{P}$ – the category of pairs (i.e., maps) from $\mathfrak{T}$. The base point will be written o, for any space X in $\mathfrak{T}$.

In this section we recall the definitions of the exact sequences in $\mathfrak{T}$ given in [3, 4]. This should assist the reader in passing to the corresponding sequences in $\mathfrak{P} = \mathfrak{P}(\mathfrak{T})$; but it will also enable us to introduce certain changes of notation which appear to us to be suitable to this broad treatment of exact sequence theory, and to make certain auxiliary remarks.

The notations, then, are those of [3] with the following exceptions. The map $A \to o$ will be written $\omega(A)$, or just ω, instead of $\underset{\circ}{\alpha}$ or $\underset{\circ}{A}$; and the map $o \to A$ will be written $\tilde{\omega}(A)$, or just $\tilde{\omega}$, instead of $\overset{\circ}{\alpha}$ or $\overset{\circ}{A}$. The groups $\Pi_n(A, \beta)$, $\Pi_n(\alpha, B)$, which are mixed constructions, using both $\mathfrak{T}$ and $\mathfrak{P}$, we will now write as $P_n(A, \beta)$, $P_n(\alpha, B)$, reserving the symbol Π for the pure constructions $\Pi_n(A, B)$, $\Pi_n(\alpha, \beta)$. Thus, by definition,

$$\begin{aligned} P_n(A, \beta) &= \Pi(\iota_n(A), \beta) \\ P_n(\alpha, B) &= \Pi(\alpha, \varrho_n(B)), \end{aligned} \tag{2.1}$$

where $\iota_n(A)$ and $\varrho_n(B)$ are the maps $A \to C\Sigma^{n-1}A$ and $E\Omega^{n-1}B \to B$ respectively, also written $\iota_n A$ and $\varrho_n B$.

The standard exact sequences are then

$$S_*(A, \beta): \cdots \to \Pi_n(A, B_1) \xrightarrow{\beta_*} \Pi_n(A, B_2) \xrightarrow{J} P_n(A, \beta) \xrightarrow{\partial} \Pi_{n-1}(A, B_1) \to \cdots$$

and

$$S^*(\alpha, B): \cdots \to \Pi_n(A_2, B) \xrightarrow{\alpha^*} \Pi_n(A_1, B) \xrightarrow{J} P_n(\alpha, B) \xrightarrow{\partial} \Pi_{n-1}(A_2, B) \to \cdots$$

In the sequence $S_*(A, \beta)$, β is a map $B_1 \to B_2$; the homomorphism J is effected by identifying $\Pi_n(A, B_2)$ with $P_n(A, \tilde{\omega}(B_2))$ and applying the obvious map $\tilde{\omega}(B_2) \to \beta$ in $\mathfrak{P}$; and ∂ is effected by restricting maps $\iota_n(A) \to \beta$ to $\Sigma^{n-1}A$ (or, equivalently, by means of the obvious map $\beta \to \omega(B_1)$ in $\mathfrak{P}$).

Let B_0 be the kernel $\beta^{-1}(o)$ of $\beta: B_1 \to B_2$. There is then an *excision* homomorphism:

$$\varepsilon: \Pi_{n-1}(A, B_0) \to P_n(A, \beta)$$

which is an isomorphism if β is a fibration. Then $S_*(A, \beta)$ yields the *absolute* sequence

$$T_*(A, \beta): \cdots \to \Pi_n(A, B_1) \xrightarrow{\beta_*} \Pi_n(A, B_2) \xrightarrow{\varepsilon^{-1}J} \Pi_{n-1}(A, B_0) \xrightarrow{\nu_*} \Pi_{n-1}(A, B_1) \to \cdots,$$

where ν embeds B_0 in B_1.

Dually let A_3 be the cokernel $A_2/\alpha A_1$ of $\alpha: A_1 \to A_2$. There is then an excision homomorphism

$$\varepsilon: \Pi_{n-1}(A_3, B) \to P_n(\alpha, B)$$

which is an isomorphism if α is a cofibration. Then $S^*(\alpha, B)$ yields the absolute sequence

$$T^*(\alpha, B)\colon \cdots \to \Pi_n(A_2, B) \xrightarrow{\alpha^*} \Pi_n(A_1, B) \xrightarrow{\varepsilon^{-1}J} \Pi_{n-1}(A_3, B) \xrightarrow{\nu^*} \Pi_{n-1}(A_2, B) \to \cdots,$$

where ν projects A_2 onto A_3.

By replacing A by a MOORE space $K'(G, m)$, $S_*(A, \beta)$ and $T_*(A, \beta)$ yield exact sequences for homotopy groups with coefficients in G; by replacing B by an EILENBERG-MACLANE space $K(G, m)$, $S^*(\alpha, B)$ and $T^*(\alpha, B)$ yield exact sequences for cohomology groups with coefficients in G. Given a short exact sequence

$$0 \to G_1 \xrightarrow{\lambda} G_2 \xrightarrow{\mu} G_3 \to 0$$

we may realize it by a *fibre sequence*

$$o \to K(G_1, m) \to K(G_2, m) \xrightarrow{\Phi} K(G_3, m)$$

or by a *cofibre sequence*

$$K'(G_1, m) \xrightarrow{\Psi} K'(G_2, m) \to K'(G_3, m) \to o\,.$$

Then $T_*(A, \Phi)$ is a coefficient sequence in the cohomology of A and $T^*(\Psi, B)$ is a coefficient sequence in the homotopy of B. These coefficient sequences may be generalized as follows. Let Θ be a *primary cohomology operation* of type (q_1, q_2, G_1, G_2) so that we may identify Θ with an element of $\Pi(K(G_1, q_1), K(G_2, q_2))$. It follows from Proposition 2.5 below that the group $P_n(A, \beta)$ depends only on the homotopy class of β so that we may write $P_n(A, \Theta)$ for the group obtained by choosing any map in the class Θ. We thus obtain the exact sequence

$$S_*(A, \Theta)\colon \cdots \to H^{q_1-n}(A; G_1) \xrightarrow{\Omega^n\Theta} H^{q_2-n}(A; G_2) \to P_n(A, \Theta) \to H^{q_1-n+1}(A; G_1) \to \cdots$$

Similarly a *homotopy operation* Θ of type (q_1, q_2, G_1, G_2) determines a group $P_n(\Theta, B)$ and an exact sequence

$$S^*(\Theta, B)\colon \cdots \to \pi_{q_1+n}(G_1; A) \xrightarrow{\Sigma^n\Theta} \pi_{q_2+n}(G_2; A) \to P_n(\Theta, B) \to \pi_{q_1+n-1}(G_1; A) \to \cdots$$

Now let $\lambda\colon B_2 \to B_2'$ be a map. We may then regard the pair-map $(1, \lambda)$ as a map $\beta \to \lambda\beta$ in $\mathfrak{P}$,

$$\begin{array}{ccc} B_1 & \xrightarrow{1} & B_1 \\ \beta \big\downarrow & \Longrightarrow & \big\downarrow \lambda\beta \\ B_2 & \xrightarrow{\lambda} & B_2', \end{array}$$

and this induces a map from $S_*(A, \beta)$ to $S_*(A, \lambda\beta)$. By applying the 5-lemma we immediately infer

Proposition 2.2. If $\lambda : B_2 \to B_2'$ is a homotopy equivalence, then

$$(1, \lambda)_* : P_n(A, \beta) \cong P_n(A, \lambda\beta), \qquad n > 1.$$

Similarly

Proposition 2.3. If $\lambda : B_1' \to B_1$ is a homotopy equivalence, then

$$(\lambda, 1)_* : P_n(A, \beta\lambda) \cong P_n(A, \beta), \qquad n > 1.$$

Dually, we have

Proposition 2.2*. If $\varkappa : A_1' \to A_1$ is a homotopy equivalence, then

$$(\varkappa, 1)^* : P_n(\alpha, B) \cong P_n(\alpha\varkappa, B), \qquad n > 1.$$

Proposition 2.3*. If $\varkappa : A_2 \to A_2'$ is a homotopy equivalence, then

$$(1, \varkappa)^* : P_n(\varkappa\alpha, B) \cong P_n(\alpha, B), \qquad n > 1.$$

Since any map β may be factorized as $\beta_0\lambda$, where β_0 is a fibration and λ a homotopy equivalence it follows from Proposition 2.3 that $P_n(A, \beta) \cong$ $\cong P_n(A, \beta_0) \cong \Pi_{n-1}(A, B_0)$, where B_0 is the fibre of β_0. Thus every relative group $P_n(A, \beta)$ is an absolute group: from this we may infer, for example,

Corollary 2.4. The universal coefficient theorem and the coefficient sequence apply to the relative homotopy group[2]) $\pi_n(G; \beta)$.

Dually any map α may be factored as $\varkappa\alpha_0$, where α_0 is a cofibration and $\varkappa$ a homotopy equivalence. Then (Proposition 2.3*), $P_n(\alpha, B) \cong P_n(\alpha_0, B) \cong$ $\cong \Pi_{n-1}(A_0, B)$, where A_0 is the cofibre of α_0, and so every relative group $P_n(\alpha, B)$ is an absolute group. Thus

Corollary 2.4*. The universal coefficient theorem and the coefficient sequence apply to the relative cohomology groups $H^n(\alpha; G)$.

As a further corollary of 2.3 we infer[3])

Proposition 2.5. If $\beta_0 \simeq \beta_1 : B_1 \to B_2$ then $P_n(A, \beta_0) \cong P_n(A, \beta_1)$, $n > 1$.

Proof. Let $\beta : B_1 \times I \to B_2$ be the homotopy and let $u_i : B_1 \to B_1 \times I$ be given by $u_i(b) = (b, i)$, $i = 0, 1$. Then $\beta u_i = \beta_i$ and u_i is a homotopy equivalence. Thus

$$P_n(A, \beta_0) = P_n(A, \beta u_0) \cong P_n(A, \beta) \cong P_n(A, \beta u_1) = P_n(A, \beta_1).$$

Proposition 2.5*. If $\alpha_0 \simeq \alpha_1 : A_1 \to A_2$ then $P_n(\alpha_0, B) \cong P_n(\alpha_1, B)$, $n > 1$.

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[2]) For the definition of the relative homotopy groups (with coefficients) and relative cohomology groups see [4].

[3]) It has been pointed out by C. R. CURJEL that 2.5 also holds if $n = 1$. – We write $\simeq$ for "homotopic to", and I for the unit interval $0 \leqslant t \leqslant 1$.

3. Natural homotopy constructions

Our object in this section is to establish certain natural homotopy constructions which will be valuable in proving the exactness of the homotopy sequences in $\mathfrak{P}$ and in making applications.

We first systematize the notion of the *(based) mapping cylinder*. The category $\mathfrak{P}$ has as objects maps γ in $\mathfrak{T}$ and as maps "pair maps" $(u, v) : \gamma \to \gamma'$ such that the diagram

$$\begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "\gamma"'] & \cdot \arrow[d, "\gamma'"] \\ \cdot \arrow[r, "v"] & \cdot \end{tikzcd}$$

is commutative. We may write $\Phi = (u, v)$,

$$\begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\overset{\Phi}{\Longrightarrow}"] & \cdot \arrow[d, "\gamma'"] \\ \cdot \arrow[r, "v"'] & \cdot \end{tikzcd}$$

for emphasis. We also introduce the category $\mathfrak{L}$ of "triples". An object of $\mathfrak{L}$ is a sequence

$$[\gamma_1, \gamma_2] = \cdot \xrightarrow{\gamma_1} \cdot \xrightarrow{\gamma_2} \cdot$$

of maps in $\mathfrak{T}$ and a map in $\mathfrak{L}$ is a triple of maps $(u, v, w) : [\gamma_1, \gamma_2] \to$ $\to [\gamma_1', \gamma_2']$ such that the diagram

$$\begin{tikzcd} \cdot \arrow[r, "\gamma_1"] \arrow[d, "u"] & \cdot \arrow[r, "\gamma_2"] \arrow[d, "v"] & \cdot \arrow[d, "w"] \\ \cdot \arrow[r, "\gamma_1'"'] & \cdot \arrow[r, "\gamma_2'"'] & \cdot \end{tikzcd}$$

is commutative. The based mapping cylinder is then a covariant functor $M : \mathfrak{P} \to \mathfrak{L}$. Thus if $\gamma : A \to B$ is an object in $\mathfrak{P}$ then $M(\gamma)$ is the sequence

$$A \xrightarrow{\gamma_1} M_\gamma \xrightarrow{\gamma_2} B\,, \tag{3.1}$$

where M_γ is obtained from $(A \times I) \vee B$ by making the identifications

$$(a, 1) = \gamma a\,,$$

$$(o, t) = o\,, \quad 0 \leqslant t \leqslant 1\,,$$

and $\gamma_1 a = (a, o)$, $\gamma_2(a, t) = \gamma a$, $\gamma_2 b = b$. If $\Phi = (u, v) : \gamma \to \gamma'$, then

$$M(\Phi) = (u, M_\Phi, v) : M(\gamma) \to M(\gamma'),$$

$$\begin{tikzcd}
A \arrow[r, "\gamma_1"] \arrow[d, "u"] & M_\gamma \arrow[r, "\gamma_2"] \arrow[d, "M_\Phi"] & B \arrow[d, "v"] \\
A' \arrow[r, "\gamma_1'"] & M_{\gamma'} \arrow[r, "\gamma_2'"] & B'
\end{tikzcd} \tag{3.2}$$

where $M_\Phi(a, t) = (ua, t)$, $M_\Phi(b) = vb$.

Now the map γ_1 in (3.1) is certainly a cofibration; we shall improve this result now by showing that the homotopy extensions from A to M_γ may be constructed canonically.

Suppose given a diagram

$$\begin{tikzcd}
M_\gamma \arrow[dr, "g_0"] & \\
A \arrow[u, "\gamma_1"] \arrow[r, "f_t"'] & Q
\end{tikzcd} \tag{3.3}$$

with $g_0 \gamma_1 = f_0$. We then call $\Delta = (f_t, \gamma, g_0)$ a *triangle* and a homotopy $g_t : M_\gamma \to Q$ with $g_t \gamma_1 = f_t$ a *lift* of Δ. We prove

Proposition 3.4 (Naturality of lifts). There is a function L, defined on the set of triangles, with $L(\Delta)$ a lift of Δ, and satisfying the conditions

(i) if $k : Q \to R$ is a map in $\mathfrak{T}$ then $L(kf_t, \gamma, kg_0) = kL(f_t, \gamma, g_0)$;
(ii) if $\Phi : (u, v) : \gamma' \to \gamma$ is a map in $\mathfrak{P}$ then $L(f_t u, \gamma', g_0 M_\Phi) = L(f_t, \gamma, g_0) M_\Phi$.

Proof. Let $\varkappa : I \times I \to I \times 0 \cup \dot{I} \times I$ be a fixed retraction. For any triangle $\Delta = (f_t, \gamma, g_0)$ define $H : A \times (I \times 0 \cup \dot{I} \times I) \to Q$ by

$$\begin{aligned} H(a, t, 0) &= g_0(a, t), \\ H(a, 0, t) &= f_t a, \\ H(a, 1, t) &= g_0 \gamma a, \end{aligned}$$

and then define $g_t : M_\gamma \to Q$ by

$$\begin{aligned} g_t(a, t_0) &= H(a, \varkappa(t_0, t)) \\ g_t b &= g_0 b. \end{aligned}$$

Certainly g_t lifts $\varDelta$ and we set $L(\varDelta) = g_t$.

To prove (i), let $\overline{H}$, $\overline{g}_t$ be constructed as above from the triangle (kf_t, γ, kg_0). Then $\overline{H} = kH$ so that $\overline{g}_t = kg_t$.

To prove (ii), let $\overline{H}$, $\overline{g}_t$ be constructed as above from the triangle $(f_t u, \gamma', g_0 M_\varPhi)$. Then it is easy to verify that $\overline{H} = H(u \times 1)$, where 1 is the identity on $I \times 0 \cup \dot{I} \times I$, so that

$$\overline{g}_t(a', t_0) = \overline{H}(a', \varkappa(t_0, t)) = H(ua', \varkappa(t_0, t)) = g_t(ua', t_0) = g_t M_\varPhi(a', t_0),$$
$$\overline{g}_t b' = g_0 M_\varPhi b' = g_t M_\varPhi b',$$

whence $\overline{g}_t = g_t M_\varPhi$ and (ii) is proved.

A case of special interest is that in which $\gamma = \omega(A) : A \to o$; then $\gamma_1 = \iota_1(A) : A \to CA$ and a map $\varPhi : \omega(A') \to \omega(A)$ is just a map $u : A' \to A$. Moreover $M_\varPhi$ is $Cu : CA' \to CA$. If we write (f_t, A, g_0) for $(f_t, \omega(A), g_0)$ we have

Corollary 3.5. The function L, restricted to the triangles (f_t, A, g_0) satisfies

(i) if $k : Q \to R$ is a map in $\mathfrak{T}$ then $L(kf_t, A, kg_0) = kL(f_t, A, g_0)$;

(ii) if $u : A' \to A$ is a map in $\mathfrak{T}$ then $L(f_t u, A', g_0 Cu) = L(f_t, A, g_0) Cu$.

Our second naturality theorem concerns nullhomotopic maps $\varPhi = (f_0, g_0)$

$$\begin{array}{ccc} X & \xrightarrow{f_0} & A \\ \Big\downarrow{\scriptstyle \iota_1(X)} & & \Big\downarrow{\scriptstyle \alpha} \\ CX & \xrightarrow{g_0} & B \end{array}$$

with $\alpha f_0 = 0$[4]). More precisely we consider nullhomotopies (f_t, g_t) of such maps and call them *admissible*.

Proposition 3.6. There is a function N, defined on the set of admissible nullhomotopies (f_t, g_t) whose value $N(f_t, g_t)$ is a pair $(h, \overline{g}_t)$ consisting of a map $h : CX \to A$ and a homotopy $\overline{g}_t$ of g_0 with $\overline{g}_t \iota_1 = o$, $\alpha h = \overline{g}_1$. The function N has the following properties

(i) if $\varPhi = (u, v) : \alpha \to \alpha'$ then $N(uf_t, vg_t) = (uh, v\overline{g}_t)$;

(ii) if $\xi : X' \to X$ then $N(f_t \xi, g_t C\xi) = (hC\xi, \overline{g}_t C\xi)$;

(iii) if $f_0 = 0$ then $h\iota_1 = 0$.

[4]) We use the symbol 0 for the zero map $Y \to Z$, $0(y) = o$, for any Y, Z.

Proof. Given f_t, g_t we define

$$\bar{g}_t : CX \to B$$

by
$$\begin{aligned} \bar{g}_t(x, t_0) &= g_{2tt_0}(x, t_0(1-t)), & 0 \leqslant t_0 \leqslant \tfrac{1}{2}, \\ &= g_t(x, t_0 - (1-t_0)t), & \tfrac{1}{2} \leqslant t_0 \leqslant 1; \end{aligned}$$

and we define $h : CX \to A$ by

$$\begin{aligned} h(x, t_0) &= f_{2t_0}(x), & 0 \leqslant t_0 \leqslant \tfrac{1}{2}, \\ &= o \quad , & \tfrac{1}{2} \leqslant t_0 \leqslant 1. \end{aligned}$$

It is a straight-forward matter to verify that $N(f_t, g_t) = (h, \bar{g}_t)$ is a function satisfying conditions (i), (ii) and (iii).

Finally we need

Proposition 3.7. Any map $\Phi = (u, u') : C\xi \to \alpha$ is nullhomotopic.

Proof. We have the diagram

$$\begin{tikzcd} CX \arrow[r, "u"] \arrow[d, "C\xi"] & A \arrow[d, "\alpha"] \\ CX' \arrow[r, "u'"] & A' \end{tikzcd}$$

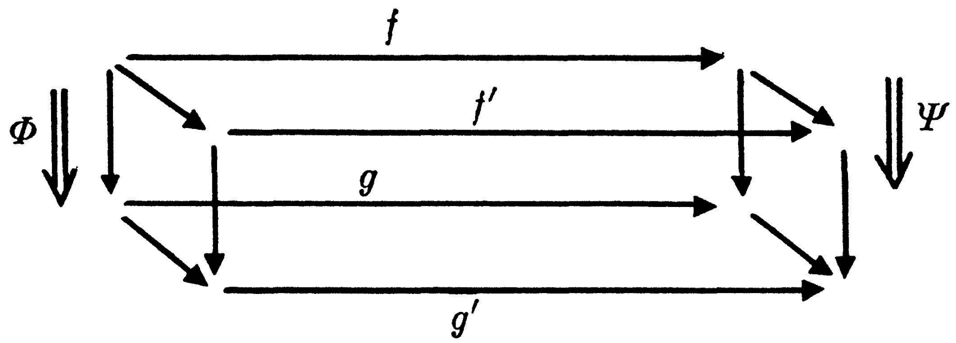
and we define a nullhomotopy by

$$\begin{aligned} u_t(x, t_0) &= u(x, t_0 + t(1-t_0)), & x \in X, \\ u'_t(x', t_0) &= u'(x', t_0 + t(1-t_0)), & x' \in X'. \end{aligned}$$

We leave to the reader the formulation of the propositions dual to those enunciated in this section.

4. The exact sequences in $\mathfrak{P}(\mathfrak{T})$

We now proceed, by direct analogy with the corresponding notions in section 2, to describe the basic exact sequences $S_*(\alpha, \Psi)$ and $S^*(\Phi, \beta)$ in $\mathfrak{P}$. To do this we have of course to introduce the category $\mathfrak{P}^2 = \mathfrak{P}(\mathfrak{P})$, whose objects Φ are maps in $\mathfrak{P}$ and whose maps are pairs of maps of $\mathfrak{P}$ (and hence quadruples of maps of $\mathfrak{T}$) satisfying certain evident commutativity relations. Thus a map from Φ to Ψ, say, is a quadruple $\begin{pmatrix} f & f' \\ g & g' \end{pmatrix}$, representing the diagram



We will be particularly concerned with the case $\Phi = \iota_n(\alpha)$; then if $\alpha : A \to A'$, Φ is the map $(\iota_n(A), \iota_n(A')) : \Sigma^{n-1}\alpha \to C\Sigma^{n-1}\alpha$. The group $P_n(\alpha, \Psi)$ is, by definition, the group $\Pi(\iota_n(\alpha), \Psi)$,

$$P_n(\alpha, \Psi) = \Pi(\iota_n(\alpha), \Psi), \tag{4.1}$$

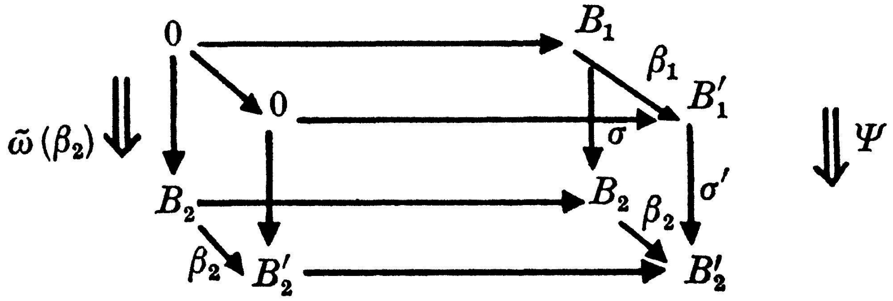
and, dually,

$$P_n(\Phi, \beta) = \Pi(\Phi, \varrho_n(\beta)). \tag{4.1*}$$

We now describe the sequence $S_*(\alpha, \Psi)$; we suppose $\Psi : \beta_1 \to \beta_2$ so that the sequence reads

$$S_*(\alpha, \Psi) : \cdots \to \Pi_n(\alpha, \beta_1) \xrightarrow{\Psi_*} \Pi_n(\alpha, \beta_2) \xrightarrow{J} P_n(\alpha, \Psi) \xrightarrow{\partial} \Pi_{n-1}(\alpha, \beta_1) \to \cdots;$$

the homomorphism ∂ is induced by restricting maps $\iota_n(\alpha) \to \Psi$ to $\Sigma^{n-1}(\alpha)$, and the homomorphism J by identifying $\Pi_n(\alpha, \beta_2)$ with $P_n(\alpha, \tilde{\omega}(\beta_2))$ and applying the evident map $\tilde{\omega}(\beta_2) \to \Psi$



We have now described the sequence and introduced all necessary notations. We prove

Theorem 4.2. *The sequence $S_*(\alpha, \Psi)$ is exact.*

Proof. It is clearly sufficient to look at the stretch

$$\Pi_1(\alpha, \beta_1) \xrightarrow{\Psi_*} \Pi_1(\alpha, \beta_2) \xrightarrow{J} P_1(\alpha, \Psi) \xrightarrow{\partial} \Pi(\alpha, \beta_1) \xrightarrow{\Psi_*} \Pi(\alpha, \beta_2)$$

and prove exactness at the three middle sets.

Exactness at $\Pi_1(\alpha, \beta_2)$.

We identify $\Pi_1(\alpha, \beta_i)$ with $P_1(\alpha, \tilde{\omega}(\beta_i))$, $i = 1, 2$. Then if $\begin{pmatrix}\omega & \omega\\ f & f'\end{pmatrix}$ represents $\varkappa \in \Pi_1(\alpha, \beta_1)$, $J\Psi_*\varkappa$ is represented by $\begin{pmatrix}0 & 0\\ \sigma f & \sigma' f'\end{pmatrix} : \iota_1(\alpha) \to \Psi$. But

$$\begin{pmatrix}0 & 0\\ \sigma f & \sigma' f'\end{pmatrix} = \begin{pmatrix}0 & 0\\ f & f'\end{pmatrix}\begin{pmatrix}1 & 1\\ \sigma & \sigma'\end{pmatrix},$$

the factorization being through the identity map of β_1. It follows from 3.7 that $\begin{pmatrix}0 & 0\\ f & f'\end{pmatrix} \simeq 0$ so that $J\Psi_* = 0$.

Now let $\lambda \in \Pi_1(\alpha, \beta_2)$ with $J\lambda = 0$ and let λ be represented by $\begin{pmatrix}\omega & \omega\\ g_0 & g_0'\end{pmatrix}$; thus there is a nullhomotopy $\begin{pmatrix}f_t & f_t'\\ g_t & g_t'\end{pmatrix}$ of $\begin{pmatrix}0 & 0\\ g_0 & g_0'\end{pmatrix} : \iota_1(\alpha) \to \Psi$. By 3.6 we obtain homotopies $\bar{g}_t, \bar{g}_t'$ of g_0, g_0' and maps $h : CA \to B_1$, $h' : CA' \to B_1'$ such that $\bar{g}_t\iota = 0$, $\bar{g}_t'\iota = 0$, $\sigma h = \bar{g}_1$, $\sigma' h' = \bar{g}_1'$. Moreover, by 3.6 (i, ii),

$$N(\beta_1 f_t, \beta_2 g_t) = (\beta_1 h, \beta_2 \bar{g}_t),$$
$$N(f_t'\alpha, g_t' C\alpha) = (h' C\alpha, \bar{g}_t' C\alpha);$$

but $\beta_1 f_t = f_t'\alpha$, $\beta_2 g_t = g_t' C\alpha$, so that

$$\beta_1 h = h' C\alpha, \tag{4.3}$$

$$\beta_2 \bar{g}_t = \bar{g}_t' C\alpha. \tag{4.4}$$

It follows from (4.4) that $\begin{pmatrix}\omega & \omega\\ \bar{g}_t & \bar{g}_t'\end{pmatrix}$ is a homotopy of $\begin{pmatrix}\omega & \omega\\ g_0 & g_0'\end{pmatrix}$. Thus λ is represented by $\begin{pmatrix}\omega & \omega\\ \bar{g}_1 & \bar{g}_1'\end{pmatrix}$. From (4.4) and 3.6 (iii) we infer that

$$\begin{pmatrix}\omega & \omega\\ \bar{g}_1 & \bar{g}_1'\end{pmatrix} = \begin{pmatrix}0 & 0\\ \sigma & \sigma'\end{pmatrix}\begin{pmatrix}\omega & \omega\\ h & h'\end{pmatrix}$$

so that $\lambda = \Psi_*(\varkappa)$, where $\varkappa$ is represented by $\begin{pmatrix}\omega & \omega\\ h & h'\end{pmatrix}$.

Exactness at $\Pi_1(\alpha, \Psi)$.

The relation $\partial J = 0$ is trivial. Suppose now that $\mu \in P_1(\alpha, \Psi)$ with $\partial\mu = 0$. Thus if μ is represented by $\begin{pmatrix}f_0 & f_0'\\ g_0 & g_0'\end{pmatrix} : \iota_1(\alpha) \to \Psi$, there is a nullhomotopy (f_t, f_t') of (f_0, f_0'). We apply 3.5 to "extend" f_t, f_t' to homotopies g_t, g_t' of g_0, g_0'. Thus

$$g_t = L(\sigma f_t, A, g_0), \quad g_t' = L(\sigma' f_t', A', g_0').$$

Then, by 3.5 (i), $\beta_2 g_t = L(\beta_2\sigma f_t, A, \beta_2 g_0)$ and, by 3.5 (ii), $g'_t C\alpha = L(\sigma' f'_t\alpha, A, g'_0 C\alpha)$; but $\sigma' f'_t\alpha = \sigma'\beta_1 f_t = \beta_2\sigma f_t$ and $g'_0 C\alpha = \beta_2 g_0$. Thus

$$\beta_2 g_t = g'_t C\alpha$$

so that $\begin{pmatrix} f_t & f'_t \\ g_t & g'_t \end{pmatrix}$ is a homotopy of $\begin{pmatrix} f_0 & f'_0 \\ g_0 & g'_0 \end{pmatrix}$. This means that $\begin{pmatrix} 0 & 0 \\ g_1 & g'_1 \end{pmatrix}$ also represents μ so that $\mu = J\lambda$ where λ is represented by

$$\begin{pmatrix} \omega & \omega \\ g_1 & g'_1 \end{pmatrix}.$$

Exactness at $\Pi(\alpha, \beta_1)$

Let $(f, f') : \alpha \to \beta_1$. Then to assert exactness at $\Pi(\alpha, \beta_1)$ is just to assert that the map $(\sigma f, \sigma f') : \alpha \to \beta_2$ is nullhomotopic if and only if it may be extended to $C\alpha$; this, of course, is clearly true. This completes the proof of the theorem. –

The dual sequence is, explicitly

$$S^*(\Phi, \beta) : \cdots \to \Pi_n(\alpha_2, \beta) \xrightarrow{\Phi^*} \Pi_n(\alpha_1, \beta) \xrightarrow{J} P_n(\Phi, \beta) \xrightarrow{\partial} \Pi_{n-1}(\alpha_2, \beta) \to \cdots$$

Theorem 4.2*. *The sequence* $S^*(\Phi, \beta)$ *is exact.*

We mention certain immediate and obvious consequences of theorems 4.2 and 4.2*.

Proposition 4.3. The map $\Psi : \beta_1 \to \beta_2$ induces isomorphisms $\Pi_n(\alpha, \beta_1) \cong \Pi_n(\alpha, \beta_2)$ for all n if and only if $P_n(\alpha, \Phi) = 0$, all n.

Proposition 4.4. If Θ_1, Θ_2 are homotopy equivalences then $P_n(\Theta_1\Phi\Theta_2, \beta) \cong P_n(\Phi, \beta)$, $n > 1$.

Proposition 4.5. If $\Phi \simeq \Phi'$ then $P_n(\Phi, \beta) \cong P_n(\Phi', \beta)$, $n > 1$.

The reader may provide the duals of these propositions.

5. Fibrations and excision

In this section we prove excision theorems corresponding to those in $\mathfrak{T}$ (see section 2). In the notation of the preceding section let B_0, B'_0 be the kernels of σ, σ', so that β_1 induces $\beta_0 : B_0 \to B'_0$ and let ν, ν' embed B_0, B'_0 in B_1, B'_1. We refer to β_0 as the kernel of Ψ. Then (ν, ν') serves to induce the excision homomorphism

$$\varepsilon : \Pi_{n-1}(\alpha, \beta_0) \to P_n(\alpha, \Psi) . \quad (5.1)$$

We call the map Ψ a *fibre map* if it has the lifting homotopy property: the precise definition simply translates the standard definition from $\mathfrak{T}$ to $\mathfrak{P}$. If Ψ is a fibre map β_0 is called the *fibre* of Ψ.

Theorem 5.2. If Ψ *is a fibre map* ε *is an isomorphism.*

Proof. The proof, of course, closely resembles that of the corresponding theorem in $\mathfrak{T}$ and need only be sketched. We need only look at the case $n = 1$.

ε *is epimorphic.* Consider a map $\begin{pmatrix} f_0 & f_0' \\ g_0 & g_0' \end{pmatrix}: \iota_1(\alpha) \to \Psi$. By 3.7 there is a nullhomotopy of (g_0, g_0'); since Ψ is a fibre map we may cover this nullhomotopy with a homotopy of (f_0, f_0'). Thus there is a homotopy $\begin{pmatrix} f_t & f_t' \\ g_t & g_t' \end{pmatrix}$ with $g_1 = 0$, $g_1' = 0$. Thus (f_1, f_1') factors through (ν, ν') so that ε is onto $P_1(\alpha, \Psi)$.

ε *is monomorphic.* Suppose given $(f_0, f_0') : \alpha \to \beta_0$ and a nullhomotopy

$$\begin{pmatrix} u_t & u_t' \\ g_t & g_t' \end{pmatrix} : \iota_1(\alpha) \to \Psi ,$$

where $u_0 = \nu f_0$, $u_0' = \nu' f_0'$, $g_0 = 0$, $g_0' = 0$. We must show that $(f_0, f_0') \simeq 0$. We have then a diagram

$$\begin{tikzcd}
A \times I \arrow[rr, "U"] \arrow[dr, "\alpha \times 1"] \arrow[dd, "\iota_1 A \times 1"'] & & B_1 \arrow[dr, "\beta_1"] \arrow[dd, "\sigma" near start] & \\
& A' \times I \arrow[rr, "U'" near start] \arrow[dd, "\iota_1 A' \times 1" near start] & & B_1' \arrow[dd, "\sigma'"] \\
CA \times I \arrow[rr, "G" near end] \arrow[dr, "C\alpha \times 1"'] & & B_2 \arrow[dr, "\beta_2"'] & \\
& CA' \times I \arrow[rr, "G'"] & & B_2'
\end{tikzcd}$$

where 1 is the identity on I and U, U', G, G' are the homotopies u_t, u_t', g_t, g_t'. An easy extension of the argument of Proposition 3.7 establishes the existence of a nullhomotopy of (G, G') rel $(CA \times \dot{I}, CA' \times \dot{I})$. We define $V : A \times (I \times 0 \cup \dot{I} \times I) \to B_1$ by

$$V(a, t, 0) = U(a, t) ,$$
$$V(a, 0, t) = u_0(a) ,$$
$$V(a, 1, t) = 0 ;$$

and define V' similarly. Then (V, V') is a partial cover of the nullhomotopy of (G, G'). Since the pair $I \times I, I \times 0 \cup \dot{I} \times I$ is homeomorphic to the pair $I \times I, I \times 0$, it follows from the fact that Ψ is a fibre map that the partial cover (V, V') may be extended to a total cover $(\overline{V}, \overline{V}')$. Define

$$(W, W') : \alpha \times 1 \to \beta_1$$

by

$$W(a, t) = \overline{V}(a, t, 1), \qquad W'(a', t) = \overline{V}'(a', t, 1).$$

Finally observe that $W(a, 0) = u_0(a)$, $W(a, 1) = 0$, and similarly for W'; and that $\sigma W = 0$, $\sigma' W' = 0$ since $(\overline{V}, \overline{V}')$ covers a nullhomotopy. Thus (W, W') factors through β_0 and thereby determines a nullhomotopy of (f_0, f_0').

Corollary 5.3. If Ψ is a fibre map there is an exact sequence

$$T_*(\alpha, \Psi) : \cdots \to \Pi_n(\alpha, \beta_1) \xrightarrow{\Psi_*} \Pi_n(\alpha, \beta_2) \xrightarrow{\varepsilon^{-1}J} \Pi_{n-1}(\alpha, \beta_0) \xrightarrow{\Gamma_*} \Pi_{n-1}(\alpha, \beta_1) \to \cdots$$

Here we have written Γ for (ν, ν'); it is obvious that $\partial\varepsilon = \Gamma_*$.

We state the dual situation for completeness. The dual excision homomorphism is

$$\varepsilon : \Pi_{n-1}(\alpha_3, \beta) \to P_n(\Phi, \beta). \tag{5.1*}$$

Here A_3, A_3' are the cokernels of $\sigma_1 : A_1 \to A_2$, $\sigma_1' : A_1' \to A_2'$ and $\alpha_3 : A_3 \to A_3'$ is the map induced by α_2.

We call Φ a *cofibre map* if it has the descending homotopy (homotopy extension) property. If Φ is a cofibre map α_3 is called the *cofibre* of Φ.

Theorem 5.2*. *If Φ is a cofibre map ε is an isomorphism.*

Corollary 5.3*. If Φ is a cofibre map there is an exact sequence

$$T^*(\Phi, \beta) : \cdots \to \Pi_n(\alpha_2, \beta) \xrightarrow{\Phi^*} \Pi_n(\alpha_1, \beta) \xrightarrow{\varepsilon^{-1}J} \Pi_{n-1}(\alpha_3, \beta) \xrightarrow{\Gamma^*} \Pi_{n-1}(\alpha_2, \beta) \to \cdots$$

We close this section by an important example of a cofibre (and one of a fibre) map. Let us revert to (3.2) and consider the map $(\gamma_1, \gamma_1') : u \to M_\Phi$.

Theorem 5.4*. *The map (γ_1, γ_1') is a cofibre map.*[5]

Proof. We are given the diagram

$$\begin{tikzcd}
M_\Phi \arrow[dr, "{(g_0, g_0')}"] & \\
u \arrow[u, "{(\gamma_1, \gamma_1')}"] \arrow[r, "{(f_t, f_t')}"'] & q
\end{tikzcd}$$

[5] This consequence of 3.4 was first observed by G. A. Hunt.

and must lift this triangle to obtain (g_t, g'_t). Following 3.4 we set

$$g_t = L(f_t, \gamma, g_0), \quad g'_t = L(f'_t, \gamma', g'_0)$$

and it remains to show that $qg_t = g'_t M_\Phi$. But

$$\begin{aligned} qg_t &= L(qf_t, \gamma, qg_0), && \text{by 3.4 (i),} \\ g'_t M_\Phi &= L(f'_t u, \gamma, g'_0 M_\Phi), && \text{by 3.4 (ii),} \end{aligned}$$

and $gf_t = f'_t u, qg_0 = g'_0 M_\Phi$; thus the theorem follows.

There is, of course, a dual story in which the mapping cylinder functor is replaced by the mapping track functor. Thus given $\Phi = (u, v) : \gamma \to \gamma'$ we apply the mapping track functor E to obtain

$$\begin{tikzcd} A \arrow[r, "\gamma^2"] \arrow[d, "u"'] & E \arrow[r, "\gamma^1"] \arrow[d, "E_\Phi"] & B \arrow[d, "v"] \\ A' \arrow[r, "\gamma'^2"] & E_\gamma \arrow[r, "\gamma'^1"] & B' \end{tikzcd} \tag{3·2*}$$

and have

Theorem 5.4. The map (γ^1, γ'^1) is a fibre map.

6. Mixed sequences and weak fibrations in $\mathfrak{P}(\mathfrak{T})$

We revert to the exact sequence $S_*(\alpha, \Psi)$ and consider the special case in which $\alpha = \iota_1 A$. We define

$$P_n(A, \Psi) = P_{n-1}(\iota_1 A, \Psi) \tag{6.1}$$

and so obtain the exact sequence

$$S_*(A, \Psi) : \cdots \to P_n(A, \beta_1) \xrightarrow{\Psi_*} P_n(A, \beta_2) \xrightarrow{J} P_n(A, \Psi) \xrightarrow{\partial} P_{n-1}(A, \beta_1) \to \cdots$$

Dually we replace β by $\varrho_1 B$ in the exact sequence $S^*(\Phi, \beta)$; if we define

$$P_n(\Phi, B) = P_{n-1}(\Phi, \varrho_1 B) \tag{6.1*}$$

we obtain the exact sequence

$$S^*(\Phi, B) : \cdots \to P_n(\alpha_2, B) \xrightarrow{\Phi^*} P_n(\alpha_1 B) \xrightarrow{J} P_n(\Phi, B) \xrightarrow{\partial} P_{n-1}(\alpha_2, B) \to \cdots$$

Notice that the sets $P_n(A, \Psi)$, $P_n(\Phi, B)$ are defined for $n \geqslant 2$, have group structure if $n > 2$, abelian if $n > 3$.

We may immediately infer analogues of Propositions 2.2, 2.3. Thus if $\Theta : \beta_2 \to \beta'_2$ then $(1, \Theta) : \Psi \to \Theta \circ \Psi$ and we infer from Propositions 2.2, 2.3 and the exactness of $S_*(A, \Psi)$

Proposition 6.2. If $\Theta = (\lambda, \lambda')$ where λ, λ' are homotopy equivalences, then

$$(1, \Theta)_* : P_n(A, \Psi) \cong P_n(A, \Theta \circ \Psi), \qquad n > 2.$$

Similarly,

Proposition 6.3. If $\Theta = (\lambda, \lambda') : \beta_1' \to \beta_1$, where λ, λ' are homotopy equivalences, then

$$(\Theta, 1)_* : P_n(A, \Psi \circ \Theta) \cong P_n(A, \Psi), \qquad n > 2.$$

Dually,

Proposition 6.2*. If $\Theta = (\varkappa, \varkappa') : \alpha_1' \to \alpha_1$, where $\varkappa, \varkappa'$ are homotopy equivalences, then

$$(\Theta, 1)^* : P_n(\Phi, B) \cong P_n(\Phi \circ \Theta, B), \qquad n > 2.$$

Proposition 6.3*. If $\Theta = (\varkappa, \varkappa') : \alpha_2 \to \alpha_2'$, where $\varkappa, \varkappa'$ are homotopy equivalences, then

$$(1, \Theta)^* : P_n(\Theta \circ \Phi, B) \cong P_n(\Phi, B), \qquad n > 2.$$

We will also need the following elementary consequences of the mixed exact sequences.

Proposition 6.4. (i) If β_1 is a homotopy equivalence, then

$$J : P_n(A, \beta_2) \cong P_n(A, \Psi).$$

(ii) If β_2 is a homotopy equivalence, then

$$\partial : P_n(A, \Psi) \cong P_{n-1}(A, \beta_1).$$

Proposition 6.4*. (i) If α_2 is a homotopy equivalence, then

$$J : P_n(\alpha_1, B) \cong P_n(\Phi, B).$$

(ii) If α_1 is a homotopy equivalence, then

$$\partial : P_n(\Phi, B) \cong P_{n-1}(\alpha_2, B).$$

To prove Proposition 6.4 (i), we have only to invoke the exactness of $S_*(A, \beta_1)$ to infer that $P_n(A, \beta_1) = 0$. The other assertions are proved similarly.

We will leave the main applications to subsequent sections. In this section we are content to investigate the excision homomorphisms for the groups $P_n(A, \Psi)$, $P_n(\Phi, B)$. We will deal explicitly with the latter group but first frame the appropriate dual definitions. We say that a map $\Psi = (u, v)$ is a *weak fibre map* if each of u, v is a fibre map; dually $\Phi = (u, v)$ is a *weak* *cofibre map* if each of u, v is a cofibre map. It is easy to show that a fibre (cofibre) map is a weak fibre (cofibre) map; on the other hand, the converse is false. For example, if

$$\begin{array}{ccc} A_1 & \xrightarrow{\alpha_1} & A_1' \\ u\downarrow & & \downarrow v \\ A_2 & \xrightarrow[\alpha_2]{} & A_2' \end{array}$$

is a commutative diagram of polyhedral inclusions with $A_1 \neq A_1' \cap A_2$, then (u, v) is a weak cofibre map but not, in general, a cofibre map. However, we will prove

Theorem 6.5*. *If Φ is a weak cofibre map the excision homomorphism*

$$\varepsilon : P_{n-1}(\alpha_3, B) \to P_n(\Phi, B)$$

is an isomorphism.

Proof. We have $\Phi = (\sigma, \sigma') : \alpha_1 \to \alpha_2$ with each of σ, σ' a cofibre map. We apply the mapping cylinder functor M to σ and σ' obtaining the diagram

$$\begin{array}{ccccc} \cdot & \xrightarrow{\sigma_1} & \cdot & \xrightarrow{\sigma_2} & \cdot \\ \alpha_1\downarrow & & \downarrow\alpha & & \downarrow\alpha_2 \\ \cdot & \xrightarrow[\sigma_1']{} & \cdot & \xrightarrow[\sigma_2']{} & \cdot \end{array} \tag{6.6}$$

Notice that the map α is just M_{Φ^T}, where $\Phi^T = (\alpha_1, \alpha_2) : \sigma \to \sigma'$; we call Φ^T the *transpose* of Φ (but wait till later to develop special properties of the transpose). Let $\Phi_1 = (\sigma_1, \sigma_1') : \alpha_1 \to \alpha, \Phi_2 = (\sigma_2, \sigma_2') : \alpha \to \alpha_2$. Then σ_2, σ_2' are homotopy equivalences and $\Phi = \Phi_2\Phi_1$. Thus by Proposition 6.3*.

$$(1, \Phi_2)^* : P_n(\Phi, B) \cong P_n(\Phi_1, B) . \tag{6.7}$$

Now Φ_1 is a cofibre map (Theorem 5.4*) so that, by Theorem 5.2*,

$$\varepsilon : P_{n-1}(\alpha_0, B) \cong P_n(\Phi_1, B) , \tag{6.8}$$

where α_0 is the cofibre of Φ_1. Now the map $(1, \Phi_2) : \Phi_1 \to \Phi$ clearly induces a map $\tilde{\Phi}$ from the cofibre of Φ_1 to the cofibre of Φ,

$$\tilde{\Phi} : \alpha_0 \to \alpha_3 .$$

Moreover, the naturality of the excision homomorphism expresses itself by the commutativity relation

$$(1, \Phi_2)^* \varepsilon = \varepsilon \tilde{\Phi}^* : P_{n-1}(\alpha_3, B) \to P_n(\Phi_1, B) . \tag{6.9}$$

Thus, it remains, in the light of (6.7), (6.8) and (6.9), to prove

$$\tilde{\Phi}^* : P_{n-1}(\alpha_3, B) \cong P_{n-1}(\alpha_0, B) . \tag{6.10}$$

Consider the diagrams

$$\begin{tikzcd} A_1 \arrow[r, "\sigma_1"] \arrow[d, "1"'] & M_\sigma \arrow[r, "\nu_1"] \arrow[d, "\sigma_2"] & A_0 \arrow[d, "\tilde{\sigma}"] \\ A_1 \arrow[r, "\sigma"'] & A_2 \arrow[r, "\nu"'] & A_3 \end{tikzcd} \qquad \begin{tikzcd} A_1' \arrow[r, "\sigma_1'"] \arrow[d, "1"'] & M_{\sigma'} \arrow[r, "\nu_1'"] \arrow[d, "\sigma_2'"] & A_0' \arrow[d, "\tilde{\sigma}'"] \\ A_1' \arrow[r, "\sigma'"'] & A_2' \arrow[r, "\nu'"'] & A_3' \end{tikzcd}$$

where ν_1, ν are projections onto the cofibres of σ_1, σ (and similarly for the second diagram). Then $\tilde{\Phi} = (\tilde{\sigma}, \tilde{\sigma}') : \alpha_0 \to \alpha_3$ and (6.10) follows from Propositions 2.2*, 2.3* once we have proved that $\tilde{\sigma}, \tilde{\sigma}'$ *are homotopy equivalences.* In fact we will prove the stronger statement

Proposition 6.11*. The map $(1, \sigma_2)$ is a homotopy equivalence in $\mathfrak{P}$.

Proof. We define $g_0 : A_2 \to M_\sigma$ by $g_0(a_2) = a_2$, $a_2 \in A_2$ and $f_t : A_1 \to M_\sigma$ by $f_t(a_1) = (a_1, 1 - t)$, $a_1 \in A_1$. Then $f_0 = g_0 \sigma$ so that, σ being a cofibre map, there exists $g_t : A_2 \to M_\sigma$ with $f_t = g_t \sigma$. Then $(1, g_1) : \sigma \to \sigma_1$, since $g_1 \sigma(a_1) = f_1 a_1 = (a_1, 0) = \sigma_1 a_1$. We show that $(1, g_1)$ is a homotopy inverse of $(1, \sigma_2)$.

First $(1, \sigma_2)(1, g_1) = (1, \sigma_2 g_1) : \sigma \to \sigma$. Now $\sigma_2 g_t \sigma(a_1) = \sigma_2 f_t(a_1) = \sigma_2(a_1, 1 - t) = \sigma(a_1)$. Thus we have a homotopy $(1, \sigma_2 g_t) : \sigma \to \sigma$ and $\sigma_2 g_0 = 1$ so that $(1, \sigma_2)(1, g_1) \simeq 1$.

Second $(1, g_1)(1, \sigma_2) = (1, g_1 \sigma_2) : \sigma_1 \to \sigma_1$. We may easily find a homotopy $g_1 \sigma_2 \simeq 1 : M_\sigma \to M_\sigma$; namely, $\Theta_t : M_\sigma \to M_\sigma$, where

$$\left.\begin{aligned} \Theta_t(a_1, u) &= (a_1, u + 2t(1 - u)) \\ \Theta_t(a_2) &= a_2 \end{aligned}\right\} \quad 0 \leqslant t \leqslant \tfrac{1}{2}$$

$$\left.\begin{aligned} \Theta_t(a_1, u) &= (a_1, 2 - 2t) \\ \Theta_t(a_2) &= g_{2t-1} a_2 \end{aligned}\right\} \quad \tfrac{1}{2} \leqslant t \leqslant 1 .$$

Then $\Theta_0 = 1$, $\Theta_1 = g_1 \sigma_2$; but the embedded subspace A_1 does not stay fixed during the homotopy but slides down the mapping cylinder and up again. We thus wish to replace Θ_t by a homotopy which leaves A_1 (identified with $A_1 \times 0 \subset M_\sigma$) pointwise fixed. Let $\Theta : M_\sigma \times I \to M_\sigma$ be the homotopy Θ_t. Then $\Theta \mid A_1 \times I$ is given by

$$\begin{aligned} \Theta(a_1, t) &= (a_1, 2t)\,, \quad 0 \leqslant t \leqslant \tfrac{1}{2} \\ &= (a_1, 2-2t)\,, \quad \tfrac{1}{2} \leqslant t \leqslant 1\,. \end{aligned}$$

Then, obviously, $\Theta | A_1 \times I \simeq \Gamma : A_1 \times I \to M_\sigma$ rel $A_1 \times \dot{I}$, where $\Gamma(a_1, t) = a_1$. Let $M : A_1 \times I \times I \to M_\sigma$ be the homotopy. Define $\Theta_0 : M_\sigma \times (I \times 0 \cup \dot{I} \times I) \to M_\sigma$ by $\Theta_0(b, t, 0) = \Theta(b, t)$, $b \in M_\sigma$, $\Theta_0(b, i, t) = \Theta(b, i)$, $i = 0, 1$. Clearly

$$\Theta_0 \mid A_1 \times (I \times 0 \cup \dot{I} \times I) = M \mid A_1 \times (I \times 0 \cup \dot{I} \times I)\,.$$

We now use the facts (i) that the pair $(I \times I, I \times 0 \cup \dot{I} \times I)$ is homeomorphic to the pair $(I \times I, I \times 0)$; and (ii) that $\sigma_1 \times 1 : A_1 \times I \subset M_\sigma \times I$ is a cofibre map, to infer that Θ_0 admits an extension $\Theta_1 : M_\sigma \times I \times I \to M_\sigma$ with $\Theta_1 \mid A_1 \times I \times I = M$. If we define

$$\Theta'_t : M_\sigma \to M_\sigma$$

by $\Theta'_t(b) = \Theta_1(b, t, 1)$, $b \in M_\sigma$, then $\Theta'_0 = \Theta_0 = 1$, $\Theta'_1 = \Theta_1 = g_1\sigma_2$ and $\Theta'_t(a_1) = \Theta_1(a_1, t, 1) = M(a_1, t) = \Gamma(a_1, t) = a_1$. Thus $(1, \Theta'_t)$ is a homotopy from 1 to $(1, g_1\sigma_2)$ and the proof of the proposition is complete. With it we also complete the proof of the theorem.

Corollary 6.12*. If Φ is a weak cofibre map there is an exact sequence

$$T^*(\Phi, B) : \cdots \to P_n(\alpha_2, B) \xrightarrow{\Phi^*} P_n(\alpha_1, B) \xrightarrow{\varepsilon^{-1}J} P_{n-1}(\alpha_3, B) \xrightarrow{\Gamma^*} P_{n-1}(\alpha_2, B) \to \cdots$$

We will not enunciate the duals explicitly beyond

Corollary 6.12. If Ψ is a weak fibre map there is an exact sequence

$$T_*(A, \Psi) : \cdots \to P_n(A, \beta_1) \xrightarrow{\Psi_*} P_n(A, \beta_2) \xrightarrow{\varepsilon^{-1}J} P_{n-1}(A, \beta_0) \xrightarrow{\Gamma_*} P_{n-1}(A, \beta_1) \to \cdots$$

We close this section by pointing out that the groups $P_n(A, \Psi)$ constitute a natural generalization of the triad homotopy groups. Indeed if we have $A = S^0$ and

$$\begin{tikzcd} Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\overset{\Longrightarrow}{\Psi}"] & Y_1 \arrow[d] \\ Y_2 \arrow[r] & X \end{tikzcd}$$

where all maps are inclusions and $Z = Y_1 \cap Y_2$, then $P_n(A, \Psi) = \pi_n(X; Y_1, Y_2)$, and the exact sequence $S_*(A, \Psi)$ is one of the exact sequences of the triad. To obtain the other exact sequence we consider the transpose Ψ^T of Ψ and invoke the isomorphism

$$P_n(A, \Psi) \cong P_n(A, \Psi^T) \tag{6.13}$$

which will be proved at the beginning of the next section (see Proposition 7.4).

7. Transposition and the exact sequences of a triple

We referred briefly in the previous section to the notion of the *transpose*[6]) of a map in $\mathfrak{P}$. If $\Psi = (\sigma, \sigma') : \beta_1 \to \beta_2$ then the transpose of Ψ is the map $\Psi^T = (\beta_1, \beta_2) : \sigma \to \sigma'$. Clearly there is a $1-1$ correspondence between maps $\Phi \to \Psi$ and maps $\Phi^T \to \Psi^T$, the correspondence being achieved by the transposition $\begin{pmatrix} f & f' \\ g & g' \end{pmatrix} \to \begin{pmatrix} f & g \\ f' & g' \end{pmatrix}$. This correspondence plainly induces an isomorphism

$$\tau_0 : \Pi_n(\Phi, \Psi) \cong \Pi_n(\Phi^T, \Psi^T)\,. \tag{7.1}$$

Now consider the special case in which $\Phi = \iota_1(\iota_1 A)$; it will be sufficient to look at the case $n = 0$. In detail, then, $\Phi = \iota_1(\iota_1 A)$ is the map

$$\Phi \Downarrow \quad \begin{array}{ccc} A & \xrightarrow{\iota_1 A} & CA \\ {\scriptstyle \iota_1 A}\downarrow & & \downarrow{\scriptstyle \iota_1 CA} \\ CA & \xrightarrow[C\iota_1 A]{} & CCA \end{array}$$

Let $s : CCA \to CCA$ be the homeomorphism given by $s(a, t, u) = (a, u, t)$. Then

Lemma 7.2. The map $\begin{pmatrix} 1 & 1 \\ 1 & s \end{pmatrix}$ is a equivalence from Φ to Φ^T.

Proof. It is only necessary to observe that $s \circ C\iota_1 A = \iota_1 CA$ and $s^2 = 1$.

Corollary 7.3. The map $\begin{pmatrix} 1 & 1 \\ 1 & s \end{pmatrix}$ induces an isomorphism

$$\tau_1 : \Pi_n(\Phi, \Psi) \cong \Pi_n(\Phi^T, \Psi)\,,$$

where $\Phi = \iota_1(\iota_1 A)$.

Putting together (7.1) and Corollary 7.3 we get

Proposition 7.4. There is a natural isomorphism $\tau (= \tau_0 \tau_1)$

$$\tau : P_n(A, \Psi) \cong P_n(A, \Psi^T)\,, \quad n > 2\,.$$

---

[6]) The equivalent notion in an exact category was exploited in [6].

We will agree to identify the groups $P_n(A, \Psi)$ and $P_n(A, \Psi^T)$ by means of the isomorphism τ; we remark that even if $n = 2$, τ is a $1-1$ correspondence.

We have remarked that Proposition 7.4 yields, by specialization, the second exact sequence of a triad. We also remark that it enables us to regard the sequence $S_*(A, \Psi)$ as part of a network of exact sequences. Thus we have

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & \Pi_n(A, B_1) & \to & \Pi_n(A, B_2) & \to & P_n(A, \sigma) & \to & \Pi_{n-1}(A, B_1) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & \Pi_n(A, B_1') & \to & \Pi_n(A, B_2') & \to & P_n(A, \sigma') & \to & \Pi_{n-1}(A, B_1') & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & P_n(A, \beta_1) & \to & P_n(A, \beta_2) & \to & P_n(A, \Psi) & \to & P_{n-1}(A, \beta_1) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & \Pi_{n-1}(A, B_1) & \to & \Pi_{n-1}(A, B_2) & \to & P_{n-1}(A, \sigma) & \to & \Pi_{n-2}(A, B_1) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}$$

with exact rows and columns; this we may express more succinctly by the exact triangle

$$\begin{tikzcd}
S_*(A, \beta_1) \arrow[rr, "\Psi_*"] & & S_*(A, \beta_2) \arrow[dl, "J"] \\
 & S_*(A, \Psi) \arrow[ul, "\partial"] &
\end{tikzcd}$$

(or its transpose!)

Dually we have

Proposition 7.4*. There is a natural isomorphism

$$\tau : P_n(\Phi, B) \cong P_n(\Phi^T, B) .$$

This yields an exact triangle of exact sequences

$$\begin{tikzcd}
S^*(\alpha_2, B) \arrow[rr, "\Phi^*"] & & S^*(\alpha_1, B) \arrow[dl, "J"] \\
 & S^*(\Phi, B) \arrow[ul, "\partial"] &
\end{tikzcd}$$

There is one other elementary deduction which may be drawn from Proposition 7.4 and the exactness of the sequence $S_*(A, \Psi)$, namely

Theorem 7.5. $\Psi_* : P_n(A, \beta_1) \cong P_n(A, \beta_2)$, all n, if and only if

$$\Psi_*^T : P_n(A, \sigma) \cong P_n(A, \sigma')\,, \quad \text{all } n\,.$$

The dual is obvious; we will not state it explicitly. But we remark that 7.5 and 7.5* yield results for the classical homotopy and cohomology groups.

We now turn attention again to the category $\mathfrak{L}$ of triples (see Section 3). Given a triple $[\lambda, \mu]$,

$$P \xrightarrow{\lambda} Q \xrightarrow{\mu} R\,,$$

write $\nu = \mu\lambda$; then we may define four maps in $\mathfrak{P}$, namely

$$\Psi_1 = (\lambda, 1) : \nu \to \mu\,, \qquad \Psi_2 = (1, \mu) : \lambda \to \nu\,,$$

and their transposes. We have the exact sequence

$$S_*(A, \Psi_1) : \cdots \to P_n(A, \nu) \xrightarrow{(\lambda, 1)_*} P_n(A, \mu) \xrightarrow{J} P_n(A, \Psi_1) \xrightarrow{\partial} P_{n-1}(A, \nu) \to \cdots;$$

moreover by 6.4 (ii), applied to $\Psi_1^T = (\nu, \mu) : \lambda \to 1$, we obtain an isomorphism

$$\partial : P_n(A, \Psi_1^T) \cong P_{n-1}(A, \lambda)\,. \tag{7.5}$$

Thus a unique homomorphism $\varrho : P_{n-1}(A, \lambda) \to P_{n-1}(A, \nu)$ is defined by

$$\partial = \varrho\, \partial^T \tau\,, \tag{7.6}$$

where we have written ∂^T for the boundary isomorphism (7.5).

Lemma 7.7. $\varrho = \Psi_{2*} = (1, \mu)_*$.

Proof. It is obviously sufficient to consider $\varrho : P_1(A, \lambda) \to P_1(A, \nu)$.

We consider an element $\alpha = \{(f, g)\} \in P_1(A, \lambda)$, and seek to construct an element $\xi \in P_2(A, \Psi_1^T)$ with $\partial^T \xi = \alpha$. The problem then is to provide suitable maps $CA \to R, C^2A \to R$ yielding a commutative diagram

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "\iota_1 A"] \arrow[dd, "\iota_1 A"'] & & P \arrow[dr, "\lambda"] \arrow[dd, "\nu"] & \\
& CA \arrow[rr, "g"] \arrow[dd] & & Q \arrow[dd, "\mu"] \\
CA \arrow[rr, "\iota_1 CA"] \arrow[dr, "C\iota_1 A"'] & & R \arrow[dr, "1"] & \\
& C^2A \arrow[rr] & & R
\end{tikzcd}$$

We define the map $CA \to R$ to be μg. Then $\mu g \circ \iota_1 A = \nu f$ because $g \circ \iota_1 A = \lambda f$. It remains to describe a map $u : C^2 A \to R$ satisfying $u \circ C\iota_1 A = \mu g$, $u \circ \iota_1 CA = \mu g$. This is equivalent to the problem of extending to $A \times I \times I$ a map $v : A \times (I \times I)^{\cdot} \to R$ given by

$$\begin{aligned} v(a, t, 0) &= \mu g(a, t) \quad , \\ v(a, 0, t) &= \mu g(a, t) \quad , \\ v(a, t, 1) &= v(a, 1, t) = 0 \, . \end{aligned}$$

Since the map v has an evident extension to $A \times I \times I$, such a map u may be defined. Thus

$$\partial^T \left\{ \begin{pmatrix} f & g \\ \mu g & u \end{pmatrix} \right\} = \{f, g\} \ ,$$

so that

$$\partial^T \tau \left\{ \begin{pmatrix} f & \mu g \\ g & su \end{pmatrix} \right\} = \{f, g\} \ ;$$

it follows that $\varrho(\alpha) = \partial \begin{Bmatrix} f & \mu g \\ g & su \end{Bmatrix} = \{f, \mu g\} = (1, \mu)_* \{f, g\} = (1, \mu)_* (\alpha)$, and the lemma is proved.

Corollary 7.8. *(The exact sequence of a triple.) Given a triple* $[\lambda, \mu]$ *with* $\nu = \mu \lambda$, *there is an exact sequence*

$$S_*(A ; \lambda, \mu) : \cdots \to P_n(A, \lambda) \overset{(1,\mu)_*}{\to} P_n(A, \nu) \overset{(\lambda,1)_*}{\to} P_n(A, \mu) \overset{d}{\to} P_{n-1}(A, \lambda) \to \cdots$$

The homomorphism d in the sequence $S_*(A ; \lambda, \mu)$ is, by definition, $\partial^T \tau J$. However we may easily show that

$$\partial^T \tau J = J \partial \ , \tag{7.9}$$

where, on the right hand side, ∂ is the boundary $\partial : P_n(A, \mu) \to \Pi_{n-1}(A, Q)$ and J is the homomorphism $J : \Pi_{n-1}(A, Q) \to P_{n-1}(A, \lambda)$. For (taking $n = 2$) if $\xi \in P_2(A, \mu)$ is represented by $\begin{pmatrix} \omega & \omega \\ f & g \end{pmatrix} : \iota_1 \iota_1 A \to \tilde{\omega}(\mu)$, then $J\xi$ is represented by $\begin{pmatrix} 0 & 0 \\ f & g \end{pmatrix}$, $\tau J \xi$ by $\begin{pmatrix} 0 & f \\ 0 & gs \end{pmatrix}$ and $\partial^T \tau J \xi$ by $(0, f) : \iota_1 A \to \lambda$; while $\partial \xi$ is represented by $(\omega, f) : \iota_1 A \to \tilde{\omega} Q$ and $J \partial \xi$ by $(0, f) : \iota_1 A \to \lambda$. Interpreting d then as $J \partial$ we get the final version of the sequence of a triple; notice that the exact sequence of a triple is valid without any special assumptions on the nature of the maps λ, μ, ν.

The dual sequence may be related to the same triple $[\lambda, \mu]$. We have

Corollary 7.8*. *(The dual exact sequence of a triple.) Given a triple* $[\lambda, \mu]$ *with* $\nu = \mu \lambda$, *there is an exact sequence*

$$S^*(\lambda, \mu ; B) : \cdots \to P_n(\mu, B) \overset{(\lambda,1)^*}{\to} P_n(\nu, B) \overset{(1,\mu)^*}{\to} P_n(\lambda, B) \overset{d}{\to} P_{n-1}(\mu, B) \to \cdots$$

If we take $A = S^0$ and λ, μ inclusions the sequence $S_*(A; \lambda, \mu)$ is just the classical homotopy sequence of the triple (R, Q, P). More generally we could take for A a MOORE space $K'(G, m)$ and obtain the sequence

$$\cdots \to \pi_n(G; \lambda) \xrightarrow{(1,\mu)_*} \pi_n(G; \nu) \xrightarrow{(\lambda,1)_*} \pi_n(G; \mu) \xrightarrow{d} \pi_{n-1}(G; \lambda) \to \cdots ,$$

an exact sequence of homotopy groups with coefficients for the triple. If in the triple $[\lambda, \mu]$, λ and μ are both fibre maps then $\nu = \mu\lambda$ is a fibre map. Moreover if $\lambda_0 = \lambda \mid$ (fibre of ν), regarded as a map into the fibre of μ, then λ_0 is a fibre map whose fibre coincides with the fibre of λ and $S_*(A; \lambda, \mu)$ may be identified with $T_*(A, \lambda_0)$.

In the dual case we may take B to be an EILENBERG-MACLANE complex, and apply the "singular polyhedron" functor to λ, μ, ν. Then if λ, μ, ν are inclusions we get the classical (singular) cohomology sequence of the triple (R, Q, P). Even if λ, μ, ν are not inclusions we get the sequence

$$\cdots \to H^m(\mu; G) \xrightarrow{(\lambda,1)^*} H^m(\nu; G) \xrightarrow{(1,\mu)^*} H^m(\lambda; G) \xrightarrow{d} H^{m+1}(\mu; G) \to \cdots .$$

Also we obtain the sequence $T^*(\mu^0, B)$ from $S^*(\lambda, \mu; B)$ if λ, μ are cofibre maps; here μ^0 is the cofibre map from the cofibre of λ to the cofibre of ν and the cofibre of μ^0 coincides with the cofibre of μ.

Finally we remark (compare [1] or [2]) that we may topologize the groups $H^m(\alpha; G)$ if G is a topological group and then, by regarding $H_m(\alpha; G)$ as Char $(H^m(\alpha; \text{Char } G))$, where G is discrete, we may deduce the corresponding homology sequence of a triple

$$\cdots \leftarrow H_m(\mu; G) \xleftarrow{(\lambda,1)_*} H_m(\nu; G) \xleftarrow{(1,\mu)_*} H_m(\lambda; G) \xleftarrow{d} H_{m+1}(\mu; G) \leftarrow \cdots \tag{7.10}$$

However, we will give an alternative, combinatorial, treatment for the homology sequence of a triple in the appendix (Section 9). The exact sequences of homotopy and homology groups of a triple were exploited in [5] in obtaining homotopy and homology decompositions of any arbitrary map.

8. The twisted homotopy groups $\tilde{\Pi}_n(\alpha, \beta)$

If $\beta : B_1 \to B_2$ there is plainly an induced homomorphism

$$\beta_* : P_n(\alpha, B_1) \to P_n(\alpha, B_2) . \tag{8.1}$$

It is tempting, amid the plethora of exact sequences available, when we select suitable objects from the categories $\mathfrak{T}$ and $\mathfrak{P}$, to conjecture that it should be possible to embed the homomorphism (8.1) in an exact sequence and even to suppose that the group which appears in the sequence to measure the failure of β_* to be an isomorphism is $\Pi_n(\alpha, \beta)$. This last turns out to be false; in fact we will show that the group which does appear is obtained by going into the "higher" category by means of the functions ι_1, ϱ_1 and there submitting $\iota_1(\alpha)$ or $\varrho_1(\beta)$ to a twist.

Let us write $\tilde{\iota}_1(\alpha)$, $\tilde{\varrho}_1(\beta)$ for $\iota_1(\alpha)^T$, $\varrho_1(\beta)^T$. Then, in particular

$$\varrho_1(\beta) : E\beta \to \beta ,$$

so that

$$\tilde{\varrho}_1(\beta) : \varrho_1(B_1) \to \varrho_1(B_2) .$$

Notice that ϱ_1 cannot be regarded as a *functor* from $\mathfrak{T}$ to $\mathfrak{P}$, since we would then have confused $\varrho_1(\beta)$ with $\tilde{\varrho}_1(\beta)$; we have reserved the symbols ι_1, ϱ_1 to refer to certain canonical processes for associating objects of the category $\mathfrak{P}(\mathfrak{C})$ with objects of the category $\mathfrak{C}$, for *any* suitable $\mathfrak{C}$.

We then have the exact sequence

$$S_*(\alpha, \beta) = S_*(\alpha, \tilde{\varrho}_1(\beta)) :$$
$$\cdots \to P_n(\alpha, B_1) \xrightarrow{\beta_*} P_n(\alpha, B_2) \xrightarrow{J} \tilde{\Pi}_{n-1}(\alpha, \beta) \xrightarrow{\partial} P_{n-1}(\alpha, B_1) \to \cdots ,$$

where, by definition,

$$\tilde{\Pi}_n(\alpha, \beta) = P_n(\alpha, \tilde{\varrho}_1(\beta)) . \tag{8.2}$$

We now clarify the relation of $\tilde{\Pi}_n(\alpha, \beta)$ to $\Pi_n(\alpha, \beta)$; in the course of doing so we will hope to justify our convention in assigning the dimension n rather than $(n+1)$ to the group defined in (8.2) as $P_n(\alpha, \tilde{\varrho}_1(\beta))$. We point out the following evident relations.

Proposition 8.3. $P_n(A, \varrho_1 B) \cong \Pi_n(A, B)$, $P_n(\alpha, \varrho_1(\beta)) \cong \Pi_n(\alpha, \beta)$.

Proposition 8.3*. $P_n(\iota_1 A, B) \cong \Pi_n(A, B)$, $P_n(\iota_1(\alpha), \beta) \cong \Pi_n(\alpha, \beta)$.

Proof. Since $\varrho_1 B$ is a fibre map we may apply excision to $P_n(A, \varrho_1 B)$, getting $\Pi_{n-1}(A, \Omega B) \cong P_n(A, \varrho_1 B)$; but $\Pi_{n-1}(A, \Omega B) = \Pi_n(A, B)$.

Similar arguments justify the other four statements. Thus in particular we see that the difference in the definitions of $\Pi_n(\alpha, \beta)$ and $\tilde{\Pi}_n(\alpha, \beta)$ may be described by saying that we pass from the former to the latter by twisting $\varrho_1(\beta)$.

Now $P_n(\alpha, \tilde{\varrho}_1(\beta)) = \Pi_{n-1}(\iota_1(\alpha), \tilde{\varrho}_1(\beta))$, by definition. Applying the isomorphism τ_0 (7.1) we find

$$P_n(\alpha, \tilde{\varrho}_1(\beta)) \cong \Pi_{n-1}(\tilde{\iota}_1(\alpha), \varrho_1(\beta)) = P_n(\tilde{\iota}_1(\alpha), \beta) .$$

We thus see that, if we identify groups connected by the canonical isomorphism τ_0, the definition of $\tilde{\Pi}_n(\alpha, \beta)$ is self-dual,

$$\tilde{\Pi}_n(\alpha, \beta) = P_n(\tilde{\iota}_1(\alpha), \beta) . \tag{8.2*}$$

Thus the group $\tilde{\Pi}_n(\alpha, \beta)$ also figures in the exact sequence

$S^*(\alpha, \beta) = S^*(\tilde{\iota}_1(\alpha), \beta)$:

$$\cdots \to P_n(A_2, \beta) \xrightarrow{\alpha^*} P_n(A_1, \beta) \xrightarrow{J} \tilde{\Pi}_{n-1}(\alpha, \beta) \xrightarrow{\partial} P_{n-1}(A_2, B) \to \cdots$$

In fact the sequences $S_*(\alpha, \beta)$, $S^*(\alpha, \beta)$ may be fitted into the network of exact sequences which we may represent as

$$\begin{tikzcd}
S^*(\alpha, B_1) \arrow[rr, "\beta_*"] & & S^*(\alpha, B_2) \arrow[dl, "J"] \\
& S^*(\alpha, \beta) \arrow[ul, "\partial"] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
S_*(A_2, \beta) \arrow[rr, "\alpha^*"] & & S_*(A_1, \beta) \arrow[dl, "J"] \\
& S_*(\alpha, \beta) \arrow[ul, "\partial"] &
\end{tikzcd}$$

Certain obvious consequences may be drawn from the exact sequences $S_*(\alpha, \beta)$ and $S^*(\alpha, \beta)$ and results of section 2. Rather than list them in detail we sum them up by saying that the groups $\tilde{\Pi}_n(\alpha, \beta)$ are unaffected (up to isomorphism) by composing α or β with a homotopy equivalence or by replacing α or β by homotopic maps. We know no such theorems for the groups $\Pi_n(\alpha, \beta)$. On the other hand we should record

Proposition 8.4. $\Pi_n(\alpha, \beta) \cong \tilde{\Pi}_n(\alpha, \beta)$ if $\alpha = \iota_1 A$ or $\beta = \varrho_1 B$.

For, as pointed out in Lemma 7.2, $\tilde{\iota}_1 \iota_1 A$ is equivalent to $\iota_1 \iota_1 A$. Of course, if $\alpha = \iota_1 A$ then $\Pi_n(\alpha, \beta)$ is just $P_{n+1}(A, \beta)$.

The introduction of the groups $\tilde{\Pi}_n(\alpha, \beta)$ gives the clue to the generalization of the sequence $S_*(A; \lambda, \mu)$ in which we replace A by an arbitrary map. We have no reason to believe that an exact sequence can be obtained by replacing $P_n(A, \Theta)$ by $\Pi_n(\alpha, \Theta)$, $\Theta = \lambda, \mu, \nu$, but an exact sequence can indeed be obtained by replacing $P_n(A, \Theta)$ by $\tilde{\Pi}_n(\alpha, \Theta)$. Our proof of this fact will use the mapping cylinder functor and an excision theorem for the groups $\tilde{\Pi}_n$.

Lemma 8.5. If $\alpha : A_1 \to A_2$ is a cofibration, so is $\tilde{\iota}_1(\alpha) : \iota_1 A_1 \to \iota_1 A_2$.

Proof. Let K_α be the space formed from the disjoint union $(A_1 \times I) \cup (A_2 \times 0)$ by means of the identification $(a_1, 0) \equiv (\alpha a_1, 0)$. It evidently follows from the fact that α is a cofibration that there is a map

$$r = r_\alpha : A_2 \times I \to K_\alpha \tag{8.6}$$

such that $r(a_2, 0) = (a_2, 0)$, $r(\alpha a_1, t) = (a_1, t)$. Conversely the existence of such a map r implies that α is a cofibration. For if (f_t, α, g_0) is a triangle [7])

$$\begin{tikzcd}
A_2 \arrow[dr, "g_0"] & \\
A_1 \arrow[u, "\alpha"] \arrow[r, "f_t"'] & X
\end{tikzcd}$$

we define $h : K_\alpha \to X$ by $h(a_1, t) = f_t a_1, h(a_2, 0) = g_0 a_2, a_1 \in A_1, a_2 \in A_2$; then we lift f_t by $g_t : A_2 \to X$ where $g_t(a_2) = hr(a_2, t)$.

Now suppose given the commutative diagram

$$\begin{tikzcd}
 & & X \arrow[dd, "\xi"] \\
A_1 \arrow[urr, "f_t"] \arrow[r, "\alpha"] \arrow[dd, "\iota_1 A_1"'] & A_2 \arrow[ur, "g_0"] \arrow[dd, "\iota_1 A_2"] & \\
 & & X' \\
CA_1 \arrow[urr, "f'_t"] \arrow[r, "C\alpha"'] & CA_2 \arrow[ur, "g'_0"'] &
\end{tikzcd}$$

[7]) This is an abuse of the terminology of section 3.

We have to show that we may "lift" f_t, f'_t to g_t, g'_t such that $g'_t \circ \iota_1 A_2 = \xi g_t$. Using the map $r_\alpha : A_2 \times I \to K_\alpha$ we have already defined g_t. Similarly we define

$$r_{C\alpha} : CA_2 \times I \to K_{C\alpha}$$

by

$$r_{C\alpha}(u, a_2, t) = (u, r_\alpha(a_2, t)) ;$$

notice that here we have found it convenient to regard CA_1, CA_2 as obtained by identification from $I \times A_1, I \times A_2$. Then $r_{C\alpha}$ is the map giving the cofibre structure of the map $C\alpha$, and we define $h' : K_{C\alpha} \to X'$ by

$$h'(u, a_1, t) = f'_t(u, a_1) ,$$

$$h'(u, a_2, 0) = g'_0(u, a_2) ,$$

and $g'_t : CA_2 \to X'$ by $g'_t(u, a_2) = h' r_{C\alpha}(u, a_2, t)$. A straightforward computation establishes the required commutativity relation.

Theorem 8.7. *If α is a cofibre map with cofibre A_3 then there is an excision isomorphism*

$$\varepsilon : P_n(A_3, \beta) \cong \tilde{\Pi}_n(\alpha, \beta) .$$

Proof. $\tilde{\Pi}_n(\alpha, \beta) = P_n(\tilde{\iota}_1(\alpha), \beta)$. Since α is a cofibre map so is $\tilde{\iota}_1(\alpha)$, and so there is an excision isomorphism $\varepsilon : \Pi_{n-1}(\iota_1 A_3, \beta) \cong \tilde{\Pi}_n(\alpha, \beta)$; it is only necessary to observe that the cofibre of $\tilde{\iota}_1(\alpha)$ is plainly $\iota_1 A_3$. Finally $\Pi_{n-1}(\iota_1 A_3, \beta) = P_n(A_3, \beta)$ and the theorem is proved.

Corollary 8.8. Given a triple $[\lambda, \mu]$ and a map α there is an exact sequence

$$S_*(\alpha; \lambda, \mu) : \cdots \to \tilde{\Pi}_n(\alpha, \lambda) \xrightarrow{\mu_*} \tilde{\Pi}_n(\alpha, \nu) \xrightarrow{\lambda_*} \tilde{\Pi}_n(\alpha, \mu) \xrightarrow{d} \tilde{\Pi}_{n-1}(\alpha, \lambda) \to \cdots .$$

(Here we have been deliberately imprecise in our notation for the homomorphisms.)

Proof. We first apply the mapping cylinder functor and the remark preceding 8.4 to replace the sequence $S_*(\alpha; \lambda, \mu)$ by the isomorphic sequence $S_*(\alpha_1; \lambda, \mu)$. We next apply Theorem 8.7 to replace $S_*(\alpha_1; \lambda, \mu)$ by the isomorphic sequence $S_*(A_0; \lambda, \mu)$, where A_0 is the cofibre of α_1. But the sequence $S_*(A_0; \lambda, \mu)$ is exact.

Corollary 8.8*. Given a triple $[\lambda, \mu]$ and a map β there is an exact sequence

$$S^*(\lambda, \mu; \beta) : \cdots \to \tilde{\Pi}_n(\mu, \beta) \xrightarrow{\lambda^*} \tilde{\Pi}_n(\nu, \beta) \xrightarrow{\mu^*} \tilde{\Pi}_n(\lambda, \beta) \xrightarrow{d} \tilde{\Pi}_{n-1}(\mu, \beta) \to \cdots .$$

We remark that the homomorphism d of the sequence $S_*(\alpha;\lambda,\mu)$ has simply been defined so that the diagram

$$\begin{tikzcd} \tilde{\Pi}_n(\alpha,\mu) \arrow[r, "\alpha_2^*"] \arrow[d, "d"] & \tilde{\Pi}_n(\alpha_1,\mu) \arrow[d, "d"] & P_n(A_0,\mu) \arrow[l, "\varepsilon"'] \arrow[d, "d"] \\ \tilde{\Pi}_{n-1}(\alpha,\lambda) \arrow[r, "\alpha_2^*"] & \tilde{\Pi}_{n-1}(\alpha_1,\lambda) & P_{n-1}(A_0,\lambda) \arrow[l, "\varepsilon"'] \end{tikzcd}$$

be commutative (all the horizontal maps are isomorphisms). A direct definition of d would have involved a discussion of exact sequences in $\mathfrak{P}^2(\mathfrak{T})$! It is not difficult to see that this definition of d suffices to yield the network of exact sequences expressed by the exact triangle

$$\begin{tikzcd} S_*(A_2;\lambda,\mu) \arrow[rr, "\alpha^*"] & & S_*(A_1;\lambda,\mu) \arrow[dl, "J"] \\ & S_*(\alpha;\lambda,\mu) \arrow[ul, "\partial"] & \end{tikzcd}$$

Similarly we have the exact triangle

$$\begin{tikzcd} S^*(\lambda,\mu;B_1) \arrow[rr, "\beta_*"] & & S^*(\lambda,\mu;B_2) \arrow[dl, "J"] \\ & S^*(\lambda,\mu;\beta) \arrow[ul, "\partial"] & \end{tikzcd}$$

9. Appendix: The combinatorial homology groups

In this appendix we describe how the cohomology groups and homomorphisms discussed in previous sections have their parallels for *homology groups based on chain complexes*. Our main concern is the homology sequence of a *triple*, but we begin with an observation on the usual coefficient sequence for homology. We observe, namely, that if $C = (C_n, d_n)$ is a chain complex and $\Phi: G_1 \to G_2$ a homomorphism, then we may define the homology groups of C with coefficients in Φ to be the homology groups of the chain mapping cylinder of the chain map $C \otimes G_1 \to C \otimes G_2$ induced by Φ. With this definition of $H_*(C;\Phi)$ we have the exact sequence

$$\cdots \to H_n(C;G_1) \xrightarrow{\Phi_*} H_n(C;G_2) \xrightarrow{J} H_n(C;\Phi) \xrightarrow{\partial} H_{n-1}(C;G_1) \to \cdots. \quad (9.1)$$

In particular if C is the singular chain complex of the space X, we have the notion of the singular homology group of X with coefficients in Φ and the exact sequence

$$\cdots \to H_n(X;G_1) \xrightarrow{\Phi_*} H_n(X;G_2) \xrightarrow{J} H_n(X;\Phi) \xrightarrow{\partial} H_{n-1}(X;G_1) \to \cdots. \tag{9.2}$$

Given any chain map $\Theta : C \to D$, we may define the *chain complex of* Θ to be the chain mapping cylinder of Θ. In particular if $\beta : B_1 \to B_2$ is a (continuous) map the singular chain complex of β, $C(\beta)$, is by definition the chain complex of the singular chain map induced by β and so we obtain, for any such β, an exact sequence

$$\cdots \to H_n(B_1) \xrightarrow{\beta_*} H_n(B_2) \xrightarrow{J} H_n(\beta) \xrightarrow{\partial} H_{n-1}(B_1) \to \cdots; \tag{9.3}$$

this may, of course, be generalized to arbitrary coefficient groups.

Let us now pass to the category $\mathfrak{P}^2$. Following on (6.1) and (6.1*) we might have defined

$$\pi_m(G;\Psi) = P_n(K'(G, m-n), \Psi), \tag{9.4}$$

$$H^m(\Phi;G) = P_n(\Phi, K(G, m+n-2)); \tag{9.4*}$$

and so specialize $S_*(A, \Psi)$, $S^*(\Phi, B)$ to homotopy and cohomology sequences.

We do not however stress these definitions because they are based on traditional conventions which serve to obscure the duality. However, we will now introduce the singular homology groups $H_m(\Psi)$ based on the singular chain complex $C(\Psi)$. Let Ψ be the map

$$\begin{array}{ccc} B_1 & \xrightarrow{\beta_1} & B_1' \\ \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sigma'} \\ B_2 & \xrightarrow[\beta_2]{} & B_2' \end{array} \qquad \Big\Downarrow \Psi$$

Then clearly $\Psi = (\sigma, \sigma')$ induces a chain map $\Psi_0 : C(\beta_1) \to C(\beta_2)$, and we define $C(\Psi)$ to be the chain mapping cylinder[8]) of Ψ_0. We will make this definition quite explicit, allowing ourselves to write $\beta_1, \beta_2, \sigma, \sigma'$ for the chain maps they induce. Then

$$C_m(\Psi) = C_{m-2}(B_1) \oplus C_{m-1}(B_1') \oplus C_{m-1}(B_2) \oplus C_m(B_2'), \tag{9.5}$$

[8]) This definition may plainly be generalized to any commutative square of chain maps.

and the boundary operator is

$$d(u_1, u_1', u_2, u_2') = (du_1, -\beta_1 u_1 - du_1', \sigma u_1 - du_2, \sigma' u_1' + \beta_2 u_2 + du_2') . \quad (9.6)$$

One certainly then has a homology sequence

$$\cdots \to H_n(\beta_1) \xrightarrow{\Psi_*} H_n(\beta_2) \xrightarrow{J} H_n(\Psi) \xrightarrow{\partial} H_{n-1}(\beta_1) \to \cdots . \quad (9.7)$$

It is worth remarking that a HUREWICZ homomorphism $h : \pi_n(\Psi) \to H_n(\Psi)$ is easily defined. For if we take $\Theta = \iota_1 \iota_1 S^{n-2}$, then $H_n(\Theta)$ is cyclic infinite and we may orient Θ by picking a generator $\eta \in H_n(\Theta)$. Now an element α of $\pi_n(\Psi)$ is represented by a map $a : \Theta \to \Psi$ and we define

$$h(\alpha) = a_*(\eta) .$$

However, there is no immediate generalization of the HUREWICZ isomorphism theorem to the category $\mathfrak{P}^2$. For the exactness of $S_*(A, \Psi)$ implies that $\pi_*(\Psi) = 0$ if and only if $\Psi_* : \pi_*(\beta_1) \cong \pi_*(\beta_2)$; and the exactness of (9.7) implies that $H_*(\Psi) = 0$ if and only if $\Psi_* : H_*(\beta_1) \cong H_*(\beta_2)$. But it is well-known (in the case of inclusion maps) that Ψ_* may induce homology isomorphisms but not homotopy isomorphisms. It seems possible that a HUREWICZ theorem in $\mathfrak{P}^2$ may involve homology and homotopy groups with relativized coefficients.

If we look at (9.5) and (9.6) we see that the map

$$\tau : C_m(\Psi) \to C_m(\Psi^T) ,$$

given by

$$\tau(u_1, u_1', u_2, u_2') = (-u_1, u_2, u_1', u_2') \quad (9.8)$$

is a chain-isomorphism and so induces a homology isomorphism

$$\tau_* : H_*(\Psi) \cong H_*(\Psi^T) . \quad (9.9)$$

We exploit (9.9) in studying the homology sequence of a triple $\nu = \mu\lambda$ (see (7.10)). In the light of (9.7) we have the exact sequence

$$\cdots \to H_n(\nu) \xrightarrow{(\lambda,1)_*} H_n(\mu) \xrightarrow{J} H_n(\Psi) \xrightarrow{\partial} H_{n-1}(\nu) \to \cdots , \qquad \Psi = (\lambda, 1)$$

and the isomorphism

$$\partial = \partial^T : H_n(\Psi^T) \cong H_{n-1}(\lambda) .$$

Using (9.9), we get the exact sequence

$$\cdots \to H_n(\nu) \overset{(\lambda,1)_*}{\to} H_n(\mu) \overset{\varrho}{\to} H_{n-1}(\lambda) \overset{\sigma}{\to} H_{n-1}(\nu) \to \cdots ,$$

where $\varrho = \partial^T \tau_* J$, $\sigma \partial^T \tau_* = \partial$. Given $P \overset{\lambda}{\to} Q \overset{\mu}{\to} R$, let $\partial_0 : H_n(\mu) \to H_{n-1}(Q)$ be the boundary in the exact sequence of μ and let $J_0 : H_{n-1}(Q) \to H_{n-1}(\lambda)$ be the J-homomorphism in the exact sequence of λ.

Lemma 9.10. $\varrho = J_0 \partial_0$.

Proof. The reader may verify that $\partial^T \tau_* J$ and $J_0 \partial_0$ are both induced by the chain map

$$(u, u') \to (0, u), \qquad u \in C_{n-1}(Q), \qquad u' \in C_n(R).$$

Lemma 9.11. $\sigma = -(1, \mu)_*$.

Proof. $\partial : H_n(\Psi) \to H_{n-1}(\nu)$ is induced by the chain map

$$(u_1, u_1', u_2, u_2') \to (u_1, u_1'), \qquad \begin{aligned} &u_1 \in C_{n-2}(P), \quad u_1' \in C_{n-1}(R), \\ &u_2 \in C_{n-1}(Q), \quad u_2' \in C_n(R). \end{aligned}$$

On the other hand, $(1, \mu)_* \partial^T \tau_*$ is induced by the chain map

$$(u_1, u_1', u_2, u_2') \to (-u_1, \mu u_2).$$

Now if (u_1, u_1', u_2, u_2') is a cycle it follows from (9.6) that $u_1' + \mu u_2 + du_2' = 0$. Then $(0, u_1' + \mu u_2) = d(0, -u_2')$ so that

$$\partial = -(1, \mu)_* \partial^T \tau_* ,$$

from which the lemma immediately follows.

We have now proved

Theorem 9.12. *If* $[\lambda, \mu]$ *is a triple with* $\nu = \mu\lambda$, *then there is an exact sequence*

$$\cdots \to H_n(\lambda) \overset{(1,\mu)_*}{\to} H_n(\nu) \overset{(\lambda,1)_*}{\to} H_n(\mu) \overset{\partial}{\to} H_{n-1}(\lambda) \to \cdots ,$$

where $\partial = J_0 \partial_0$ (see 9.10).

A direct proof of this theorem would, of course, have been available but we have preferred to parallel the arguments of section 7.

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(Received May 24, 1960)