§4. HODGE THEORY FOR HYPERBOLIC 3-MANIFOLDS

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the way Lebrun did. LeBrun arrives at his flat $\mathbf{CP}^1$ bundle through a foliation argument which presumably can be mimicked in the 3-manifold.

§ 4. Hodge theory for hyperbolic 3-manifolds

Apart from the topological and geometrical applications which we discussed in § 3, our Kaluza-Klein approach also has some more analytical applications.

Recall that the Hodge-star $*: \Omega^n(Y) \to \Omega^n(Y)$, on a $2n$-dimensional oriented Riemannian manifold Y, depends only on the conformal structure underlying the metric. This has two consequences:

1) The L^2-norm $\| \omega \|^2 = \int \omega \wedge *\omega$, of $\omega \in \Omega^n(Y)$, is conformally invariant.

2) The harmonic n-forms, i.e. the $\omega \in \Omega^n(Y)$ s.t. $d\omega = d^*\omega = 0$, depend only on the conformal structure of Y.

Of course conformal rescaling lies at the heart of our construction in § 2, and we shall now show how the above applies to this situation. Let X be the conformal compactification of $M \times S^1$ as in § 2. Harmonic 2-forms on X are automatically S^1-invariant because they are in one-one correspondence with the elements of $H^2(X; \mathbf{R})$ $(= H^2(M; \mathbf{R}) \oplus H^1(M, \delta M; \mathbf{R})$, see § 2). By restriction to the open subset $M \times S^1 \subset X$ and a conformal rescaling of the metric on $M \times S^1$, 2) above implies that we get S^1-invariant harmonic 2-forms on $M \times S^1$ with respect to the product metric.

An S^1-invariant form can be written as $\omega = \rho^*\alpha + \rho^*\beta \wedge d\theta$, with $\alpha \in \Omega^2(M)$, $\beta \in \Omega^1(M)$ and $\rho: M \times S^1 \to M$ the projection. A short computation shows that such S^1-invariant forms ω are harmonic iff α and β are harmonic on M. If ω is a harmonic 2-form on $M \times S^1$ arising from a form on X then it follows from proposition 2.2 that $\alpha \in \Omega^2(M)$ and $\beta \in \Omega^1(M)$ are harmonic representatives for the class $\omega \in H^2(M; \mathbf{R}) \oplus H^1(M, \delta M; \mathbf{R})$. The forms α and β have finite L^2-norm on M by 1) above.

Conversely any S^1-invariant, harmonic 2-form $\tilde{\omega}$ on $M \times S^1$ with finite L^2-norm arises in this way. By 1) above one can always consider $\tilde{\omega}$ to be an L^2-form ω on X because $\cup S_j = X \backslash M \times S^1$ has measure 0. Applying the first order elliptic operator $d \oplus d^*$ to ω gives a distributional form in $L^2_{-1}(\Lambda^*(X))$ of distributional order $\leqslant 1$, which has support in the co-dimension 2 manifold $\cup_j S_j \subset X$. The following lemma shows that this implies that $(d \oplus d^*)\omega = 0$, which proves that ω is a smooth harmonic form on X, as we claimed.

LEMMA 4.1. *Let* μ *be a distribution of order* $\leqslant 1$ *in* $L^2_{-1}(\mathbf{R}^n)$. *If* $\operatorname{supp} \mu$ *is contained in* $\mathbf{R}^{n-2}$ *then* $\mu = 0$.

Proof. Without loss of generality assume that μ is compactly supported. The structure theorem for distributions carried by submanifolds (see Hörmander [21] theorem 2.3.5) asserts that μ is a finite linear combination of distributions ν of the form $<\nu, f> = <\eta, \delta_{\mathbf{R}^{n-2}} \cdot D^k_{\mathbf{n}} \cdot f>$, where η is a compactly supported distribution on $\mathbf{R}^{n-2}$, $\delta_{\mathbf{R}^{n-2}}$ is restriction to $\mathbf{R}^{n-2}$ and $D^k_{\mathbf{n}}$ is a k-th derivative $(0 \leqslant k \leqslant 1)$ in a direction $\mathbf{n}$ normal to $\mathbf{R}^{n-2}$.

The Fourier transform $\hat{\mu}(u, x, y)$ is a smooth function on $\mathbf{R}^{n-2} \oplus \mathbf{R} \oplus \mathbf{R}$ of the form $f_0(u) + f_1(u) \cdot x + f_2(u) \cdot y$. It is easy to see from this that the L^2_{-1}-norm cannot be finite, unless $\mu = 0$. □

Denote by $\mathcal{H}^i(M)$ the vectorspace of harmonic (i.e. closed and coclosed) i-forms on M with finite L^2-norm. Summarizing the above we have proved:

THEOREM 4.2. *The natural maps* $\mathcal{H}^1(M) \to H^1(M, \delta M; \mathbf{R})$ *and* $\mathcal{H}^2(M) \to H^2(M; \mathbf{R})$ *are isomorphisms.* □

On the universal cover, *Poisson transformation* gives a one-one correspondence between closed and co-closed 1-forms on H^3 and exact one forms with hyperfunction coefficients on δH^3, and this is what we shall exploit next. If the hyperfunction one form is continuous then it is the boundary value of the one form on H^3 in the classical sense, this is special for hyperbolic space. Thus in this case Poisson transformation is solving a Dirichlet boundary value problem on $(H^3, \delta H^3)$. The Poisson transform $\mathcal{P}(\phi)$ of a continuous function ϕ on δH^3 is defined as (see e.g. Gaillard [13]):

$$\mathcal{P}(\phi)(h) = \int_{S^2} P(h, b) \cdot \phi(b) \qquad \text{with} \qquad P(h, b) = \pi^{-1}(h_3/|h-b|^2)^2 db_1 \Lambda db_2 ,$$

where $h = (h_1, h_2, h_3) \in \mathbf{R}^3_+ \cong H^3$, $h_3 > 0$ and $b = (b_1, b_2, 0) \in \mathbf{R}^2 \subset \delta H^3$. For exact one-forms $\alpha = d\phi$ we define $\mathcal{P}(\alpha) = d\mathcal{P}(\phi)$. As $\mathcal{P}(\phi)$ is harmonic, $\mathcal{P}(\alpha)$ is closed and co-closed. Using this, we can identify our L^2 cohomology as follows:

THEOREM 4.3. *Poisson transformation induces an isomorphism from* Γ-*invariant closed one-forms with hyperfunction coefficients on* δH^3 *with support* *in the limit set to closed and co-closed one forms on* H^3/Γ *with finite* L^2 *norm. Such hyperfunction one-forms are one-currents.*

Proof. An L^2 harmonic 1-form on M lifts to an invariant 1-form ω on H^3. From Gaillard [13] we know that ω is the Poisson transform of a unique closed 1-form α on $S^2 = \delta H^3$ with hyperfunction coefficients. From theorem 4.2 it follows that ω is bounded on a fundamental domain, so it is of slow growth and therefore α is a current. Now write $\alpha = d\phi$, $\omega = d\psi$ for a distribution ϕ and a function ψ. It follows that ψ is the Poisson transform of ϕ (after adding a constant). From theorem 4.2 it follows that the one form ω extends smoothly to a one form on $(H^3 \cup \delta H^3) - \Lambda$, zero on the boundary $\delta H^3 - \Lambda$. This implies that ψ is smooth on $(H^3 \cup \delta H^3) - \Lambda$. In Schlichtkrull [32], chapter 4, it is proved that under these conditions ψ converges uniformly to ϕ. But then ϕ must be constant on components of $\delta H^3 - \Lambda$ and therefore the support of α is contained in Λ.

Conversely let α be a closed 1-form with hyperfunction coefficients in S^2 with support in Λ, and let ω be its Poisson transform. We shall prove that ω, which is automatically closed and co-closed, has finite L^2 norm. As above let $\omega = d\psi$ and $\alpha = d\phi$, then ϕ is constant on components of $\delta H^3 - \Lambda$. Apart from the boundary value ϕ there is another "boundary value" ϕ', just as in the classical case there is the von Neumann boundary value problem next to the Dirichlet boundary value problem. In further analogy with the classical case the global boundary value ϕ' can be obtained from ϕ by applying a pseudo-differential operator on S^2 to it, which has a real analytic integral kernel, see Schiffmann [31]. So, ϕ and ϕ' are real analytic in $\delta H^3 - \Lambda$.

Oshima [30] theorem 5.3 shows then that locally in $\delta H^3 - \Lambda$ we have:

$$\psi(h_1, h_2, h_3) = c_1(h_1, h_2, h_3) + c_2(h_1, h_2, h_3) \cdot h_3^2 \cdot q(\log h_3),$$

with (h_1, h_2, h_3) upper half space coordinates, q a polynomial in one variable and $c_1(h_1, h_2, 0) = \phi(h_1, h_2)$, $c_2(h_1, h_2, 0) = \phi'(h_1, h_2)$. From this it follows that ω has an expansion locally bounded by $cst \cdot h_3 \cdot q(\log h_3)$.

Recall that a fundamental domain for the Γ-action on H^3 intersects δH^3 in a compact fundamental domain for the Γ-action in $\delta H^3 - \Lambda$. This together with our estimate implies readily that the L^2 norm of ω restricted to a fundamental domain is finite. □

A few remarks are in order. First of all it should be possible to give an effective bound on the distributional order of the currents α on S^2, and also if $\alpha = d\phi$ it should be possible to determine if the function ϕ (constant on components of $\delta H^3 - \Lambda$) is locally integrable. Also it should be noted that $\omega \Lambda d\theta$ is a solution on X of a p.d.e with real analytic coefficients, i.e. it is real analytic. This shows inmedeately that ω has an expansion as in the proof of theorem 4.3, without logaritmic terms.

Next we can use the above to define a simple invariant of the hyperbolic structure on M. The Hodge star of the hyperbolic 3-manifold M gives an isomorphism $*_3 : \mathscr{H}^1(M) \to \mathscr{H}^2(M)$. Both $\mathscr{H}^1(M)$ and $\mathscr{H}^2(M)$ contain an integral lattice of maximal rank coming from integral cohomology. These lattices do not generally coincide under $*_3$; in fact their intersection is empty unless the 4-manifold carries a self-dual harmonic form which represents an integral cohomology class. The relative position of the two lattices in $H^2(M; \mathbf{R})$ is described by:

4.1 $$h(M) \in GL(H^2(M; \mathbf{R}))/GL(H^2(M; \mathbf{Z}) \otimes \mathbf{Z}),$$

which is an invariant of the hyperbolic structure of M. Similar invariants are very popular in algebraic geometry. There discrete lattices in a complex vector space give rise to invariants associated to the complex structure of manifolds.

We proceed to sketch how the above theory relating solutions of elliptic p.d.e. on M to invariant solutions on X generalizes. Suppose $D : \Gamma(E) \to \Gamma(F)$ is a conformally invariant first order (possibly over-determined) elliptic operator acting on sections of the vector bundle E over X. This class of operators was studied in detail by Hitchin [18], and comprises, among others, Dirac and twistor operators on X and the operator $d + d^*$ on 2-forms which we studied above. Again restriction of S^1-invariant solutions on X to $M \times S^1$ gives solutions to a closely related geometric p.d.e. on M.

Conversely we can start with a solution on M and require that it has a finite L^2-norm on $X \backslash (\cup S_j)$. In general this is not the same as having a finite L^2-norm on M, but it is the same as having a finite weighted L^2-norm on M. The weighting function is a suitable power of the function on M which conformally rescales the hyperbolic metric on M to a metric on X. Such a function is determined up to multiplication by functions $\phi : M \to \mathbf{R}_{>0}$ which are bounded above and below. The exact value of the power needed is an inhomogeneous linear function of the conformal weight of E. The extension over the fixed surfaces S_j goes now as in lemma 4.1. We shall not make use of this in the sequel and therefore leave the details to the reader.

Remarks. 1) It would be interesting to see what kind of harmonic representatives for classes in $H^1(M; \mathbf{R})$ can be found.

2) Theorem 4.2 generalizes to identify elements of $H^j(M, \delta M; \mathbf{R})$ with L^2 harmonic forms for any oriented n-dimensional Riemannian manifold M for which a conformal compactification of $M \times S^k$ exists, for all k, provided $j < n/2$.

§ 5. MONOPOLES AND INSTANTONS

Our goal is now to exploit the compactification X of $M \times S^1$ (see § 2) to get monopoles on M from S^1-invariant instantons on X. We shall also relate the instanton number on X to various topological invariants of the monopoles on M. General background for this section can be found in Freed-Uhlenbeck [12] and Jaffe-Taubes [22]. More specifically our approach here is very similar to the one taken in Atiyah [2].

Let P be a principal $SU(2)$-bundle over X, with $c_2(P) = k \geqslant 0$. Recall that X comes naturally with a conformal structure. This enables us to talk about *instantons or anti-self-dual connections* A on P. These are defined to be the solutions of the *anti-self-duality equation*:

5.1 $$F^A = - *_4 F^A \quad (*_4 \text{ the Hodge star on } \Lambda^2(X)).$$

Here F^A is the curvature of A, a section of $\Lambda^2(X) \otimes g_P$ with $g_P = P \times_{Ad} su(2)$. The instantons are the absolute minima of the *Yang-Mills functional*:

5.2 $$YM(A) = (16\pi^2)^{-1} \int_X < F^A \wedge *F^A >$$

where $<\alpha, \beta> = -2 \cdot tr(\alpha\beta)$ is an invariant inner product on $su(2)$. For an instanton $YM(A) = k$.

Next assume that the double cover $\tilde{S}^1$ of S^1 acts on P by bundle automorphisms, covering the action on X; the double cover will be needed in order to include the spin bundles of X. Our interest will now lie in $\tilde{S}$-invariant instantons on P. To relate these to objects on M introduce the map:

$$j : M \to X : m \to i'(m, 1) \quad (\text{compare } 2.2),$$

which is a diffeomorphism onto its image. Let v be the vectorfield on P induced by the $\tilde{S}^1$-action. If we interprete an $\tilde{S}^1$-invariant connection A as a 1-form on P, then define the Higgs-field Φ to be the $su(2)$-valued function $j^*A(\frac{1}{2}v)$ on j^*P. It is easy to see that Φ is a section of j^*g_P.