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A STUDY OF THREE-DIMENSIONAL PILE-GROUPS

UNTERSUCHUNG RÄUMLICHER PFAHLGRUPPEN

ETUDE DE GROUPES DE PIEUX DANS L'ESPACE

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In his famous thesis [1]) CHR. NØKKENTVED has treated the calculation according to the elastic theory of three-dimensional pile-groups. His method is based upon the determination of three principal screw-axes of the pile group and the reduction of the acting forces to these axes. The present paper proposes a more direct method that should be easier to grasp in general and should lead to more mechanical computations without being more lengthy. A simple iteration method is also proposed that sometimes may prove to be expedient.

In the following exposition of the subject elementary vector and tensor (or matrix) algebra is applied, which greatly abbreviates and simplifies theoretical and practical treatment. To explain notation and other fundamentals a short review of vector and matrix algebra will first be given. For full proofs or further particulars the reader is referred to standard text-books [2]) [3]).

Elements of matrix algebra

A set of numbers $x_1, x_2, \ldots x_n$ is called a vector $\boldsymbol{x}$. In three-dimensional space the vector may be represented by a directed distance, whose projections upon three orthogonal coordinate axes are the three components x_1, x_2, x_3 of the vector. The scalar product of two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ is written $\boldsymbol{x}\boldsymbol{y}$ and defined as $x_1 y_1 + x_2 y_2 + x_3 y_3$. The square-root of the scalar product of a vector $\boldsymbol{x}$ by itself, $\sqrt{x_1^2 + x_2^2 + x_3^2}$, is the length of the vector. A vector of unit length is called a unit vector. It is easily demonstrated that $\boldsymbol{x}\boldsymbol{y}$ is equal to the length of $\boldsymbol{x}$ times the projection of $\boldsymbol{y}$ upon $\boldsymbol{x}$. The vector product of two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ is written $[\boldsymbol{x}\boldsymbol{y}]$ and is defined as a vector whose length is equal to the area of the parallelogram formed by $\boldsymbol{x}$ and $\boldsymbol{y}$, coinitially drawn, and whose direction is perpendicular to both $\boldsymbol{x}$ and $\boldsymbol{y}$ in such a way that $\boldsymbol{x}$, $\boldsymbol{y}$, and $[\boldsymbol{x}\boldsymbol{y}]$, consecutively drawn, form a right handed screw. Evidently $[\boldsymbol{y}\boldsymbol{x}] = -[\boldsymbol{x}\boldsymbol{y}]$. The components of $[\boldsymbol{x}\boldsymbol{y}]$ are easily found to be $x_1 y_2 - x_1 y_2$, $x_2 y_1 - x_1 y_3$, and $x_1 y_2 - x_2 y_1$. The parallelepiped formed by three coinitial vectors $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$ obviously has the volume $\boldsymbol{x}[\boldsymbol{y}\boldsymbol{z}] = \boldsymbol{y}[\boldsymbol{z}\boldsymbol{x}] = \boldsymbol{z}[\boldsymbol{x}\boldsymbol{y}]$. Another useful vector formula is $[\boldsymbol{x}[\boldsymbol{y}\boldsymbol{z}]] = \boldsymbol{y} \cdot \boldsymbol{x}\boldsymbol{z} - \boldsymbol{z} \cdot \boldsymbol{x}\boldsymbol{y}$. Other elements of vector algebra may be found in standard text-books [2]) [3]).

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1) CHR. NØKKENTVED, Beregning av Paelevaerker. Copenhagen 1923. The treatment of general three-dimensional pile-groups is excluded from the German edition, Berechnung von Pfahlrosten. Berlin 1928.

2) GEORG JOOS, Theoretische Physik. Leipzig 1943.

3) MARGENAU and MURPHY, Mathematics of Physics and Chemistry. New York 1943.

An expression of the form $\boldsymbol{y} = \boldsymbol{e} \cdot \boldsymbol{ax} + \boldsymbol{f} \cdot \boldsymbol{bx} + \boldsymbol{g} \cdot \boldsymbol{cx} = \boldsymbol{Tx}$ coordinates by „affine" or „linear" transformation a vector $\boldsymbol{y}$ to every vector $\boldsymbol{x}$. $\boldsymbol{T}$ is called a tensor. It is easily seen that the expression for $\boldsymbol{y}$ may be brought to the form

$$\boldsymbol{y} = \boldsymbol{Tx} = T_{11}\boldsymbol{i}_1 \cdot \boldsymbol{i}_1\boldsymbol{x} + T_{12}\boldsymbol{i}_1 \cdot \boldsymbol{i}_2\boldsymbol{x} + T_{13}\boldsymbol{i}_1 \cdot \boldsymbol{i}_3\boldsymbol{x} + T_{21}\boldsymbol{i}_2 \cdot \boldsymbol{i}_1\boldsymbol{x} + \ldots$$

where $\boldsymbol{i}_1$, $\boldsymbol{i}_2$, and $\boldsymbol{i}_3$ are unit vectors in the three coordinate axes of a right-handed orthogonal coordinate system. Written in component form this is equivalent to the linear system of equations

$$y_1 = T_{11}x_1 + T_{12}x_2 + T_{13}x_3, \quad y_2 = T_{21}x_1 + \quad \text{etc.}$$

The array

$$(T_{kl}) = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}$$

of the nine components or elements of $\boldsymbol{T}$ is called the matrix of $\boldsymbol{T}$. Where no misunderstanding can occur the notation T will be used for (T_{kl}). It is easily seen that the transposed tensor expression $\boldsymbol{y}' = \boldsymbol{T}'\boldsymbol{x} = \boldsymbol{a} \cdot \boldsymbol{ex} + \boldsymbol{b} \cdot \boldsymbol{fx} + \boldsymbol{c} \cdot \boldsymbol{gx}$ has the matrix elements $T'_{kl} = T_{lk}$. (T_{lk}) is called the transposed matrix of (T_{kl}). The trace (Spur) of a tensor $\boldsymbol{T}$ is defined as the sum of its main diagonal elements $T_{11} + T_{22} + T_{33}$.

If $\boldsymbol{z} = \boldsymbol{Sy}$ one has $\boldsymbol{z} = \boldsymbol{STx} = \boldsymbol{Rx}$. It can easily be proved that $\boldsymbol{R}$ is a tensor whose matrix elements R_{km} are formed by multiplying together in order and adding the elements of the kth row of (S_{kl}) and mth column of (T_{lm}) that is $R_{km} = \sum\limits_l S_{kl} T_{lm}$. The same rule is prescribed for the multiplication of two rectangular (not necessarily square) arrays or matrices S_{kl} and T_{kl}. S and T must only be conformable, that is, the number of columns in S must equal the number of rows in T. Tensor and matrix multiplication follow the associative but not the commutative law. When a matrix product is transposed the sequence of the matrices must be reversed: if $R = ST$ then $R' = T'S'$. This holds true for any number of factors.

The cofactor T^{kl} of an element T_{kl} of a square matrix is $(-I)^{k+l}$ times the determinant obtained from the array after striking out the row and column in which T_{kl} appears. According to the LAPLACE development the determinant $|T|$ of the matrix is equal to $\sum\limits_{k=1}^{3} T_{kl}\, T^{kl}$ $(l = 1, 2, 3)$. The matrix of the cofactors T^{lk} is the adjoint matrix to (T_{kl}). Note that the adjoint is formed by first finding the cofactors T^{kl} of the elements T_{kl} and then transposing the resulting matrix. The adjoint matrix of T, divided by $|T|$ is called the reciprocal matrix of T and is denoted T^{-1}. It is obviously only necessary that $|T| \neq 0$. From the properties of determinants it follows that $TT^{-1} = T^{-1}T = E$ where the unit matrix E has unity for elements along the main diagonal and all other elements zero. For every matrix $AE = EA = A$. Multiplication of a system of equations $\boldsymbol{y} = \boldsymbol{Tx}$ by $\boldsymbol{T}^{-1}$ solves the system: $\boldsymbol{T}^{-1}\boldsymbol{y} = \boldsymbol{T}^{-1}\boldsymbol{Tx} = \boldsymbol{Ex} = \boldsymbol{x}$.

Numerical examples

To make himself familiar with these vector and matrix rules the reader should check the following numerical results. The scalar product of the two vectors $\boldsymbol{x} = (x_1, x_2, x_3) = (1, 2, 3)$ and $\boldsymbol{y} = (13, 17, 19)$ is $\boldsymbol{xy} = 1 \cdot 13 + 2 \cdot 17 + 3 \cdot 19 = 104$. Their vector product is $[\boldsymbol{xy}] = (-13, 20, -9)$. The length of $\boldsymbol{x}$ is $\sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$.

The product of the two matrices

$$S = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 3 \\ 2 & 2 & 1 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 0 & 1 \\ 3 & 2 & 1 \end{pmatrix} \quad \text{is} \quad R = ST = \begin{pmatrix} 11 & 6 & 6 \\ 9 & 6 & 4 \\ 7 & 2 & 9 \end{pmatrix}$$

The product of T and the conformable one-column matrix $x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ (also denoted $x = \{1, 2, 3\}$ to save space) of the vector x above is $Tx = \{11, 3, 10\}$. One also finds $S(Tx) = \{41, 33, 38\}$ or by using R, $Rx = \{41, 33, 38\}$.

The cofactors of T_{kl} form the matrix (T^{kl}) below. From T and (T^{kl}) the determinant $|T$ may be calculated by $\sum\limits_{k \text{ or } l=1}^{3} T_{kl} T^{kl}$ in six different ways to -4. The adjoint matrix of T is (T^{lk}) and the reciprocal matrix is T^{-1}:

$$(T^{kl}) = \begin{pmatrix} -2 & 3 & 0 \\ 6 & -7 & -4 \\ 0 & -2 & 0 \end{pmatrix}, \quad (T^{lk}) = \begin{pmatrix} -2 & 6 & 0 \\ 3 & -7 & -2 \\ 0 & -4 & 0 \end{pmatrix}, \quad T^{-1} = \frac{1}{4}\begin{pmatrix} 2 & -6 & 0 \\ -3 & 7 & 2 \\ 0 & 4 & 0 \end{pmatrix}$$

Finally the reader may verify numerically that the product of the above matrices T and T^{-1} equals the unit matrix E.

Fundamental assumptions for pile groups

The first assumption in the calculation of pile groups is that the pile reactions are elastic and follow HOOKE's law. A pile j is driven to refusal through loose soil so that its end e_j (Fig. 1) bears upon a hard stratum. Its head h_j is encased in the footing of the structure. The length $h_j e_j$ of the pile is l_j, its cross section area A_j and its modulus of elasticity E_j. If the pile is shortened by a small length Δl_j an axial compressive force

$$p_0 = \frac{E_j A_j \Delta l_j}{l_j} = \lambda_j \Delta l_j \tag{1}$$

will be induced in the pile.

In a frictional pile the compressive force decreases from head to end. Then formula (1) cannot be applied. It will instead be necessary to determine, by tests or by reasoning, a ratio λ_j between pile force and the corresponding elastic displacement of the pile head. This ratio λ_j will be termed pile stiffness. Usually all piles in a pile-group are driven to the same prefixed penetration for the last hammer blow. In such a case the same stiffness λ may be applied to all piles of the same dimension in the group.

A second supposition for the analysis is that no transversal forces nor moments shall be transferred from the piles to the pile encasement. Every pile is supposed to act as a straight column, hinged at both ends and centrally loaded [4]). A generalization to account for said transversal forces and moments may be easily carried through along the same lines as given in NØKKENTVED's cited work [5]). As a rule not much is gained by the inclusion

[4]) This does not imply that a pile driven in an elastic medium must be designed as a free column. On the contrary, it has been shown by FORSSELL (Beräkning av pålar, Betong 1918) and by GRANHOLM (On the Elastic Stability of Piles Surrounded by a Supporting Medium, Stockholm 1929), that the buckling of a pile hardly ever needs to be considered, even if the pile is driven into quite loose soil (cf. TIMOSHENKO, Theory of Elastic Stability, New York 1936, p. 108). If a pile is partly free to buckle, for instance if a part of it is surrounded by water, its safety for buckling must be investigated.

[5]) PER GULLANDER in his work Teori för grundpålningar, Stockholm 1914, cf. also Theorie der Pfahlgründungen, Bautechnik 1928, p. 818, includes transversal forces from the piles.

of these transversal forces and moments [6]). For constructive and other reasons the existence of and satisfactory evaluation of these forces sometimes may be rather uncertain or ambiguous. Some building specifications therefore require that transversal forces and moments in the piles be neglected in the stress analysis.

An elastic reaction from the pressure of soil upon the sides of the pile head encasement, or upon a street piling surrounding it, may be accounted for by attaching to the encasement fictive piles with the same elastic properties and strength as the soil.

A third supposition is that the pile head encasement, which henceforth will be termed pier, is perfectly rigid. For the concrete footings now almost exclusively used this supposition closely agrees with actual conditions. On the other hand, timber pile grillages have sometimes been used as piers in such a manner that said condition of rigidity has not been complied with. For such designs the developments of this paper must necessarily be modified.

Movements, forces, and conditions of equilibrium

The acting forces will impart to the rigid pier a small translation, defined by a vector $\boldsymbol{t}$ (Fig. 1), and a small rotation of the amount r about an axis of rotation $\boldsymbol{r}$ through the point 0. The rotation is thus defined by the vector $\boldsymbol{r}$. The pile heads h_j in the pier are located by the vectors

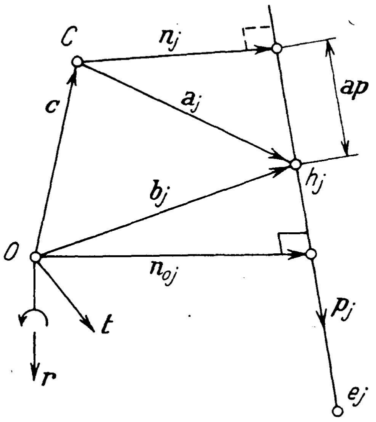

Fig. 1.
Location and movements of pile j
Lage und Bewegung von Pfahl j
Position et mouvement du pieux j.

$Oh_j = \boldsymbol{b}_j = \boldsymbol{b}$ (henceforward indices j will be omitted where no misunderstanding can occur), or by the vectors $Ch = \boldsymbol{a}$ from the point C located by the vector $OC = \boldsymbol{c}$. The total movement of the head h is given by the vector $\boldsymbol{t} + [\boldsymbol{r}\,\boldsymbol{b}]$.

Each pile is localized by the head vector $\boldsymbol{b}_j$ and the unit vector $\boldsymbol{p}_j$ along the pile axis. The projection of the movement of the pile head upon the pile axis thus is the scalar product

$$\boldsymbol{p}(\boldsymbol{t} + [\boldsymbol{r}\,\boldsymbol{b}]) = \boldsymbol{p}\,\boldsymbol{t} + [\boldsymbol{b}\,\boldsymbol{p}]\,\boldsymbol{r} = \boldsymbol{p}\,\boldsymbol{t} + [\boldsymbol{n}_0\,\boldsymbol{p}]\,\boldsymbol{r}$$

where $\boldsymbol{n}_0$ is the shortest (normal) vector from 0 to the pile axis. The last formula shows that the projection of the movement does not depend upon which point on the pile axis the vector $\boldsymbol{b}$ locates: $\boldsymbol{b}$ need not necessarily

[6]) Cf. A. Agatz, Der Kampf des Ingenieurs gegen Erde und Wasser im Grundbau, Berlin 1936, p. 193.

run to the pile head. The projection $\boldsymbol{pt} + [\boldsymbol{bp}]\boldsymbol{r}$ obviously is equal to the shortening Δl_j of the pile. This causes a compressive force (1)

$$p_0 = \lambda(\boldsymbol{pt} + [\boldsymbol{bp}]\boldsymbol{r}) \tag{2}$$

in the pile. The force by which the pile acts upon the pier may then be represented by the vector

$$-p_0\,\boldsymbol{p} = -\lambda \boldsymbol{p}(\boldsymbol{pt} + [\boldsymbol{bp}]\boldsymbol{r}) \tag{3}$$

Aside from the pile reactions, all other forces that act upon the pier may be reduced to a force $\boldsymbol{f}$ and a moment $\boldsymbol{m}$ about the point 0. The equilibrium of the pier demands that

$$\left.\begin{aligned} \boldsymbol{f} + \sum_j -p_{0j}\boldsymbol{p}_j &= 0 \\ \boldsymbol{m} + \sum_j [\boldsymbol{b}, -p_{0j}\boldsymbol{p}_j] &= 0 \end{aligned}\right\} \tag{4}$$

The sums should be extended over all the piles in the pile-group. The vector $[\boldsymbol{b}, -p_{0j}\boldsymbol{p}_j]$ is the moment about O of the force in the jth pile. Entering (3), (4) will become

$$\left.\begin{aligned} \boldsymbol{f} &= \Sigma\lambda\boldsymbol{p}\cdot\boldsymbol{pt} + \Sigma\lambda\boldsymbol{p}\cdot[\boldsymbol{bp}]\boldsymbol{r} = \boldsymbol{Tt} + \boldsymbol{Ur} \\ \boldsymbol{m} &= \Sigma\lambda[\boldsymbol{bp}]\cdot\boldsymbol{pt} + \Sigma\lambda[\boldsymbol{bp}]\cdot[\boldsymbol{bp}]\boldsymbol{r} = \boldsymbol{U}'\boldsymbol{t} + \boldsymbol{Vr} \end{aligned}\right\} \tag{5}$$

Here $\Sigma\lambda\boldsymbol{p}\cdot\boldsymbol{p} = \boldsymbol{T}$, $\Sigma\lambda\boldsymbol{p}\cdot[\boldsymbol{bp}] = \boldsymbol{U}$, $\Sigma\lambda[\boldsymbol{bp}]\cdot\boldsymbol{p} = \boldsymbol{U}'$, and $\Sigma\lambda[\boldsymbol{bp}]\cdot[\boldsymbol{bp}] = \boldsymbol{V}$ may be recognized as tensors. The tensor $\boldsymbol{U}$, for instance, may be written $\boldsymbol{U} = \Sigma\lambda(p_1\boldsymbol{i}_1 + p_2\boldsymbol{i}_2 + p_3\boldsymbol{i}_3)\cdot([bp]_1\boldsymbol{i}_1 + [bp]_2\boldsymbol{i}_2 + [bp]_3\boldsymbol{i}_3) = \Sigma\lambda p_1[bp]_1\boldsymbol{i}_1\cdot\boldsymbol{i}_1 + \Sigma\lambda p_1[bp]_2\boldsymbol{i}_1\cdot\boldsymbol{i}_2 + \Sigma\lambda p_1[bp]_3\boldsymbol{i}_1\cdot\boldsymbol{i}_3 + \Sigma\lambda p_2[bp]_1\boldsymbol{i}_2\cdot\boldsymbol{i}_1 + \ldots$ where $U_{kl} = \Sigma\lambda p_k[bp]_l$ are the matrix elements of the tensor with regard to the nine units $\boldsymbol{i}_k\cdot\boldsymbol{i}_l$.

Arraying in a rectangular matrix the components of $\sqrt{\lambda}\,\boldsymbol{p}$ or $\sqrt{\lambda}$ times the direction cosines of the piles

$$A = \begin{bmatrix} \sqrt{\lambda_a}\,p_{a1} & \sqrt{\lambda_b}\,p_{b1} & \sqrt{\lambda_c}\,p_{c1}\ldots \\ \sqrt{\lambda_a}\,p_{a2} & \sqrt{\lambda_b}\,p_{b2} & \sqrt{\lambda_c}\,p_{c2}\ldots \\ \sqrt{\lambda_a}\,p_{a3} & \sqrt{\lambda_b}\,p_{b3} & \sqrt{\lambda_c}\,p_{c3}\ldots \end{bmatrix} = A_{a,b,c,\ldots} \tag{6}$$

and the components of the vectors $\sqrt{\lambda}\,[\boldsymbol{bp}]$ in another rectangular matrix

$$B = \begin{bmatrix} \sqrt{\lambda_a}\,[bp]_{a1} & \sqrt{\lambda_b}\,[bp]_{b1} & \sqrt{\lambda_c}\,[bp]_{c1}\ldots \\ \sqrt{\lambda_a}\,[bp]_{a2} & \sqrt{\lambda_b}\,[bp]_{b2} & \sqrt{\lambda_c}\,[bp]_{c2}\ldots \\ \sqrt{\lambda_a}\,[bp]_{a3} & \sqrt{\lambda_b}\,[bp]_{b3} & \sqrt{\lambda_c}\,[bp]_{c3}\ldots \end{bmatrix} = B_{a,b,c,\ldots} \tag{7}$$

one observes that the following matrix products form sums of proportional matrices of the third order

$$AA' = \sum_j \lambda_j \begin{bmatrix} p_{j1}^2 & p_{j1}p_{j2} & p_{j1}p_{j3} \\ p_{j2}p_{j1} & p_{j2}^2 & p_{j2}p_{j3} \\ p_{j3}p_{j1} & p_{j3}p_{j2} & p_{j3}^2 \end{bmatrix} = T_a + T_b + T_c + \ldots = T \tag{8}$$

$$AB' = \sum_j \lambda_j \begin{bmatrix} p_{j1}[bp]_{j1} & p_{j1}[bp]_{j2} & p_{j1}[bp]_{j3} \\ p_{j2}[bp]_{j1} & p_{j2}[bp]_{j2} & p_{j2}[bp]_{j3} \\ p_{j3}[bp]_{j1} & p_{j3}[bp]_{j2} & p_{j3}[bp]_{j3} \end{bmatrix} = U_a + U_b + U_c + \ldots = U \tag{9}$$

$$B\,A' = U_a' + U_b' + U_c' + \ldots = U' \tag{10}$$

$$B\,B' = \sum_j \lambda_j \begin{bmatrix} [bp]_{j1}[bp]_{j1} & [bp]_{j1}[bp]_{j2} & [bp]_{j1}[bp]_{j3} \\ [bp]_{j2}[bp]_{j1} & [bp]_{j2}[bp]_{j2} & [bp]_{j2}[bp]_{j3} \\ [bp]_{j3}[bp]_{j1} & [bp]_{j3}[bp]_{j2} & [bp]_{j3}[bp]_{j3} \end{bmatrix} = V_a + V_b + V_c + \ldots = V \tag{11}$$

One also observes that each term of these sums represents the contribution of the individual piles to the matrices T, U, U' and V, respectively.

The matrix U' is the transposed matrix U. Their common trace is zero, for

$$U_{11} + U_{22} + U_{33} = \Sigma\lambda(p_1[bp]_1 + p_2[bp]_2 + p_3[bp]_3) = \Sigma\lambda\,\boldsymbol{p}[\boldsymbol{bp}] = \Sigma\lambda\,\boldsymbol{b}[\boldsymbol{pp}] = 0.$$

The unit vectors $\boldsymbol{p}$ are perpendicular to their respective vectors $\boldsymbol{n}$, or $\boldsymbol{np} = 0$. Hence

$$[\boldsymbol{p}[\boldsymbol{np}]] = \boldsymbol{n}\cdot\boldsymbol{pp} - \boldsymbol{p}\cdot\boldsymbol{np} = \boldsymbol{n}, \qquad [\boldsymbol{n}[\boldsymbol{pn}]] = \boldsymbol{p}\cdot\boldsymbol{nn} - \boldsymbol{n}\cdot\boldsymbol{pn} = \boldsymbol{p}\cdot n^2,$$

$$[\boldsymbol{rn}] = [\boldsymbol{r}[\boldsymbol{p}[\boldsymbol{np}]]] = \boldsymbol{p}\cdot\boldsymbol{r}[\boldsymbol{np}] - [\boldsymbol{np}]\cdot\boldsymbol{rp}$$

and, consequently, as $[\boldsymbol{np}] = [\boldsymbol{bp}]$,

$$\Sigma\lambda\boldsymbol{p}\cdot[\boldsymbol{bp}]\boldsymbol{r} = \Sigma\lambda[\boldsymbol{bp}]\cdot\boldsymbol{pr} + [\boldsymbol{r}\Sigma\lambda\boldsymbol{n}_0] \tag{12}$$

$$\boldsymbol{Ur} = \boldsymbol{U'r} + [\boldsymbol{r}\Sigma\lambda\boldsymbol{n}_0] = \boldsymbol{U'r} + \boldsymbol{Dr} \tag{13}$$

where $\boldsymbol{D}$ is an antimetric (skew-symmetric) tensor with the matrix

$$(D_{kl}) = \begin{bmatrix} 0 & \Sigma\lambda n_{03} & -\Sigma n_{02} \\ -\Sigma\lambda n_{03} & 0 & \Sigma\lambda n_{01} \\ \Sigma\lambda n_{02} & -\Sigma\lambda n_{01} & 0 \end{bmatrix} \tag{14}$$

as may easily be verified by expanding the expressions $[\boldsymbol{r}\Sigma\lambda\boldsymbol{n}_0]$ and $\boldsymbol{Dr}$ in component form.

$\boldsymbol{T}$ and $\boldsymbol{V}$ obviously are symmetric: $T_{kl} = T_{lk}$, $V_{kl} = V_{lk}$. Their traces are $\Sigma\lambda(p_1^2 + p_2^2 + p_3^2) = \Sigma\lambda$ and $\Sigma\lambda([bp]_1[bp]_1 + [bp]_2[bp]_2 + [bp]_3[bp]_3) = \Sigma\lambda[\boldsymbol{bp}][\boldsymbol{bp}] = \Sigma\lambda[\boldsymbol{n}_0\boldsymbol{p}][\boldsymbol{n}_0\boldsymbol{p}] = \Sigma\lambda\boldsymbol{n}_0[\boldsymbol{p}[\boldsymbol{n}_0\boldsymbol{p}]] = \Sigma\lambda n_0^2$.

Solution of the equilibrium equations

The equations (5)

$$\left.\begin{aligned} \boldsymbol{f} &= \boldsymbol{Tt} + \boldsymbol{Ur} \\ \boldsymbol{m} &= \boldsymbol{U't} + \boldsymbol{Vr} \end{aligned}\right\} \tag{5}$$

are solved by multiplying the first by $\boldsymbol{U'T}^{-1}$ and the second by $\boldsymbol{UV}^{-1}$

$$\boldsymbol{U'T}^{-1}\boldsymbol{f} = \boldsymbol{U't} + \boldsymbol{U'T}^{-1}\boldsymbol{Ur}$$

$$\boldsymbol{UV}^{-1}\boldsymbol{m} = \boldsymbol{UV}^{-1}\boldsymbol{U't} + \boldsymbol{Ur}$$

subtracting

$$\boldsymbol{f} - \boldsymbol{UV}^{-1}\boldsymbol{m} = (\boldsymbol{T} - \boldsymbol{UV}^{-1}\boldsymbol{U'})\boldsymbol{t} = \boldsymbol{T}_0\boldsymbol{t}$$

$$\boldsymbol{m} - \boldsymbol{U'T}^{-1}\boldsymbol{f} = (\boldsymbol{V} - \boldsymbol{U'T}^{-1}\boldsymbol{U})\boldsymbol{r} = \boldsymbol{V}_0\boldsymbol{r}$$

and multiplying by $\boldsymbol{T}_0^{-1}$ and $\boldsymbol{V}_0^{-1}$, respectively,

$$\left.\begin{aligned} \boldsymbol{t} &= \boldsymbol{T}_0^{-1}\boldsymbol{f} - \boldsymbol{T}_0^{-1}\boldsymbol{UV}^{-1}\boldsymbol{m} \\ \boldsymbol{r} &= -\boldsymbol{V}_0^{-1}\boldsymbol{U'T}^{-1}\boldsymbol{f} + \boldsymbol{V}_0^{-1}\boldsymbol{m} \end{aligned}\right\} \tag{15}$$

The transposed matrix to $UV^{-1}U'$ is $U''V^{-1\prime}U' = UV^{-1}U'$ owing to the symmetry of V^{-1}. Hence $UV^{-1}U'$ is symmetric. By the same argument $U'T^{-1}U$ is symmetric. Consequently T_0, V_0, $T_0{}^{-1}$, and $V_0{}^{-1}$ are symmetric. The transposed matrix of $T_0{}^{-1}UV^{-1}$ is $V^{-1}U'T_0{}^{-1} = V_0{}^{-1}U'T^{-1}$, because

$$V_0 V^{-1} U' = U' T^{-1} T_0, \quad (V - U' T^{-1} U) V^{-1} U' = U' T^{-1} (T - UV^{-1} U'),$$
$$- U' T^{-1} U V^{-1} U' = - U' T^{-1} U V^{-1} U',$$

but $T_0{}^{-1}UV^{-1}$ and $V_0U'T^{-1}$ are as a rule unsymmetric, even if $U = U'$.

The course of the practical computations for a given pile-group is indicated by the formulas. Pile symbols, stiffnesses λ, location $\boldsymbol{b}$ and slope $\boldsymbol{p}$ in any suitable coordinate system is entered in a table, where also the vectors $[\boldsymbol{b}\boldsymbol{p}]$ are computed for each pile. From this table the matrices T, U, and V are determined and from these, by matrix arithmetic, V^{-1}, UV^{-1}, $UV^{-1}U'$, $T_0 = T - UV^{-1}U'$, T_0^{-1}, $T_0^{-1}UV^{-1}$ and T^{-1}, $U'T^{-1}$, $U'T^{-1}U$, $V_0 = V - U'T^{-1}U$, V_0^{-1}, $V_0^{-1}U'T^{-1}$. This performed, the translation $\boldsymbol{t}$ and rotation $\boldsymbol{r}$ of the pier may be directly computed from (15) for any set of external forces, $\boldsymbol{f}$, $\boldsymbol{m}$, and substituted in (2) which yields the pile forces.

The elements of the primary matrices T, U, and V, the adjoint matrices and the matrix products are most conveniently evaluated by multiplications and cumulative additions or subtractions on an ordinary calculating machine.

Example 1

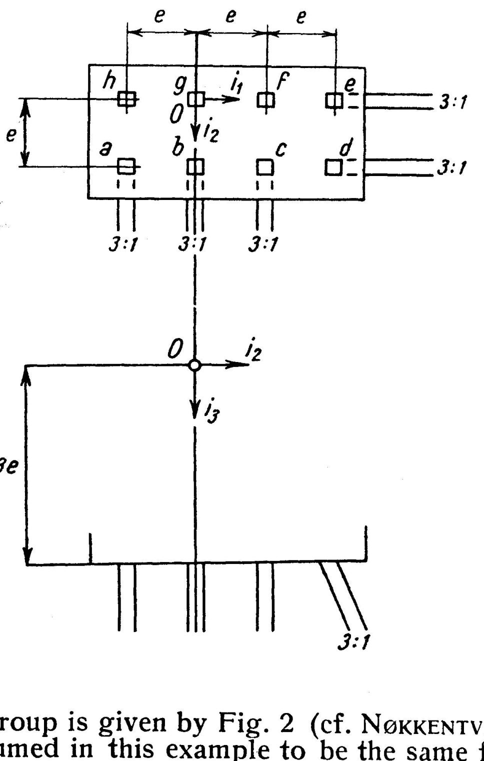

Fig. 2.
Pile foundation
Pfahlfundation
Fondation sur pieux (pilotis).

A pile group is given by Fig. 2 (cf. NØKKENTVED, loc. cit., p. 143). The pile stiffness λ will be assumed in this example to be the same for all piles. NØKKENTVED's assumption $\nu = 1$ is slightly different, since his ν is also a function of the pile batter.

Locate the origin O at the height $3e$ above the bottom of the pier at pile g. The vectors $\boldsymbol{b}$ are drawn in the plane $b_3 = 0$. Table 1 lists the piles and calculates the vectors $[\boldsymbol{b}\boldsymbol{p}]$.

Table 1.

Pile	λ_i	b			p			$[bp] = [n_0 p]$			$[n_0 p]^2 = n_0^2$
		b_1	b_2	b_3	p_1	p_2	p_3	$[bp]_1$	$[bp]_2$	$[bp]_3$	
a	1	-1				0,316	0,949		9,949	-0,316	1
b	1					0,316	0,949				
c	1	1				0,316	0,949		-0,949	+0,316	1
d	1	1	1		0,316		0,949	0,949	-0,949	-0,316	1,9
e	1	1			0,316		0,949		-0,949		0,9
f	1	1					1		-1		1
g	1						1				
h	1	-1					1		1		1
Mult.	$\cdot\lambda$	$\cdot e$	$\cdot e$	$\cdot e$				$\cdot e$	$\cdot e$	$\cdot e$	$\cdot e^2$

The primary matrices are calculated from Table 1:

$$T = (\Sigma \lambda\, p_k\, p_l) = \lambda \begin{pmatrix} 0,2 & 0 & 0,6 \\ 0 & 0,3 & 0,9 \\ 0,6 & 0,9 & 7,5 \end{pmatrix}, \qquad T^{-1}\,|T| = \lambda^2 \begin{pmatrix} 1,44 & 0,54 & -0,18 \\ 0,54 & 1,14 & -0,18 \\ -0,18 & -0,18 & 0,06 \end{pmatrix}$$

$|T| = 0,18\,\lambda^3$ (figured three ways)

$$U = (\Sigma \lambda\, p_k [bp]_l) = \lambda e \begin{pmatrix} 0,3 & -0,6 & -0,1 \\ 0 & 0 & 0 \\ 0,9 & -1,8 & -0,3 \end{pmatrix}, \qquad D = U - U' = \lambda e \begin{pmatrix} 0 & -0,6 & -1,0 \\ 0.6 & 0 & 1,8 \\ 1,0 & -1,8 & 0 \end{pmatrix}$$

$$V = (\Sigma \lambda\, [bp]_k [bp]_l) = \lambda e^2 \begin{pmatrix} 0,9 & -0,9 & -0,3 \\ -0,9 & 5,6 & -0,3 \\ -0,3 & -0,3 & 0,3 \end{pmatrix}, \qquad V^{-1}\,|V| = \lambda^2 e^4 \begin{pmatrix} 1,59 & 0\,36 & 1,95 \\ 0,36 & 0,18 & 0,54 \\ 1,95 & 0,54 & 4,23 \end{pmatrix}$$

$|V| = 0,522\,\lambda^3 e^6$ (figured three ways)

$$V^{-1} = \frac{1}{\lambda e^2} \begin{pmatrix} 3.046 & 0,690 & 3,736 \\ 0,690 & 0,345 & 1,035 \\ 3,736 & 1,035 & 8,103 \end{pmatrix}, \qquad U V^{-1} = \frac{1}{e} \begin{pmatrix} 0,1262 & -0,1045 & -0,3105 \\ 0 & 0 & 0 \\ 0,3786 & -0,3135 & -0,9315 \end{pmatrix}$$

$$U V^{-1} U' = \lambda \begin{pmatrix} 0,1316 & 0 & 0,3948 \\ 0 & 0 & 0 \\ 0,3948 & 0 & 1,1845 \end{pmatrix}, \qquad T_0 = T - U V^{-1} U' = \lambda \begin{pmatrix} 0,0684 & 0 & 0,2052 \\ 0 & 0,3000 & 0,9000 \\ 0,2052 & 0,9000 & 6,3155 \end{pmatrix}$$

$$T_0^{-1}\,|T_0| = \lambda^2 \begin{pmatrix} 1,0847 & 0,1847 & -0,0616 \\ 0,1847 & 0,3899 & -0,0616 \\ -0,0616 & -0,0616 & 0,0205 \end{pmatrix}, \qquad |T_0| = 0,0615\,\lambda^3 \quad \text{(figured three ways)}$$

$$T_0^{-1} = \frac{1}{\lambda} \begin{pmatrix} 17,64 & 3,00 & -1,00 \\ 3,00 & 6,34 & -1.00 \\ -1,00 & -1,00 & 0,333 \end{pmatrix}, \qquad T_0^{-1} U V^{-1} = \frac{1}{\lambda e} \begin{pmatrix} 1,85 & -1,53 & -4,55 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$T^{-1} = \frac{1}{\lambda} \begin{pmatrix} 8 & 3 & -1 \\ 3 & 6\tfrac{1}{3} & -1 \\ -1 & -1 & \tfrac{1}{3} \end{pmatrix}, \qquad U' T^{-1} = e \begin{pmatrix} 1,5 & 0 & 0 \\ -3,0 & 0 & 0 \\ -0,5 & 0 & 0 \end{pmatrix}$$

$$U' T^{-1} U = \lambda e^2 \begin{pmatrix} 0,45 & -0,90 & -0,15 \\ -0.90 & 1\,80 & 0,30 \\ -0,15 & 0,30 & 0,05 \end{pmatrix}, \qquad V_0 = V - U' T^{-1} U = \lambda e^2 \begin{pmatrix} 0,45 & 0 & -0,15 \\ 0 & 3,80 & -0,60 \\ -0,15 & -0,60 & 0,25 \end{pmatrix}$$

$$V_0^{-1}\,|V_0| = \lambda^2 e^4 \begin{pmatrix} 0,59 & 0,09 & 0,57 \\ 0,09 & 0,09 & 0.27 \\ 0,57 & 0,27 & 1,71 \end{pmatrix}, \qquad |V_0| = 0,1800\,\lambda^3 e^6 \quad \text{(figured three ways)}$$

$$V_0^{-1} = \frac{1}{\lambda e^2} \begin{pmatrix} 3.278 & 0.5 & 3,167 \\ 0,5 & 0.5 & 1,5 \\ 3.167 & 1,5 & 9,5 \end{pmatrix}, \qquad V_0^{-1} U' T^{-1} = \frac{1}{\lambda e} \begin{pmatrix} 1.833 & 0 & 0 \\ -1,5 & 0 & 0 \\ -4,5 & 0 & 0 \end{pmatrix}$$

A partial check upon the correctness of the matrices T, U, and V is the value of their traces, $\Sigma\lambda$, 0, and $\Sigma\lambda n_0^2$, respectively. Checks upon calculations are furnished by the determinants, each figured three ways, by the symmetry of $UV^{-1}U'$ and $U'T^{-1}U$ and by the fact that $T_0^{-1}UV^{-1}$ and $V_0^{-1}U'T^{-1}$ have transposed matrices. Equations (15) may now be written down directly in component form

$$\begin{aligned}
\lambda t_1 &= 17{,}64 f_1 + 3{,}00 f_2 - 1{,}00 f_3 - 1{,}83\, m_1/e + 1{,}5\, m_2/e + 4{,}5\, m_3/e \\
\lambda t_2 &= 3{,}00 f_1 + 6{,}34 f_2 - 1{,}00 f_3 \\
\lambda t_3 &= -1{,}00 f_1 - 1{,}00 f_2 + 0{,}33 f_3 \\
\lambda e r_1 &= -1{,}83 f_1 \qquad\qquad + 3{,}28\, m_1/e + 0.5\, m_2/e + 3{,}17\, m_3/e \\
\lambda e r_2 &= 1{,}5\ f_1 \qquad\qquad + 0{,}5\ m_1/e + 0{,}5\, m_2/e + 1{,}5\ m_3/e \\
\lambda e r_3 &= 4{,}5\ f_1 \qquad\qquad + 3{,}17\, m_1/e + 1{,}5\, m_2/e + 9{,}5\ m_3/e
\end{aligned}$$

From these formulas the translation and rotation may be easily figured for any external load f, $\boldsymbol{m}$. Substitution in (2) yields all the pile forces.

Thus the force in the pile a (fig. 2)

$$\boldsymbol{b} = -e\boldsymbol{i}_1, \quad \boldsymbol{p} = 0{,}316\,(\boldsymbol{i}_2 + 3\,\boldsymbol{i}_3), \quad [\boldsymbol{bp}] = 0{,}316\, e\,(3\,\boldsymbol{i}_2 - \boldsymbol{i}_3)$$

will be $\boldsymbol{p}\lambda\boldsymbol{t} + [\boldsymbol{bp}]\frac{1}{e}\lambda e\boldsymbol{r} = 0{,}316\,(3{,}00 f_1 + 6{,}34 f_2 - 1{,}00 f_3 - 3{,}00 f_1 - 3{,}00 f_2 + 1{,}00 f_3 + 4{,}5 f_1 + 1{,}5\, m_1/e + 1{,}5\, m_2/e + 4{,}5\, m_3/e - 4{,}5 f_1 - 3{,}17\, m_1/e - 1{,}5\, m_2/e - 9{,}5\, m_3/e) = 0{,}316\,(3{,}34 f_2 - 1{,}67\, m_1/e - 5\, m_3/e)$.

A similar formula may be written down for every pile in the group. Each formula clearly shows which movable external forces should be included to cause maximum force in the pile. From the formula influence lines or influence surfaces for live loads may be easily drawn.

Coordinate transformations. Pile-center

The computations of Example 1 show that the direct calculation of the translation and rotation is comparatively simple. The question arises whether changes of origin or coordinate transformations may expedite the procedure. In his treatment of pile groups Nøkkentved determines three principal screw axes and employs one principal axis transformation. Still his computations corresponding to the example just figured become rather extensive.

Even if a suitable transformation can be found that admits of simple relations between forces and moments in a certain coordinate system, there still remains to determine this reference system, to express the external forces in it, to calculate the movements, and, eventually, to refigure the resulting expressions for the movement of the pier to a suitable form for use with formula (2). The combined work may very well exceed that of the direct calculation shown.

However, equation (12) or (13) suggests a change of origin in order to make U and U' symmetric and equal, namely to remove the origin O to such a point C that $\Sigma\lambda\boldsymbol{n} = \Sigma\lambda\boldsymbol{n}_c = 0$. Also the skew-symmetric tensor $\boldsymbol{D}$ will then become null. To the point C in space where $\Sigma\lambda\boldsymbol{n} = 0$ the term pile-group center or, in short, pile-center will be applied. At the pile-center the scalar point function

$$\Sigma\lambda[\boldsymbol{np}]^2 = \Sigma\lambda\{\boldsymbol{n}^2\boldsymbol{p}^2 - (\boldsymbol{np})^2\} = \Sigma\lambda n^2$$

has a minimum. For if C (Fig. 1) is regarded as a movable point located by the point vector $\boldsymbol{c}$ from the fixed origin O, one sees that

$$\boldsymbol{n} = \boldsymbol{a} - \boldsymbol{ap}\cdot\boldsymbol{p} = \boldsymbol{b} - \boldsymbol{c} - (\boldsymbol{b}-\boldsymbol{c})\,\boldsymbol{p}\cdot\boldsymbol{p} = \boldsymbol{n}_0 - (\boldsymbol{c} - \boldsymbol{cp}\cdot\boldsymbol{p}) \qquad (16)$$

where

$$\boldsymbol{n}_0 = \boldsymbol{b} - \boldsymbol{bp}\cdot\boldsymbol{p} \qquad (17)$$

By common vector analysis formulas [2] [3]),

$$\operatorname{grad} \boldsymbol{n}^2 = 2\,[\boldsymbol{n} \operatorname{rot} \boldsymbol{n}] + 2\boldsymbol{n} \operatorname{grad} \cdot \boldsymbol{n} = -2\boldsymbol{n} \operatorname{grad} \cdot \boldsymbol{c} + 2\boldsymbol{n} \operatorname{grad} \cdot \boldsymbol{c}\boldsymbol{p} \cdot \boldsymbol{p} = -2\boldsymbol{n}$$

hence grad $\Sigma\lambda n^2 = -2\Sigma\lambda \boldsymbol{n}$. The condition for an extreme value of the function $\Sigma\lambda\,[\boldsymbol{n}\boldsymbol{p}]^2 = \Sigma\lambda \boldsymbol{n}^2$, that its gradient is zero, is satisfied at the point where $\Sigma\lambda \boldsymbol{n} = 0$. This extreme value obviously must be a minimum.

The pile-center is thus located by the equation

$$\Sigma\lambda \boldsymbol{n} = \Sigma\lambda \boldsymbol{n}_0 - (\boldsymbol{c}\Sigma\lambda - \boldsymbol{c}\Sigma\lambda \boldsymbol{p} \cdot \boldsymbol{p}) = 0 \tag{18}$$

or, in tensor form,

$$(\Sigma\lambda \boldsymbol{E} - \boldsymbol{T})\,\boldsymbol{c} = \boldsymbol{F}\boldsymbol{c} = \Sigma\lambda \boldsymbol{n}_0 \tag{19}$$

whose solution

$$\boldsymbol{c} = \boldsymbol{F}^{-1}\Sigma\lambda \boldsymbol{n}_0 \tag{20}$$

yields the position of the point C. The vector $\Sigma\lambda \boldsymbol{n}_0$ is most easily obtained from the matrix (14): $D = U - U'$.

Example 2

Find the pile-center of the pile-group of Example 1. Matrix D: evaluated in Example 1, yields $\Sigma\lambda \boldsymbol{n}_0 = \lambda \boldsymbol{e}(1{,}8\,\boldsymbol{i}_1 + 1{,}0\,\boldsymbol{i}_2 - 0{,}6\,\boldsymbol{i}_3)$. From T and $\Sigma\lambda = 8\,\lambda$ is obtained

$$\Sigma\lambda E - T = F = \lambda \begin{pmatrix} 7{,}8 & 0 & -0{,}6 \\ 0 & 7{,}7 & -0{,}9 \\ -0{,}6 & -0{,}9 & 0{,}5 \end{pmatrix}, \qquad F^{-1}\,|F| = \lambda^2 \begin{pmatrix} 3{,}04 & 0{,}54 & 4{,}62 \\ 0{,}54 & 3{,}54 & 7{,}02 \\ 4{,}62 & 7{,}02 & 60{,}06 \end{pmatrix}$$

$|F| = 20{,}9\,\lambda^3$ (figured three ways), $\boldsymbol{c} = \boldsymbol{F}^{-1}\Sigma\lambda \boldsymbol{n}_0 = \{(3{,}04 \cdot 1{,}8 + 0{,}54 \cdot 1 - 4{,}62 \cdot 0{,}6)\,\boldsymbol{i}_1 + (0{,}54 \cdot 1{,}8 + 3{,}54 \cdot 1 - 7{,}02 \cdot 0{,}6)\,\boldsymbol{i}_2 + (4{,}62 \cdot 1{,}8 + 7{,}02 \cdot 1 - 60{,}06 \cdot 0{,}6)\,\boldsymbol{i}_3\}\,\lambda^3 e/|F| = e\,(0{,}155\,\boldsymbol{i}_1 + 0{,}014\,\boldsymbol{i}_2 - 0{,}989\,\boldsymbol{i}_3)$.

Ascertainment of the pile-center in regularly spaced pile-groups.

In many cases of regular pile arrangement the pile-center can be wholly or partly ascertained by considerations of symmetry or of the minimum of $\Sigma\lambda n^2$. For instance, if the pile-group is composed of a number of sub-groups of mutually parallel piles in such a manner that the center-of-gravity lines of the individual sub-groups intersect at one point, the pile-center will be located at that point. For in each sub-group $\Sigma\lambda n^2$ will be a minimum along its center-of-gravity line, and for all the piles $\Sigma\lambda n^2$ will be a minimum at the intersection of these lines. This is the characteristic of the pile-center. If the pile-center can be established by such a conclusion, the first origin O obviously should be immediately located at the pile-center to avoid the computations for a change of origin.

Advantages gained by establishing the pile-center.

By choosing the origin at the pile-center the number of different matrices in (5) will be reduced from four to three symmetric. The numerical computations will be somewhat facilitated thereby, but not so very much, since the matrices $T_0^{-1}UV^{-1}$ and $V_0^{-1}U'T^{-1}$ of the solution (15) will generally be unsymmetric and unequal anyhow.

Further, by one principal axis transformation it is possible to diagonalize by well-known methods any one of the three matrices $T, U = U'$, or V. By a linear transformation it is even possible to diagonalize any two of these three matrices. Each transformation will require rather much numerical work, including the solution of a characteristic equation of the third degree. When that is performed, the subsequent computations will be simplified, even if the total computation is not. Particularly, a diagonalization of U may save work.

Addition of a fictive pile

However, with the object of saving numerical work, another course may be tried. It is sometimes possible to add to the original pile-group a fictive pile in such a way that the resulting pile-group will be more easily calculated. The original external forces $\boldsymbol{f}$, $\boldsymbol{m}$ induce in the fictive pile a compressive force $p_0 = q$ cf. (2), to be evaluated. To the original external forces then is added along the unit vector $\boldsymbol{p}_f$ of the fictive pile a force q, which neutralizes the force from the fictive pile upon the pier.

A force $= 1$ along the axis $\boldsymbol{p}_f$ of the fictive pile, applied as an external force to the pier, will cause in the fictive pile a force Q. A force equal to q, applied along $\boldsymbol{p}_f$ as an external force will thus induce in the fictive pile a force Qq. This force Qq is neutralized by adding another external force, Qq, along $\boldsymbol{p}_f$. The latter force causes in the fictive pile a force Q^2q, etc. In practical cases, and if the fictive pile is suitably placed, the geometric series will converge. Consequently, to counteract all the pile forces in the fictive pile the original external force system $\boldsymbol{f}$, $\boldsymbol{m}$ must be completed with the force $q + Qq + Q^2q + \ldots = q/(1-Q)$ acting along the axis $\boldsymbol{p}_f$ of the fictive pile. The forces in the other, real piles obviously should be computed from the completed external force system $\boldsymbol{f} + \boldsymbol{p}_f\, q/(1-Q)$, $\boldsymbol{m} + [\boldsymbol{b}_f\, \boldsymbol{p}_f]\, q/(1-Q)$.

Each pile, fictive or real, contributes to the matrix of $\boldsymbol{U} = \Sigma \lambda \boldsymbol{p} \cdot [\boldsymbol{b}\boldsymbol{p}]$ (and also to the matrices of $\boldsymbol{T}$ and $\boldsymbol{V}$) with a matrix with proportional rows and columns, cf. (9),

$$\Delta U = \begin{pmatrix} \lambda p_1 [bp]_1 & \lambda p_1 [bp]_2 & \lambda p_1 [bp]_3 \\ \lambda p_2 [bp]_1 & \lambda p_2 [bp]_2 & \lambda p_2 [bp]_3 \\ \lambda p_3 [bp]_1 & \lambda p_3 [bp]_2 & \lambda p_3 [bp]_3 \end{pmatrix}$$

Conversely, it is easy to find a pile that yields a given proportional matrix increment ΔU. Of the quantities determining the pile, λ or the length of $\boldsymbol{b}$ may be put equal to a prefixed value λ or $|b|$, respectively. The direction cosines p_1, p_2, and p_3 of the piles obviously must be chosen to agree with the factors of proportionality between the rows and to agree with the condition $\boldsymbol{p}\boldsymbol{p} = 1$. It only remains to determine $[bp]_1$, $[bp]_2$ and $[bp]_3$ equal to the three numbers d_1, d_2 and d_3 respectively. The vector $\boldsymbol{b}$ that runs from the origin to the pile axis may be drawn in any plane through the origin. If it is drawn in the plane $b_3 = 0$, the two other components of $\boldsymbol{b}$ are determined by $[bp]_1 = b_2 p_3 = d_1$, $[bp]_2 = -b_1 p_3 = d_2$, and the dependent equation $[bp]_3 = b_1 p_2 - b_2 p_1 = d_3$.

Example 3

The matrix U of Example 1 is, by coincidence, proportional. It is easily seen that U becomes null if a fictive pile $\lambda_f = \lambda$, $\boldsymbol{b}_f = -2e\boldsymbol{i}_1 - e\boldsymbol{i}_2$, $\boldsymbol{p}_f = 0{,}316\,(\boldsymbol{i}_1 + 3\,\boldsymbol{i}_3)$, $[\boldsymbol{b}_f\boldsymbol{p}_f] = 0{,}316\,(-3\,\boldsymbol{i}_1 + 6\,\boldsymbol{i}_2 + \boldsymbol{i}_3)$ is introduced. The primary matrices become

$$T = \lambda \begin{pmatrix} 0{,}3 & 0 & 0{,}9 \\ 0 & 0{,}3 & 0{,}9 \\ 0{,}9 & 0{,}9 & 8{,}4 \end{pmatrix}, \qquad T^{-1}\,|T| = \lambda^2 \begin{pmatrix} 1{,}71 & 0{,}81 & -0{,}27 \\ 0{,}81 & 1{,}71 & -0{,}27 \\ -0{,}27 & -0{,}27 & 0{,}09 \end{pmatrix}, \qquad |T| = 0{,}27\,\lambda^3$$

$$V = \lambda\, e^2 \begin{pmatrix} 1{,}8 & -2{,}7 & -0{,}6 \\ -2{,}7 & 9{,}2 & 0{,}3 \\ -0{,}6 & 0{,}3 & 0{,}4 \end{pmatrix}, \qquad V^{-1}|V| = \lambda^2 e^4 \begin{pmatrix} 3{,}59 & 0{,}90 & 4{,}71 \\ 0{,}90 & 0{,}36 & 1{,}08 \\ 4{,}71 & 1{,}08 & 9{,}27 \end{pmatrix}, \qquad |V| = 1{,}206\,\lambda^3 e^6$$

$|T|$ and $|V|$ have been figured in three ways. As $U = 0$ the solution of the equation system (5) is simply $\boldsymbol{t} = \boldsymbol{T}^{-1}\boldsymbol{f}$, $\boldsymbol{r} = \boldsymbol{V}^{-1}\boldsymbol{m}$, or

$$\left.\begin{aligned} \lambda \boldsymbol{t} = {} & (6\tfrac{1}{3} f_1 + 3 f_2 - f_3)\, \boldsymbol{i}_1 + \\ & + (3 f_1 + 6\tfrac{1}{3} f_2 - f_3)\, \boldsymbol{i}_2 + \\ & + (-f_1 - f_2 + \tfrac{1}{3} f_3)\, \boldsymbol{i}_3 \end{aligned}\right\} \qquad \left.\begin{aligned} \lambda e^2 \boldsymbol{r} = {} & (2{,}977\, m_1 + 0{,}746\, m_2 + 3{,}905\, m_3)\, \boldsymbol{i}_1 + \\ & + (0{,}746\, m_1 + 0{,}299\, m_2 + 0{,}896\, m_3)\, \boldsymbol{i}_2 + \\ & + (3{,}905\, m_1 + 0{,}896\, m_2 + 7{,}687\, m_3)\, \boldsymbol{i}_3 \end{aligned}\right\}$$

A force $= 1$ in the direction of the positive axis of the fictive pile

$$\boldsymbol{f} = 0{,}316\,(\boldsymbol{i}_1 + 3\,\boldsymbol{i}_3), \quad \boldsymbol{m} = [\boldsymbol{b}\boldsymbol{f}] = 0{,}316\, e\,(-3\,\boldsymbol{i}_1 + 6\,\boldsymbol{i}_2 + \boldsymbol{i}_3)$$

yields $\quad \lambda \boldsymbol{t} = 0{,}316 \cdot 3\tfrac{1}{3}\, \boldsymbol{i}_1, \quad \lambda e \boldsymbol{r} = 0{,}316\,(-\,0{,}550\, \boldsymbol{i}_1 + 0{,}452\, \boldsymbol{i}_2 + 1{,}348\, \boldsymbol{i}_3)$

and, by (2), $\quad Q = 0{,}316^2 \cdot 3\tfrac{1}{3} + 0{,}316^2\,(1{,}650 + 2{,}712 + 1{,}348) = 0{,}9043$

The external forces $\boldsymbol{f}$, $\boldsymbol{m}$ cause in the fictive pile a force (2)

$$q = 0{,}316 \cdot 3\tfrac{1}{3} f_1 + 0{,}316\,(-\,0{,}550\, m_1 + 0{,}452\, m_2 + 1{,}348\, m_3)/e$$

This force multiplied by $1/(1-Q) = 10.45$ is cancelled by adding $\Delta \boldsymbol{f} = 10{,}45\, q \boldsymbol{p}_f$ to the applied forces and $\Delta \boldsymbol{m} = 10{,}45\, q\, [\boldsymbol{b}_f \boldsymbol{p}_f]$ to the applied moments.

$$\begin{aligned} \Delta f_1 &= 3{,}48 f_1 - 0{,}575\, m_1/e + 0{,}472\, m_2/e + 1{,}409\, m_3/e \\ \Delta f_3 &= 10{,}45 f_1 - 1{,}724\, m_1/e + 1{,}417\, m_2/e + 4{,}226\, m_3/e \\ \Delta m_1 &= -10{,}45 f_1 e + 1{,}724\, m_1 - 1{,}417\, m_2 - 4{,}226\, m_3 \\ \Delta m_2 &= 20{,}9\, f_1 e - 3{,}449\, m_1 + 2{,}834\, m_2 + 8{,}452\, m_3 \\ \Delta m_3 &= 3{,}48 f_1 e - 0{,}575\, m_1 + 0{,}472\, m_2 + 1{,}409\, m_3 \end{aligned}$$

Addition to the corresponding forces and moments in the above expressions for $\lambda \boldsymbol{t}$ and $\lambda e^2 \boldsymbol{r}$ yields

$$\begin{aligned} \lambda t_1 &= 17{,}9 f_1 + 3 f_2 - f_3 - 1{,}92\, m_1/e + 1{,}57\, m_2/e + 4{,}70\, m_3/e \\ \lambda t_2 &= 3 f_1 + 6\tfrac{1}{3} f_2 - f_3 \\ \lambda t_3 &= -f_1 - f_2 + \tfrac{1}{3} f_3 \\ \lambda e r_1 &= -1{,}93 f_1 + 3{,}29\, m_1/e + 0{,}49\, m_2/e + 3{,}13\, m_3/e \\ \lambda e r_2 &= 1{,}57 f_1 + 0{,}49\, m_1/e + 0{,}51\, m_2/e + 1{,}53\, m_3/e \\ \lambda e r_3 &= 4{,}67 f_1 + 3{,}13\, m_1/e + 1{,}53\, m_2/e + 9{,}59\, m_3/e \end{aligned}$$

The discrepancies from the result of Example 1 are attributable to the use of a limited number of places.

More than one fictive pile

It may be useful or necessary in some instances to add to the pile-group a number of fictive piles. If, for example, three such piles f, g, and h are added, a force $= 1$ along the axis $\boldsymbol{p}_f$ of f will produce in the fictive pile f a force Q_{ff}, and in the fictive piles g and h forces Q_{gf} and Q_{hf}, respectively. An external force $= 1$ in the pile axis $\boldsymbol{p}_g$ will produce in the fictive piles the forces Q_{fg}, Q_{gg}, and Q_{hg}, etc. The original external forces $\boldsymbol{f}$; $\boldsymbol{m}$ will induce in the fictive piles compressive forces q_f, q_g, and q_h. These will be neutralized by introducing additional external forces q_f, q_g, and q_h acting along the positive unit vectors $\boldsymbol{p}_f$, $\boldsymbol{p}_g$, and $\boldsymbol{p}_h$ in the respective fictive pile axes. These latter forces will produce in the three fictive piles the new compressive forces

$$Q_{ff}\, q_f + Q_{fg}\, q_g + Q_{fh}\, q_h, \quad Q_{gf}\, q_f + Q_{gg}\, q_g + Q_{gh}\, q_h, \quad Q_{hf}\, q_f + Q_{hg}\, q_g + Q_{hh}\, q_h$$

These forces may be regarded as the components of the product $\boldsymbol{Q q}$ of a tensor $\boldsymbol{Q}$ with the matrix

$$Q = \begin{pmatrix} Q_{ff} & Q_{fg} & Q_{fh} \\ Q_{gf} & Q_{gg} & Q_{gh} \\ Q_{hf} & Q_{hg} & Q_{hh} \end{pmatrix} \tag{21}$$

and a vector $\boldsymbol{q}$ with the components q_f, q_g, and q_h. Both the tensor and the vector may suitably be referred to a coordinate system with the axis unit vectors $\boldsymbol{p}_f$, $\boldsymbol{p}_g$, and $\boldsymbol{p}_h$. The components of $\boldsymbol{Q q}$, that is, the last mentioned compressive forces in the piles, are again counteracted by adding to the external forces the forces $(Qq)_f\,\boldsymbol{p}_f$, $(Qq)_g\,\boldsymbol{p}_g$, $(Qq)_h\,\boldsymbol{p}_h$ acting along the respective fictive pile axes. These forces will induce in the fictive piles compressive forces that obviously are equal to the components of the vector $\boldsymbol{Q}^2\boldsymbol{q}$. These forces are again applied as external forces, and the process is continued ad infinitum, if it is convergent. The sums of the external forces applied in the axes of the fictive piles f, g, and h, thus are equal to the components of the vector

$$\boldsymbol{q} + \boldsymbol{Q q} + \boldsymbol{Q}^2 \boldsymbol{q} + \boldsymbol{Q}^3 \boldsymbol{q} + \ldots = (\boldsymbol{E} + \boldsymbol{Q} + \boldsymbol{Q}^2 + \ldots)\,\boldsymbol{q} = \boldsymbol{N q} \tag{22}$$

This NEUMANN series may be formally treated as a geometric series [7]. Frontal multiplication with the matrix Q and subtraction yields $E = N - QN = (E - Q)N$, $(E - Q)^{-1} = N$

$$\boldsymbol{N q} = (\boldsymbol{E} - \boldsymbol{Q})^{-1}\,\boldsymbol{q} \tag{23}$$

By this formula, which evidently must hold for any number of fictive piles added, it is possible to determine the forces by which the external force system must be completed to balance the forces set up in the fictive piles [8].

The components of $\sqrt{\lambda}\,\boldsymbol{p}$ or $\sqrt{\lambda}$ times the direction cosines of the fictive piles may be arrayed in a rectangular matrix $A_{f,\,g,\,h,\,\ldots}$ as in (6), and the components of the vectors $\sqrt{\lambda}\,[\boldsymbol{b p}]$ in another rectangular matrix $B_{f,\,g,\,h,\,\ldots}$, as in (7). One observes that the following matrix products form sums of proportional matrices of the third order, cf. Equations (8) to (11),

$$\begin{aligned} AA' &= T_f + T_g + T_h + \ldots = \Delta T \\ AB' &= U_f + U_g + U_h + \ldots = \Delta U \\ BA' &= U_f' + U_g' + U_h' + \ldots = \Delta U' \\ BB' &= V_f + V_g + V_h + \ldots = \Delta V \end{aligned}$$

the terms of which represent the increments by the individual fictive piles to the matrices of T, U, U' and V. By the addition of fictive piles the

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[7]) Cf. COURANT-HILBERT, Methoden der mathematischen Physik, I, 2nd. Ed., Berlin 1931, p. 8.

[8]) The method of iteration just demonstrated resembles in some aspects the methods of successive approximation for the solution of statically indeterminate problems. It must be possible by simple matrix methods to establish the coincidence of these methods with the classical methods of solution. For example, the NEUMANN series of (22) represents the complete course of successive approximations in a statical problem. The total result of the infinite number of approximations is given in finite form by (23) which should duplicate the direct classical solution. In cases of rapid convergence it may be more practical to use approximations of the type (22) instead of a cumbersome direct solution (23).

matrix T thus will change to $T + AA'$, U to $U + AB'$, U' to $U' + BA'$, and V to $V + BB'$. By a suitable choice of fictive piles, which can be made in several ways, these resulting matrices and, consequently, the ensuing computations may be considerably simplified.

The most practicable choice of fictive piles depends much upon the original pile constellation and should most conveniently be judged from case to case.

In cases when a definite choice of fictive piles is not apparent the simplification of the matrices may, for instance, be undertaken with the first aim of making the matrix U equal to null. To that end it eppears to be expedient to start out with a change of origin to the pile-center as previously explained, and exemplified in Example 2. Then fictive piles are added to make the new matrix U, $U + AB'$ equal to null. This will not move the pile-center, for a null matrix is symmetric, and a symmetric matrix U is an indication that the origin is at the pile-center, cf. (12) or (13).

The choice of fictive piles can simply be made in such a manner that the matrix A is made equal to a square unit matrix, whence B will equal the matrix $-U'$, making $U + AB' = U - EU = 0$. The individual fictive piles are fully determined from A and B according to (6) and (7) and the method explained above for one fictive pile. This decomposition of the matrix $-U$ may many times be as serviceable as any other decomposition. The continued procedure consists in determining the new values of T, $T + AA' = T + E$ and of V, $V + BB'$. The solution of the equations (5), $\boldsymbol{t} = (\boldsymbol{T} + \boldsymbol{E})^{-1}\boldsymbol{f}$, $\boldsymbol{r} = (\boldsymbol{V} + \boldsymbol{BB}')^{-1}\boldsymbol{m}$ can be effected either directly, or by means of a separate (orthogonal) principal axis transformation for each equation, or by means of a single linear transformation, which diagonalizes both equations at the same time, by well known methods. Finally the corrections for the forces in the fictive piles are added as previously demonstrated, (22) or (23).

Using more finesse the fictive piles may instead be chosen so as to simplify the changed matrices $T + AA'$ and $V + BB'$ also. Instead of making A equal to a unit matrix it may, for instance, be given such a form that $V + BB'$ becomes a multiple of $T + AA'$, at the same time as $U + AB'$ is made null. A single orthogonal principal axis transformation then diagonalizes both $T + AA'$ and $V + BB'$.

However, even such a theoretically simple procedure involves far more numerical work than the direct computation, as in Example 1. The application of fictive piles seems practicable mainly when the structure of the original pile-group clearly signals that a definite simplification, for instance simple or double symmetry, may be won by the addition of a relatively small number of fictive piles. In such a case no mathematical intricacies are needed to determine the fictive piles.

The writer by no means wishes to create the impression that all possibilities of treating pile-groups by matrix methods have been exhausted in this paper. On the contrary, developments and amplifications may be anticipated in several directions, the pursuit and exploration of which may lead to serviceable, practical methods.

Summary

Elastic theory calculations of plane pile-groups (Gullander, Hultin) are easily carried through according to Nøkkentved's methods. However, the corresponding calculations of spatial pile-groups become so tedious and unsurveyable that they must be considered as practically prohibitive. This paper aims at showing how matrix treatment of such pile-groups clearly interprets the theoretical interrelations and makes possible systematical and practical computations according to (15), cf. Example 1.

Also demonstrated is how it sometimes may be possible to add to an actual pile-group one or more *fictive* piles in such a manner that the resulting pile-group may be more expediently analyzed, e. g. by resulting symmetry. Then the forces that the original external forces f, m induce in the fictive piles may be evaluated and *added* to f, m to neutralize the reactions upon the pier from the same fictive piles. The *added* forces cause in the fictive piles *further* reactions which are again calculated and added etc. This iteration often converges, and it is theoretically and by examples demonstrated in the paper that the finite summation of the corresponding geometric matrix series (NEUMANN series) yields the exact solution of the pile-group problem. In other words, this iteration approaches as a limit the direct solution according to (15) of the actual original pile-group.

Zusammenfassung

Elastizitätstheoretische Berechnungen von ebenen Pfahlgruppen (GULLANDER, HULTIN) sind mit Hilfe der Methode von NØKKENTVED leicht durchzuführen. Dagegen gestalten sich die entsprechenden Berechnungen von räumlichen Pfahlgruppen so mühsam und unübersichtlich, daß sie praktisch undurchführbar sind. Die vorliegende Abhandlung bezweckt zu zeigen, wie die Matrizenrechnung die theoretische Behandlung solcher Pfahlgruppen klar und einfach behandelt und die systematische und praktische Berechnung nach Gl. (15) im Beispiel Nr. 1 möglich macht.

Es wird ebenfalls gezeigt, daß es Fälle gibt, wo es möglich ist, zu einer bestehenden Pfahlgruppe einen oder mehrere fiktive Pfähle so hinzuzufügen, daß die sich daraus ergebende Pfahlgruppe rascher untersucht werden kann, z. B. durch die sich ergebende Symmetrie. Dann können die Kräfte, die die ursprünglichen äußern Kräfte f, m in die fiktiven Pfähle einführen, bewertet werden und zu f, m hinzugefügt werden, um die Reaktionen, die von den gleichen fiktiven Pfählen auf die Quaimauer ausgeübt werden, unwirksam zu machen. Die zusätzlichen Kräfte verursachen in den fiktiven Pfählen weitere Reaktionen, die wiederum berechnet und hinzugefügt werden etc. Diese Iteration konvergiert oft, und es wird in der vorliegenden Arbeit sowohl theoretisch wie durch Beispiele gezeigt, daß die Endsumme der entsprechenden geometrischen Matrizen-Reihe (NEUMANN'sche Reihe) die genaue Lösung des Pfahlgruppenproblems ergibt. Mit andern Worten bildet diese Iteration eine sukzessive Approximation der direkten Lösung der Gl. (15) der gegebenen ursprünglichen Pfahlgruppe.

Résumé

Dans le domaine de la théorie de l'élasticité, les calculs des groupes de pieux ordonnés dans un plan (GULLANDER, HULTIN) peuvent être effectués facilement par la méthode de NØKKENTVED. Par contre, les calculs correspondants pour un groupe spatial de pieux s'avèrent si pénibles et si compliqués qu'ils ne rentrent pas en ligne de compte pour la pratique. Le présent travail a pour but de montrer que le calcul de matrices permet de traiter très clairement le problème théorique et permet d'effectuer systématiquement le calcul pratique ainsi que l'indique l'équation (15).

On démontre également qu'il est quelquefois possible d'ajouter aux groupes donnés de pieux un ou plusieurs pieux fictifs de telle sorte que le groupe ainsi augmenté permet un calcul plus expéditif, du fait p. ex. de l'introduction d'une symétrie. Les réactions dues aux forces extérieures f, m introduites dans les pieux fictifs peuvent être évaluées et ajoutées à f, m, afin de neutraliser les réactions que ces mêmes pieux opèrent sur la jetée. Les forces ajoutées produisent dans les pieux fictifs de nouvelles réactions qui sont de nouveaux calculées et ajoutées, etc. Ce calcul par itération converge souvent et il est démontré aussi bien théoriquement que par des exemples que la somme des séries de matrices géométriques correspondantes contient la solution exacte du problème du groupe de pieux. En d'autres termes, ce calcul par itération approche par approximation successive la solution directe donnée par l'équation (15) du groupe de pieux.