The probability of failure and safety of structural section loaded with a multi-dimensional force-combination

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La probabilité de rupture et la sécurité d'un élément de structure chargé avec une combinaison multidimensionnelle de forces

Die Versagenswahrscheinlichkeit und die Sicherheit eines mit einer vieldimensionalen Kraftkombination belasteten Bauteiles

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The problem of reliability has been discussed in several papers during recent years and, as we know, many methods have been developed to solve this question. These solutions usually aim at determining the probability of failure of the observed structure.

As far as the writer is aware, the calculation methods are all quite approximate, and the mathematical difficulties have prevented the development of more exact solutions.

However, the character of the problem, means that there is a need for a mathematically satisfactory design method. The purpose of research in this subject is to take rational account of the irregularities of material, dimensions, loads and calculations, e.g. by a number called "safety factor".

If the calculation method by which the safety factor is determined is very approximative, we are actually obliged to use a complementary factor to eliminate all the unreliabilities which are included in the calculation of this factor. This is, of course, not desirable.

The development of the computers in the last years has made it possible to solve more complicated mathematical problems and to reach a higher degree of exactness of results than before. The following work presents an attempt to solve the probability of failure of a structural element using a computer, by a method which the writer supposes to be general enough and to contain a number of approximations, which gives a sufficient exactness for practical purposes.

This paper is to a great extent partial abbreviation of a larger study, supposed by the Scandinavian Building Institutes. The study forms part of a joint Scandinavian project and will be published by the State Building Research Institute in Denmark.

1. The necessity of a general kind of frequency-function.

A central problem in the calculation of the probability of failure is the combining of several known fr.f.(frequencyfunctions) which are connected with each other by some known function. The result of such combinations is a new fr.f., which cannot generally be determined exactly. On the other hand, also the form of the initial distributions is in most cases unknown and to be estimated from the sample.

In addition to these aspects it is necessary to avoid the errors caused by small samples. We will return this later.

To comply with the requirements mentioned above, the following fr.f. has been chosen for use in the one-dimensional case:

$$(1) \qquad f(x) = e^{\sum\limits_{k=o}^{n} a_k \cdot x^k}$$

The parameter a_o will be determined so that

$$F(\infty) = \int_{-\infty}^{+\infty} f(x) \cdot dx = 1$$ where x represents an arbitrary quantity, which has an influence on the probability of failure, e.g. a property of a material, a dimension of a structure or a load.

Without paying more attention to the following question, we need only mention that, e.g.,

- the normal distribution
- the log-normal distribution
- the first asymptotic distribution of the extreme value
- the Weibull-distribution

all converge toward (1) with increasing n.

For the distribution function we use

$$(2) \qquad F(x) = e^{-e^{\sum\limits_{k=o}^{n} a_k{}^k}}$$

and in the multi-dimensional cases anologically to (1) and (2)

$$\begin{cases} (3) \quad f(x) = e^{\sum_{k_1=o}^{n_1} x_1^{k_1} \sum_{k_2=o}^{n_2} x_2^{k_2} \cdots \sum_{k_r=o}^{n_r} x_r^{k_r} \cdot a_{k_1 k_2 \ldots k_r}} \\ \\ (4) \quad F(x) = e^{-e^{\sum_{k_1=o}^{n_1} x_1^{k_1} \sum_{k_2=o}^{n_2} x_2^{k_2} \cdots \sum_{k_r=o}^{n_r} x_r^{k_r} \cdot a_{k_1 k_2 \ldots k_r}}} \end{cases}$$

With increasing $n_1 \ldots n_r$-values in (3) and (4) we can estimate multi-dimensional samples with arbitrary moments and also define distributions with very varying forms.

2. Estimation of the parameters of the various distributions.

For large samples we use either of two estimation methods, both well known from the statistical literature. The simpler is the method of moments, introduced by K Pearson, and the more developed is the method of maximum likelihood introduced by R.A. Fisher. In this connection, it is not sensible to explain either of these methods.

For small samples we use the following, more complicated method of estimation.

We first assume that the parent population has a general fr.f. $f(x, a_o \ldots a_m)$ where the parameters $a_1 \ldots a_m$ are assumed to be unknown. The parameter a_o is a function of $a_1 \ldots a_m$ so that $F(\infty) = 1$. The sample values of x are $x_1 \ldots x_n$.

We then study the situation after one value of the sample, x_1 has been found. In this case the fr.f. of a parameter combination can be represented by

$$(5) \qquad g_1(a_o \ldots a_m) = \frac{f(x_1, a_o \ldots a_m)}{\int_{R_m} f(x_1, a_o \ldots a_m) \cdot da_1 \ldots da_m}$$

The result has been found by examining a conditional frequency function of $a_1 \ldots a_m$, relative to the hypothesis $x = x_1$. We assume then that before any values of the sample are known the fr.f. $f(x, a_o \ldots a_m)$ is represented by an m + 1-dimensional fr.f. where m-dimensional marginal distribution in the space $a_1 \ldots a_m$ is rectangular.

If we then assume that we take n values from the same unknown population of the form $f(x,a_o\ldots a_m)$, we again get a conditional distribution

$$(6)\qquad g_n(a_o\ldots a_m) = \frac{\prod_{k=1}^{n} f(x_k,a_o\ldots a_m)}{\int_{R_m}\left(\prod_{k=1}^{n} f(x_k,a_o\ldots a_m)\right) da_1\ldots da_m}$$

Function (6) represents the combined distribution of parameters $a_o\ldots a_m$ on the basis of the sample $x_1\ldots x_n$. If we now define the distribution of the value x_{n+1}, we evidently obtain a fr.f. of this value:

$$(7)\quad h(x) = \frac{\int_{R_m} f(x,a_o\ldots a_m)\cdot\prod_{k=1}^{n} f(x_k,a_o\ldots a_m)\cdot da_1\ldots da_m}{\int_{R_m}\prod_{k=1}^{n} f(x_k,a_o\ldots a_m)\cdot da_1\ldots da_m}$$

The formula (7) can now be applied to arbitrary types of distributions. It has the advantage that the mistakes which can be made using the method of moments or the method of maximum likelihood with small samples can be avoided.

4. Capacity of a structural element.

The failure of a structural element can be defined by one or several inequalities (9), assuming that this element is loaded with a k-dimensional combination of forces and moments.

These inequalities can be illustrated in a k-dimensional space R_k so that the different types of failure each form a k-dimensional set of points in R_k, which have an infinite volume and are formed as sectors.

These sets are limited in relation to each other by k-1 dimensional hyper-surfaces, and each set is divided into two subsets, the first containing all the points which cause failure and the second containing all the combinations by which failure does not occur.

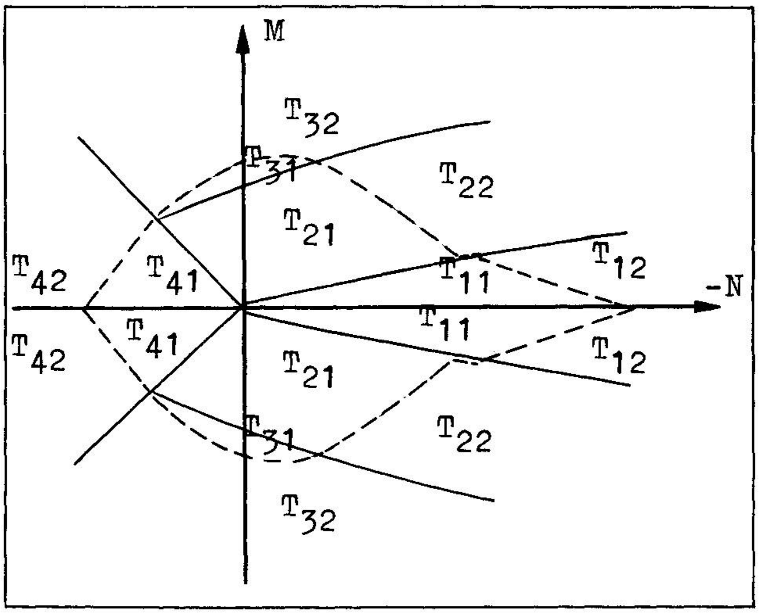
We get the equations:

$$(8)\quad \begin{cases} \sum_{j=1}^{r} T_j = R_k \text{ where } T_i\cdot T_j = 0 \text{ when } j \neq i \\ T_j = T_{j1} + T_{j2} \quad \text{where } T_{j1}\cdot T_{j2} = 0 \end{cases}$$

$$(8)\quad \begin{aligned} &\sum_{j=1}^{r} T_{j1} = U_1 \\ &\sum_{j=1}^{r} T_{j2} = U_2 = U_1^* \end{aligned}$$

where the set $\sum_{j=1}^{r} T_{j2} = U_2$ represents the points in the space $R_k(S_1 \ldots S_k)$ which cause failure of the element and the complementary set U_1, the points where no failure is produced. Here $S_1 \ldots S_k$ represent the external forces.

These parts are also represented in fig. 1, which shows an example of the different possibilities of failure by a rectangular reinforced concrete element.



Usually on the basis of empirical studies and statics we can write

$$(9)\quad \begin{cases} g_1(x_1 \ldots x_n, S_1 \ldots S_k) \leqq 0 \\ \ldots \\ g_r(x_1 \ldots x_n, S_1 \ldots S_k) \leqq 0 \end{cases}$$

where every inequality gives one type of a condition of failure. Here $x_1 \ldots x_n$ represent the internal properties of the element and $S_1 \ldots S_k$ the external forces. Anyhow, every inequality requires a group of supplementary conditions which separate the different types of failure from each other.

In this way from (8) and (9) we get as the complete condition of failure

$$(10)\quad \begin{cases} (g_1 \leqq 0 \wedge g_{11} \leqq 0 \wedge \ldots \wedge g_{1m_1} \leqq 0) \\ \vee(g_2 \leqq 0 \wedge g_{21} \leqq 0 \wedge \ldots \wedge g_{2m_2} \leqq 0) \\ \ldots \\ \vee(g_r \leqq 0 \wedge g_{r1} \leqq 0 \wedge \ldots \wedge g_{rm_r} \leqq 0) \end{cases}$$

We have already been able to define the fr.f. of the factors $x_1 \ldots x_n$. These can usually be considered as independent, and so we can write:

$$(11) \quad f(x_1 \ldots x_n) = f_1(x_1) \ldots f_n(x_n)$$

Using the quantities $S_1 \ldots S_k$ as parameters for every combination of $S_1 \ldots S_k$ we get the probability of failure through the integration:

$$(12) \quad h(S_1 \ldots S_k) = P(42) = \int_{(10)} f(x_1 \ldots x_n) \cdot dx_1 \ldots dx_n$$

where P(10) indicates the probability that (10) is valid and the region of the integration signifies the part of the space R_k where the inequalities (10) are valid.

Without further consideration of the question of the integration above, it may be noted that there are simplifying methods to solve the integral (12) so that it is not necessary to operate in n dimensions.

In this way we have been able to determine the function (12) to represent the probability of failure of the known structural element as a function of the k-dimensional combination of forces. The next problem is to define the fr.f. of the external forces which load this element.

4. Transformation of the loads into forces and moments.

By the determination of the probability of failure there is a fundamental difference between the invariable and variable loads, since the variable loads are considered as inconstant with time, and the invariable loads are considered to retain their size during the life time of the construction. The difference in the calculation is that the forces and moments caused by the invariable loads are of direct importance, while the variable loads and the forces caused by them are not of interest in themselves, but only the corresponding extreme values appearing during the lifetime of the construction.

By both types of loads we have to change the fr.f. of the loads into fr.f. of the forces. This will be done in both cases in a similar way, which will be presented below.

In most cases the mutual dependence of the loads and the forces can be given in the following form:

$$(13) \quad \begin{cases} a_{11} \cdot q_1 + \ldots + a_{1m} \cdot q_m = S_1 \\ \ldots \\ a_{k1} \cdot q_1 + \ldots + a_{km} \cdot q_m = S_k \end{cases} \quad \text{or } A \cdot q = S$$

The parameters $a_{11}\dots a_{km}$ can usually be considered as constants. If this is not the case, the solution will have a complementary complication, which will be explained later. In principle we have three different cases; $m < k$, $m = k$, $m > k$. We assume here that the rank of matrix A is m, or in the last case k.

Without the deduction of the following formulas, we have as the fr.f. of $S_1\dots S_k$ in the three different cases:

$m = k$

$$(14)\quad f_s(S_1\dots S_k) = \left[f_{q_1}(q_1 = c_{11}\cdot S_1+\dots+c_{1k}\cdot S_k)\dots f_{q_k}(q_k = c_{k1}\cdot S_1+\dots+c_{kk}\cdot S_k)\right]\cdot \frac{1}{\begin{vmatrix} a_{11}\dots a_{1k} \\ \dots \\ a_{k1}\dots a_{kk}\end{vmatrix}}$$

Here c is a reciprocal matrix of A.

$k < m$

$$(15)\quad f_s(S_1\dots S_k) = \int_{R_{m-k}} f_{q_1}(q_1=c_{11}\cdot S_1+\dots+c_{1m}\cdot q_m)\dots f_{q_k}(q_k = c_{k1}\cdot S_1+\dots+c_{km}\cdot q_m)\cdot f_{q_{k+1}}(q_{k+1})\dots f_{q_m}(q_m)\cdot \frac{1}{\begin{vmatrix} a_{11}\dots a_{1k} \\ \dots \\ a_{k1}\dots a_{kk}\end{vmatrix}}\, dq_{k+1}\dots dq_m$$

$k > m$

$$(16)\quad f_s(S_1\dots S_k) = \left[f_{q_1}(q_1=c_{11}\cdot S_1+\dots+c_{1k}\cdot S_k)\cdot\dots\cdot f_{q_k}(q_k= c_{k1}\cdot S_1+\dots+c_{kk}\cdot S_k)\right]\cdot \frac{1}{\begin{vmatrix} a_{11}\dots a_{1k} \\ \dots \\ a_{k1}\dots a_{kk}\end{vmatrix}}$$

The difference between (14) and (16) is that the fr.f. given in (16) is limited in the degenerate part of the space R_m, where $S_{k+1}\dots S_m$ have the values:

$$(17)\quad \begin{cases} S_{m+1} = c_{m+1,1}\cdot S_1+\dots+c_{m+1,m}\cdot S_m \\ \dots \\ S_k = c_{k1}\cdot S_1+\dots+c_{km}\cdot S_m \end{cases}$$

5. Definition of the probability of failure by a structural element.

We have now in R_k two different fr.f. for external forces which have been found in the way explained in 4. We also have the fr.f. of those internal quantities of the element, which are independent:

$$(18)\quad f_y(y_1\ldots y_m)\cdot dy_1\ldots dy_m = fy_1(y_1)\ldots fy_m(y_m)\cdot dy_1\ldots dy_m$$

The exact solution of the probability of failure, which is our goal, can be obtained by integrating all the possibilities by which the sum of the forces produced by the variable and invariable loads at some time during the life-time of the construction exceeds the capacity of the structural element.

This probability can be found by the following formula:

$$(19)\quad P(y<S) = \int_{R_m} f_y(y) \int_{R_k} f_{S_g}(S_g)\cdot \left\{1-\left[\int_{T_k} f_{S_p}(S_p)\ dS_{p1}\ldots dS_{pk}\right]^N\right\}$$
$$\cdot dS_{g1}\ldots dS_{gk}\cdot dy_1\ldots dy_m$$

In this formula the set T_k gives the k-dimensional set defined in the following way:

T_k is the set of combinations which form the complementary set to (10), actually the set U_1 in (8). The difference is, however, that $x_1\ldots x_n$ have been changed into $y_1\ldots y_m$ by gradual integration, and the values $S_1\ldots S_k$ in (9) are represented by $S_{g_1} + S_{p_1},\ldots S_{g_k}+S_{p_k}$.

The value N gives the relation between the life-time of the construction and the interval which has been used to define the d.f. of the variable loads in an arbitrary moment.

We assume that T_k is a set of points which fulfil the following reguirement:

$$(20)\quad g(g_1(S_{g_1}+ S_{p_1},\ldots,S_{g_k}+ S_{p_k}),\ g_2(y_1\ldots y_m)) > 0$$

Writing

$$(21)\quad f_{S_{pe}}(S_{p_1}\ldots S_{p_k}) = N\left[\int_{(20)} f_{S_p}(S_{p_1}\ldots S_{p_k})\cdot dS_{p_1}\ldots dS_{p_k}\right]^{N-1}$$
$$\cdot f_{S_p}(S_{p_1}\ldots S_{p_k})$$

Through a rather complicated deviation, we get the probability of failure (19) in the following relatively simple form:

$$(22)\quad P(y<S) = \int_{R_k} f_{S_g+S_{pe}}(S_g + S_{pe})\cdot h(S_g + S_{pe})\cdot d(S_g + S_{pe})$$

where $-f_{S_g+S_{pe}}(S_g + S_{pe})$ is the k-dimensional fr.f. of the sum of forces caused by invariable loads and the extremevalue of variable loads.

$-h(S_g + S_{pe})$ is the function from (12).

6. Definition of the probability of failure by a structure.

To define the probability of failure by a structure is a much more complicated question than the reliability of a single element of this structure. Work on this branch has already begun, and some of the main aspects, which seem to be important, are as follows:

- whether the material of the structure is brittle or tough
- the number of different possibilities of structure failure
- the number of critical sections by different types of failure
- the interdependence of the capacity of these sections.

7. Determination of the method of design the structural element.

In 5. we have been able to find a method of determining the probability of failure of a structural element. However, this does not give us the necessary information, as to what methods we should use to determine the right dimensions of this element. Because we strive for a certain, suitable probability of failure $P_1(S_q > S_y)$, we write (22) in the form

$$(23) \quad P_1(S_q > S_y) = \int_{R_k} f_{S_q}({}^{S}q/\alpha^k) \cdot h(S_q) \cdot {}^1/\alpha^k \cdot dS_q$$

and solve the value which corresponds to the probability $P_1(S_q > S_y)$ which has been chosen in the beginning of the calculation. For this value we can usually use $10^{-6} - 10^{-8}$.

The value α gives us the possibility to see, what nominal values $x_1 \ldots x_n$, $q_1 \ldots q_m$, $p_1 \ldots p_m$ we have to use in the calculation to find structures, which have the probability of failure $P_1(S_q > S_y)$. After this we maybe have the possibility of finding such methods of calculation, which are simple enough to use for an engineer who does not know the statistical basis of these methods, and at the same time achieve the same probability of failure in various parts of the structure. This should also be our goal.

Symbols:

x - quantities, which have influence on the probability of failure.

q - loads

S - forces and moments loading the structural element

S_g- forces and moments loading the structural element, caused by invariable loads.

S_p- forces and moments loading the structural element, caused by invariable loads.

S_{pe}-forces and moments loading the structural element, caused by extreme values of variable loads.

S_q- forces and moments loading the structural element, caused by total load.

S_y- forces and moments representing the capacity of the structural element.

α - a scale coefficient

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SUMMARY

A method to determine the probability of failure by different structural elements is presented, based on the use of a computer. It treats a general case where the element is loaded with a multidimensional combination of forces and moments. The paper has four mainthemes: Estimation of the parameters of the various distributions; Capacity of a structural element; transformation of the loads into forces and moments; and definition of the probability of failure.

RESUME

On présente une méthode pour déterminer la probabilité de rupture causée par différents éléments de structure et basée sur l'emploi d'un ordinateur. La méthode traite le cas général de l'élément chargé par une combinaison multidimensionelle de forces et de moments. Cet article a quatre thèmes principaux: l'estimation des paramètres de différentes distributions, la résistance d'un élément de structure, la transformation des charges en forces et en moments et la définition de la probabilité de rupture.

ZUSAMMENFASSUNG

Man hat eine Methode für die Bestimmung der Versagenswahrscheinlichkeit bei verschiedenen Konstruktionselementen dargelegt. Die Theorie fusst auf der Anwendung elektronischer Rechenmaschinen. Ein allgemeiner Fall, wo das Element mit einer multidimensionalen Kombination von Kräften und Momenten belastet ist, wird behandelt. Der Artikel ist in vier Hauptthemen aufgeteilt: Schätzung der Parameter der verschiedenen Verteilungen, die Tragfähigkeit des Konstruktionselementes, die Transformation der Lasten in Kräfte und Momente und die Bestimmung der Versagenswahrscheinlichkeit.