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Generalization of a geometrical theorem of Euler

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In his first derivation of the equation of continuity in hydrodynamics[1]) Euler calculated the change in area of an infinitesimal triangle and the change in volume of an infinitesimal tetrahedron when their vertices are carried through an infinitesimal motion in time dt. He discards terms $O(dt^2)$ in the displacements, but his subsequent calculation of area and volume is algebraically exact. For the volume change ΔV he obtains

$$\frac{\Delta V}{V} = I_1 dt + I_2 dt^2 + I_3 dt^3 , \tag{1}$$

where I_1, I_2, I_3 are the sum of the one-, two-, and three-rowed principal minors of the velocity gradient matrix and for the area change an analogous formula. Since terms $O(dt^2)$ have been neglected in the displacements, the formula (1) is in general false. Of course, the leading term is all that is required to derive the continuity equation, and the leading term is correct. Euler's subsequent derivations of the continuity equation neglect terms $O(dt^2)$ consistently, as previously had d'Alembert's in special cases, and it is a trivial matter nowadays to construct a rigorous proof making no outward use of infinitesimals at all.

Now usually when an error is found in the work of Euler, after a lapse of time sufficient for it to become remarked habitually in the histories, Euler is found to have been right after all, if once rightly understood. The present error is noticed here for the first time; and, to break with custom, so is the proper reinterpretation. While (1) is indeed false in Euclidean kinematics, it turns out to represent a correct result in the theory of affine motions.

Let x be a vector in n-space, and consider the transformations

$$\bar{x} = (At + I)x , \tag{2}$$

where A is an $n \times n$ matrix whose entries are elements of a field, I is

[1]) Principia motus fluidorum, Novi Comment. Acad. Sci. Petrop. *6* (1756—1757) 271—311 (1761). This paper was written c. 1752.

the unit matrix, and t is an indeterminate. If we wish to, we may choose to interpret (2) as a continuous family of homogeneous strains.

Introduce the Euclidean definition of volume. That is, define the volume of a paralleletope whose $n + 1$ vertices are 0 and x_i, $i = 1, 2, \ldots, n$, by the formula

$$V \equiv det\ x_i \ . \tag{3}$$

Then follows

$$\begin{aligned} \overline{V} &= det\ \overline{x}_i = det\ [(At + I)x_i] \ , \\ &= det(At + I)\ det\ x_i \ , \\ &= V\ det(At + I) \ . \end{aligned} \tag{4}$$

By the expansion of the secular determinant follows

$$\frac{\overline{V}}{V} = \sum_{i=0}^{n} I_i t^i \ ,$$

where $I_i =$ the sum of the i-rowed principal of A, or, equivalently, the ith elementary symmetric function of the proper numbers of A. It is this result which Euler proved by explicit calculation in the cases $n = 2,3$. Thus Euler is the discoverer of the secular expansion. From the manner in which he obtained the I_r it is clear though not actually proved that they are indeed invariants.

From (5) follows

$$\frac{1}{r!\,V}\frac{d^r\overline{V}}{dt^r}\bigg|_{t=0} = \begin{cases} I_r \ , \ r = 1 \ , \ 2 \ , \ \ldots \ , \ n \ , \\ 0 \ , \ r \geqq n + 1 \ , \end{cases} \tag{6}$$

yielding a geometrical interpretation for the principal invariants I_r of an arbitrary matrix.

By means of a known theorem[2]), when t and the entries of A are complex numbers from (6) and (5) we get

$$J_r = \frac{(-1)^r}{(r-1)!}\frac{d^r \log \overline{V}}{dt^r}\bigg|_{t=0} \ , \tag{7}$$

$$\log \frac{\overline{V}}{V} = \sum_{r=1}^{\infty} \frac{(-)^{r-1} J_r}{r} t^r \ , \tag{8}$$

where $J_r =$ the sum of the rth powers of the proper numbers of A. It is not difficult to show that the reciprocal of the radius of convergence of (8) is the greatest among the absolute values of the proper numbers of A. When A is Hermitian, several corollaries regarding the signs of the successive derivatives of log V may be read off from (7).

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[2]) E. g. § 132 of *Burnside* and *Panton,* Theory of Equations, 2nd ed., Dublin (1886).