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Goldstone Bosons in a Finite Volume: the Partition Function to Three Loops [1]

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Abstract. A system of Goldstone bosons – stemming from a symmetry breaking $O(N) \to O(N-1)$ – in a finite volume at finite temperature is considered. In the framework of dimensional regularization, the partition function is calculated to the 3-loop level for 3 and 4 dimensions, where the Polyakov method for the measure of the path integral is applied.

Although the underlying theory is the non-linear σ-model, it will be shown that the 3-loop result is renormalizable in the sense that all the singularities can be absorbed by the coupling constants occurring so far. In finite volume this property is highly non-trivial. Thus the method for the measure is confirmed. In addition we show that – to the considered order – it coincides with the Faddeev-Popov measure. This is also true for the maximal generalization of Polyakov's measure: none of the additional invariant terms that can be added contributes to the dimensionally regularized system.

The occurring phenomenological Lagrangian describes for example 2-flavor chiral QCD as well as the classical Heisenberg model, but there are also points of contact with the Higgs model, superconductors etc. In addition the finite size corrections to the susceptibility might improve the interpretation of Monte Carlo results on the lattice.

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1 Introduction

If a continuous symmetry is spontaneously broken, the Goldstone bosons (GB) dominate the low energy behavior of the system. The interaction among the Goldstone modes is strongly constrained by symmetries. This represents a universal feature of all models exhibiting spontaneous symmetry breaking [23].

In the present work we choose $O(N)$ as the symmetry to be broken down to $O(N-1)$. Of course this can also be applied to symmetry groups (locally) isomorphic to $O(N)$, such as $SU(2) \times SU(2) \sim O(4)$. Thus it describes QCD with two flavors and broken chiral symmetry.

The underlying theory will be the non-linear σ-model, which includes for dimension $d > 2$ all invariant terms. Thus it is not renormalizable because it contains an infinite number of coupling constants. [3] To any order in the low energy expansion, however, a "perturbative renormalization" can be realized, i.e. the coupling constants occurring to that order are able to absorb all the singularities. This property provides us with a non-trivial check of the results and the applied methods.

We are going to use a "magnetic" language, so the model most suitable to our terminology is the classical Heisenberg model for ferromagnets below the critical temperature.

Particularly in soft pion physics the method of low energy effective Lagrangians seems to be more efficient than the historical way (current algebra, Ward identities, etc.) [3, 4, 5, 7, 8, 15]. For unknown reasons two quark flavors are very light compared to the scale of the theory. If their masses would vanish (chiral limit) the QCD Lagrangian would exhibit an $SU(2)_R \times SU(2)_L$ symmetry. This symmetry spontaneously breaks down to $SU(2)_{R+L}$, creating GBs which are identified with the pions (π^+, π^0, π^-).[4] Their properties reveal the hidden symmetry. They can be analyzed by replacing $\mathcal{L}_{QCD}$ with its quark and gluon fields by an effective Lagrangian with pseudoscalar meson fields [5, 6]. One constructs the most general $\mathcal{L}$ in terms of GB fields consistent with the symmetry of the model. It is generally assumed – although not strictly proved – that the low energy predictions do *only* contain this information [22] . Then all quantum field theories generating this type of GBs are covered. Ward identities constrain the expansion of the Greens function in powers of the momenta and the external fields. They also imply that the interaction among the GB modes of low momenta is weak, and to any finite order of the low energy expansion there occur only a finite number of coupling constants [4, 22].

In the standard model the influence of the gauge and fermion fields on the scalar sector

[3]Different is the 2-dimensional case: there the model *is* renormalizable. Many recent papers concentrate on this case. We also refer to it in appendix E.

[4]For $SU(3) \times SU(3)$ the GBs are identified with the eight lightest mesons (π, K, η). This case is not described by an $O(N)$ symmetry; the groups involved in the breakdown $SU(M) \times SU(M) \rightarrow SU(M)$ are locally isomorphic to an orthogonal group only for $M = 2$.

is weak. [5] Concerning the upper bound of the Higgs mass m_H we can consider the $SU(2)$ sector separately and study an $O(4)$ scalar field theory [2]. On the other hand the $O(4)$ model is inadequate for questions involving very weak scalar interactions, such as the lower bound of m_H; there the gauge and Yukawa couplings can not be neglected any more.

In the $O(4)$ model the tree level yields the relation $m_H^2 \propto \lambda_r$, where λ_r is the renormalized self coupling. Thus m_H enters as a free parameter. Theoretical information about it can be gained, however, making use of the fact that the cutoff Λ is unremovable. Even if we set the bare $\lambda = \infty$, for any given ratio $\Lambda/m_H > O(1)$ λ_r runs away from infinity fast enough to fix an upper bound for m_H/m_W that can be determined numerically. For decreasing Λ/m_H the latter rises, so choosing the smallest, physically acceptable value for Λ/m_H, we obtain an absolute upper bound for m_H [12, 13]. But its numerical evaluation is charged with significant finite size effects due to massless GB [10]. We are going to present more precise analytical results about finite size effects of that kind.

Generally our results are suitable for comparison with data of lattice MC simulations (in particular concerning the conclusions about infinite volume), finite size properties of ferromagnets etc. About the link to bosonic strings, see [21].

But of course the confirmation of the perturbative renormalizability of the results we get with the Polyakov method for the measure has not at least a theoretical and technological meaning. This is particularly of interest in view of the symmetry groups $SU(N)$, where for $N > 2$ no other applicable treatment of the measure is known.

We also show that the leading term, which corresponds to Polyakov's definition of the functional measure, can be completed by an arbitrary linear combination of further terms obeying the symmetries of the system: it turns out that in dimensional regularization – to the considered order – all the contributions of non-leading measure terms vanish. Only if we include power divergences such a generalization becomes necessary, e.g. for the invariance of the partition function under some field transformations.

In *section 2* we introduce the model and its parameters and derive the effective action up to the third order in the derivative expansion for dimensions $d = 3$ and $d = 4$.

In *section 3* we describe Polyakov's definition of the measure occurring in the path integral and apply it to determine the measure to the second order. We also confirm the result of this method with the measure of Faddeev and Popov.

The 1-loop calculation of the partition function is given comprehensively in *section 4*. The extension of this calculation to 3 loops is described for $d = 3$ and $d = 4$ in *sections 5* and *6*, respectively.

In *section 7* we show explicitly that the results of sections 5 and 6 can be renormalized

[5] Here only a very heavy top quark could be a trouble-maker [16].

perturbatively; we determine the constraints on the counter terms.

Conclusions and five *appendices* about rather technical aspects are added. There we discuss the results with a generalized singularity structure that includes the leading power divergences and show that perturbative renormalizability still holds. In finite volume this is highly non-trivial, hence it provides us with a sensitive consistency check. We also discuss the explicit form of the generalized measure, the link to a massive expansion and the conclusions about the renormalizable case $d = 2$ as well as the reduction to quantum mechanics ($d = 1$).

2 The non-linear σ-model

In two dimensions, the non-linear σ-model can be characterized by the Lagrangian

$$\mathcal{L}^{(sy)} = \frac{F^2}{2} \partial_\mu \vec{S} \partial_\mu \vec{S} \tag{2.1}$$

where $\vec{S}(x)$ is an N-component scalar field subject to the constraint

$$\sum_{\alpha=0}^{N-1} S^\alpha(x) S^\alpha(x) = 1 \tag{2.2}$$

The model represents a renormalizable, asymptotically free two-dimensional field theory which is invariant under global $O(N)$ rotations of the vector $\vec{S}(x)$. One may introduce a term which explicitly breaks the $O(N)$ symmetry by coupling the system to an external "magnetic field" $\vec{H}$,

$$\mathcal{L}^{(sb)} = -\Sigma(\vec{H}\vec{S}) \tag{2.3}$$

For dimension $d > 2$, the Lagrangian specified in eqs (2.1) and (2.3) does, however, not make sense as it stands because the constraint (2.2) generates derivative couplings which are not renormalizable (the coupling constant F carries the dimension $[\text{mass}]^{d/2-1}$). Accordingly, for $d > 2$, the term "non-linear σ-model" does not refer to the Lagrangian (2.1), (2.3) but to the following more general construction. One considers the set of all possible $O(N)$-invariant couplings of the field $\vec{S}(x)$, allowing for arbitrarily many derivatives. Ordering the infinite series of vertices according to the number of derivatives, the first few terms are [6]

$$\begin{aligned} \mathcal{L}^{(sy)} &= \frac{F^2}{2} \partial_\mu \vec{S} \partial_\mu \vec{S} + \frac{1}{2} g_4^{(1)} \partial^2 \vec{S} \partial^2 \vec{S} + \frac{1}{4} g_4^{(2)} (\partial_\mu \vec{S} \partial_\mu \vec{S})^2 \\ & \frac{1}{4} g_4^{(3)} (\partial_\mu \vec{S} \partial_\nu \vec{S})^2 + \frac{1}{2} g_6^{(1)} \partial_\mu \partial^2 \vec{S} \partial_\mu \partial^2 \vec{S} + \dots \end{aligned} \tag{2.4}$$

where the dots stand for further terms involving six or more derivatives of the fields. The corresponding generalization of the symmetry breaking term (2.3) involves derivatives of the

[6] The index of g is the number of derivatives.

field $\vec{S}(x)$ as well as higher powers of the magnetic field. Assuming $\vec{H}$ to be constant, the first terms are [7]

$$\begin{aligned}-\mathcal{L}^{(sb)} &= \Sigma(\vec{H}\vec{S}) + h_{2,0}^{(1)}(\vec{H}\vec{S})^2 + h_{2,0}^{(2)}(\vec{H}\vec{H}) + h_{1,2}^{(1)}(\vec{H}\vec{S})(\partial_\mu\vec{S}\partial_\mu\vec{S}) + h_{1,2}^{(2)}(\vec{H}\partial^2\vec{S}) \\ &+h_{1,4}^{(1)}(\vec{H}\vec{S})(\partial_\mu\vec{S}\partial_\mu\vec{S})^2 + h_{1,4}^{(2)}(\vec{H}\vec{S})(\partial_\mu\vec{S}\partial_\nu\vec{S})^2 + h_{1,4}^{(3)}(\vec{H}\vec{S})(\partial^2\vec{S}\partial^2\vec{S}) \\ &+h_{2,2}^{(1)}(\vec{H}\vec{H})(\partial_\mu\vec{S}\partial_\mu\vec{S}) + h_{2,2}^{(2)}(\vec{H}\vec{S})^2(\partial_\mu\vec{S}\partial_\mu\vec{S}) + h_{2,2}^{(3)}(\vec{H}\vec{S})(\vec{H}\partial^2\vec{S})\ldots \qquad (2.5)\end{aligned}$$

In the following, we study the model defined by

$$\mathcal{L} = \mathcal{L}^{(sy)} + \mathcal{L}^{(sb)} \qquad (2.6)$$

in $d = 3$ and $d = 4$. More specifically, we consider the properties of the corresponding partition function Z in a finite volume, which can also be understood as an IR regularization. We introduce a rectangular box $L_1L_2\ldots L_d = V$ (L_i/L_k not large) imposing periodic boundary conditions :

$$\begin{aligned} \mathrm{Z} &= \mathrm{N}\int[d\vec{S}]e^{-\int_V \mathcal{L}dx} \\ \vec{S}(x) &= \vec{S}(x+\bar{\mathrm{n}})\ , \quad \bar{\mathrm{n}} = (\mathrm{n}_1L_1, \mathrm{n}_2L_2, \ldots, \mathrm{n}_dL_d)\ , \quad \mathrm{n}_\mu \in \mathbb{Z}\end{aligned}$$

$[d\vec{S}]$ is the ordinary measure of the path integral and N is an $\vec{H}$-independent normalization constant (which also requires renormalization, see below). One component can also be taken to be imaginary time-like, so the corresponding side of the Euclidean box, say L_d, represents the inverse, finite temperature of the system: $L_d = \frac{1}{T}$. From this interpretation we see that all the coupling constants must be independent of L_d, and due to the permutational symmetry of the Euclidean axis they can't depend on $L_1, \ldots, L_{d-1}$ either [8, 14].

As shown in [7, 14], the partition function can be expanded in inverse powers of the size of the box

$$L \doteq V^{1/d}$$

In our consideration of the free energy, L^{-1} takes the role of the energy in soft scattering amplitudes.

We consider a small magnetic field of the magnitude

$$H \doteq |\vec{H}| = O(V^{-1}) \qquad (2.7)$$

so the leading term of the symmetry breaking part of the action is of order $O(1)$.

It has been shown that for any $d > 2$, the leading order in the large volume expansion of the partition function only involves the two coupling constants Σ and F, called "magnetization" and "pion decay constant", respectively. (The latter denotation can be understood by noting that F^2 is the residue of the GB pole in the current correlation function at zero external field.)

[7] The indices of h are the power of H and the number of derivatives.

Statistical physics sometimes introduces a "helicity modulus" Υ defined as the increase of the free energy when the external field is slowly rotated: $\Delta f = \frac{1}{2}\Upsilon\alpha^2$, ($\alpha \doteq$ rotation angle/distance). It turns out that Υ coincides with F^2 [14].

The remaining terms in eqs (2.4) and (2.5), which involve additional derivatives or higher powers of the magnetic field, only show up at non-leading order. Moreover, the large volume expansion can be worked out perturbatively: to a given order in the expansion in powers of $1/L$, only graphs involving a limited number of loops contribute. In the present work, we extend the results of [7, 14] by considering the large volume expansion of the partition function up to and including terms of order $(L^{2-d})^3$ [8]. As we will see, this requires a perturbative evaluation to three loops (the loop propagator fixes the ordering in magnitudes of L^{2-d}). Higher dimensions require more coupling constants from the second order on, e.g. the terms in (2.5) with coefficients $h_{2,0}^{(i)}$ contribute to the action as L^{-d}, so they are classified differently for $d = 3$ and $d = 4$.

The GB mass is given by $m^2 = \frac{\Sigma H}{F^2}$, [9] such that $mL << 1$, i.e. the GB modes feel the boundary conditions strongly. [10] On the other hand this must not be true for other mass scales as they are given here e.g. by the mass of the σ-particle (or the ρ-particle in QCD): there L is much larger than the Compton wave length.

We note that for small H, Σ and F control the finite size effects, the GB mass and the correlation functions.

In the limit of purely spontaneous symmetry breaking, $H \to 0$, the model contains zero modes. They correspond to space-independent fields, $\vec{S}(x) =$ const. , for which only the symmetry breaking part of the action is different from zero – for weak magnetic fields the action reduces to $\Sigma \vec{H}\vec{S}V$. In the region $H = O(V^{-1})$ we are studying here, the direction of the vector $\vec{S}$ does therefore not strongly favor the direction of $\vec{H}$; in the absence of a magnetic field, all directions become equally likely. The standard perturbative expansion of the model – where the field $\vec{S}$ is expanded around the direction of the magnetic field – is not applicable here. As pointed out in [7], the problem can be solved by using collective variables. The general field configuration is represented as

$$\vec{S}(x) = \Omega^{-1}\vec{\pi}(x) \doteq \Omega^{-1}(\pi^0, \underline{\pi}) \quad , \quad \vec{H} = (H, \underline{0}) \tag{2.8}$$

where $\Omega \in O(N)$ is a global rotation associated with the zero modes (or quasi zero modes for $H > 0$) while the vector $\vec{\pi}(x)$ describes the non-zero modes. The condition

$$\int_V \pi^i(x)dx = 0 \; ; \qquad i = 1, \dots, N-1 \tag{2.9}$$

insures that this vector fluctuates around the direction $(1, \underline{0})$, such that π^0 can be expanded

[8]Actually we expand in powers of the small dimensionless quantity L^{2-d}/F^2. This is what we really mean when – for brevity – we just count the powers of L^{2-d}.

[9]This is not the case for $d = 2$, where the GB mass exhibits a gap at $H \to 0$.

[10]$m << 1/L$ – that corresponds to rule (2.7) and $d > 2$ – characterizes the so-called "ϵ-expansion", in contrast to the "chiral expansion" where $mL = O(1)$.

as [11]

$$\pi^0 = 1 - \frac{1}{2}\underline{\pi}^2 - \sum_{k=2}^{\infty} \frac{(2k-3)!!}{2^k k!} (\underline{\pi}^2)^k$$

Thus for $H \to 0$ the expectation values are:

$$< \pi^0 >= 1 \quad , \quad < \underline{\pi} >= \underline{0}$$

i.e. the $N-1$ vector field $\underline{\pi}$ represents (small) transversal excitations of $\vec{S}$ around its longitudinal component π^0 (parallel to $\vec{H}$).

It turns out that also the leading term of $\mathcal{L}^{(sy)}$ – the kinetic term $\frac{F^2}{2}\partial_\mu \underline{\pi} \partial_\mu \underline{\pi}$ – contributes to the action in order $O(1)$ (see [14]), so the power counting rule (2.7) is completed by

$$\partial_\mu \propto L^{-1} \ , \quad \pi^i(x) \propto L^{1-d/2} \ ; \quad i = 1, \dots, N-1 \tag{2.10}$$

Inserting the decomposition (2.8) in the action and using the rules (2.7), (2.10), we obtain a series of the form

$$S = \int_V \mathcal{L} dx = \sum_{\ell=0}^{\infty} \{ S_\ell + \tilde{S}_\ell(H) \}$$

where S_ℓ and $\tilde{S}_\ell$ stem from $\mathcal{L}^{(sy)}$ and $\mathcal{L}^{(sb)}$, respectively, and the sum $S_\ell + \tilde{S}_\ell(H)$ collects all contributions of order $L^{(2-d)\ell}$. Accordingly, the partition function becomes

$$\mathrm{Z} = \mathrm{N} \int [d\vec{S}] e^{-\{S_0 + \tilde{S}_0(H)\}} \left[1 - \sum_{\ell \geq 1} (S_\ell + \tilde{S}_\ell(H)) + \frac{1}{2} \{ \sum_{\ell \geq 1} (S_\ell + \tilde{S}_\ell(H)) \}^2 \dots \right]$$

In this work, we are interested only in the H-dependence of Z, so we absorb in the normalization constant N an overall H-independent factor. Therefore, to calculate the partition function to $O((L^{2-d})^3)$, we need to evaluate S_0, S_1, S_2 and $\tilde{S}_0(H) \dots \tilde{S}_3(H)$.

First we simplify $\mathcal{L}^{(sy)}$ given by (2.4): the transformation

$$\vec{S} \to \frac{\vec{S} + \alpha F^{-4/(d-2)} \partial^2 \vec{S}}{|\vec{S} + \alpha F^{-4/(d-2)} \partial^2 \vec{S}|} \tag{2.11}$$

does not change the form of the Lagrangian but only the coupling constants [14]. This allows us to choose the (dimensionless) parameter α in such a manner that $g_4^{(1)}$ vanishes. Then only the terms with $F^2, g_4^{(2)}, g_4^{(3)}$ and $g_6^{(1)}$ contribute to the order we want to consider.

If we apply the counting rule (2.7) in $\mathcal{L}^{(sb)}$, only the terms with the coupling constants $\Sigma, h_{2,0}^{(1)}, h_{2,0}^{(2)}, h_{1,2}^{(1)}, h_{1,2}^{(2)}$ and $h_{1,4}^{(3)}$ remain relevant in eq. (2.5).

The transformation

$$\vec{S} \to \frac{\vec{S} + \beta \Sigma F^{-2d/(d-2)} \vec{H}}{|\vec{S} + \beta \Sigma F^{-2d/(d-2)} \vec{H}|} \tag{2.12}$$

[11] In the literature the component π^0 is often called σ, naming the model.

enables us to put also $h^{(2)}_{1,2} = 0$ without disturbing $g^{(1)}_4 = 0$, since the latter is independent of $\vec{H}$. [12]

In addition with the transformation

$$\vec{S} \to \frac{\vec{S} + \lambda \Sigma F^{-2(d+2)/(d-2)} (\vec{H}\vec{S}) \partial^2 \vec{S}}{|\vec{S} + \lambda \Sigma F^{-2(d+2)/(d-2)} (\vec{H}\vec{S}) \partial^2 \vec{S}|} \tag{2.13}$$

we also succeed in making $h^{(3)}_{1,4}$ vanish. Else it would occur for $d = 4$. There, this last transformation manipulates $\vec{S}$ only in the 3^{rd} order, so it maintains the coupling constants up to the 2^{nd} order, in particular $g^{(1)}_4 = h^{(2)}_{1,2} = 0$.

All the three transformations performed above do not change $\vec{S}$ in the leading order, so the leading couplings F^2 and Σ remain untouched. The rest of the coupling constants, which correspond to our special choice of α, β and λ, we denote by $g^{(2)\prime}_4$ etc. In appendix B we give the explicit transformation formula (B.14), which prove that these simplifications are possible.

Now $S = \int \mathcal{L}(\vec{\pi}, \Omega \vec{H}) dx$ shall be expanded: [13] [14]

$$\begin{aligned} S \cong \int \Big[& \frac{F^2}{2} (\partial_\mu \underline{\pi} \partial_\mu \underline{\pi} + \partial_\mu \pi^0 \partial_\mu \pi^0) - \Sigma H \Omega^{00} \pi^0 \\ & + h^{(1)\prime}_{1,2} H (\Omega^{00} \pi^0 + \Omega^{0i} \pi^i)(\partial_\mu \underline{\pi} \partial_\mu \underline{\pi} + \partial_\mu \pi^0 \partial_\mu \pi^0) - h^{(1)\prime}_{2,0} H^2 (\Omega^{00} \pi^0 + \Omega^{0i} \pi^i)^2 \\ & - h^{(2)\prime}_{2,0} H^2 + \frac{1}{4} g^{(2)\prime}_4 (\partial_\mu \underline{\pi} \partial_\mu \underline{\pi})^2 + \frac{1}{4} g^{(3)\prime}_4 (\partial_\mu \underline{\pi} \partial_\nu \underline{\pi})^2 + \frac{1}{2} g^{(1)\prime}_6 (\partial_\mu \partial^2 \underline{\pi} \partial_\mu \partial^2 \underline{\pi}) \Big] dx \end{aligned}$$

If we insert π^0, omit once more the $O(N)$ symmetrical terms of 3^{rd} order and choose dimensionless coupling constants $k_1 \dots k_6$, the expansion of the action becomes:

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[12] Actually the elimination of $h^{(2)}_{1,2}$ is motivated only by the possibility of a space dependent magnetic field $\vec{H}(x)$. In our case where $\vec{H} = const.$ this term does not contribute to the action and transformation (2.12) can be used to make $h^{(1)}_{2,0}$ or $h^{(2)}_{2,0}$ vanish. We will recall this remark when counting the degrees of freedom of the counter terms associated with the non-leading coupling constants in section 7 and appendix B.

[13] The region of spatial integration is always V, unless we indicate something else.

[14] Throughout this work $\sim, \simeq, \cong$ mean: to an accuracy of $1^{st}, 2^{nd}, 3^{rd}$ order in (L^{2-d}), respectively.

$$
\begin{array}{lll}
S_0 & = \frac{F^2}{2}\int \partial_\mu \underline{\pi}\partial_\mu \underline{\pi} dx - \Sigma H V \Omega^{00} & \\
S_1 & = \frac{F^2}{2}\int (\underline{\pi}\partial_\mu \underline{\pi})^2 dx + \frac{\Sigma H \Omega^{00}}{2}\int \underline{\pi}^2 dx & \\
S_2 & = \frac{F^2}{2}\int \underline{\pi}^2(\underline{\pi}\partial_\mu \underline{\pi})^2 dx + \frac{\Sigma H \Omega^{00}}{8}\int (\underline{\pi}^2)^2 dx & d=3 \\
S_2 & = \frac{F^2}{2}\int \underline{\pi}^2(\underline{\pi}\partial_\mu \underline{\pi})^2 dx + \frac{\Sigma H \Omega^{00}}{8}\int (\underline{\pi}^2)^2 dx & \\
& + \frac{\Sigma k_1 H \Omega^{00}}{F^2}\int \partial_\mu \underline{\pi}\partial_\mu \underline{\pi} dx - \frac{\Sigma^2 H^2 V [k_2(\Omega^{00})^2 + k_3]}{F^4} & \\
& + \frac{1}{4}\int [k_4(\partial_\mu \underline{\pi}\partial_\mu \underline{\pi})^2 + k_5(\partial_\mu \underline{\pi}\partial_\nu \underline{\pi})^2 + 2\frac{k_6}{F^2}(\partial_\mu \partial^2 \underline{\pi})^2] dx & d=4 \\
S_3(H) & = \frac{\Sigma H \Omega^{00}}{16}\int (\underline{\pi}^2)^3 dx + \frac{\Sigma k_1 H \Omega^{00}}{F^4}\int \partial_\mu \underline{\pi}\partial_\mu \underline{\pi} dx - \frac{\Sigma^2 H^2 V}{F^6}[k_2(\Omega^{00})^2 + k_3)] & d=3 \\
S_3(H) & = \frac{\Sigma H \Omega^{00}}{16}\int (\underline{\pi}^2)^3 dx + \frac{\Sigma k_1 H \Omega^{00}}{F^2}\Big[\int (\underline{\pi}\partial_\mu \underline{\pi})^2 dx - \frac{1}{2}\int \underline{\pi}^2 (\partial_\mu \underline{\pi})^2 dx\Big] & \\
& + \frac{\Sigma^2 H^2 k_2}{F^4}\Big[(\Omega^{00})^2 \int \underline{\pi}^2 dx - \Omega^{0i}\Omega^{0k}\int \pi^i \pi^k dx\Big] & d=4
\end{array}
$$

Concerning the sign flip of k_1, we follow the convention given in [14]. We define the non-leading coupling constants, however, consequently dimensionless, like the transformation parameters α, β, λ before. This is achieved by multiplying with the suitable power of F, which is the only dimension-carrying coupling constant.

In addition the magnetic field is always accompanied by the (dimensionless) constant Σ.

Actually for $d = 4$ there is also one term of order $5/2$, namely:

$$\frac{\Sigma H \Omega^{0i}}{F^2} k_1 \int \pi^i (\partial_\mu \underline{\pi})^2 dx \quad .$$

But from the contraction rules it is obvious that this term will not contribute to our three loop result. So we omit it, although this is not justified in general: if we wanted to calculate to five loops we would have to include this term, since there its square in the exponential expansion *does* contribute.

3 The measure

In the presence of derivative couplings, the step from the classical Lagrangian to the quantum theory is not straightforward. In the framework of the canonical quantization procedure, the problem manifests itself, e.g., in the fact that the interaction Hamiltonian does not coincide with the negative interaction Lagrangian.

In the path integral formulation of quantum theory, the issue concerns the measure, i.e. the volume element in the space of field configurations over which we are to integrate. In particular the finite spatial volume we consider here causes non-trivial contributions.

In the following, we construct the measure by means of the method introduced by Polyakov in his analysis of the path integrals for bosonic strings [18]. Polyakov introduces a metric on the space of classical field configurations and defines the measure as the volume element induced by this metric.

In our case, where the classical configurations are characterized by the field $\vec{S}(x)$, the metric is a quadratic form involving the difference $d\vec{S}$ between two neighboring configurations,

$$ds^2 = \int\int dxdy \sum_{i,k} K_{i,k}(x,y;\vec{S},\vec{H})\ dS_i(x)dS_k(y)$$

The most important requirement to be imposed on the metric is locality: the support of the kernel $K_{i,k}$ must be concentrated at $x = y$. In addition, the metric must respect the symmetries of the Lagrangian – Euclidean invariance as well as the invariance under global $O(N)$ rotations in the isospin space.

The ansatz

$$ds^2 \doteq \frac{1}{V}\int d\vec{S}(x)d\vec{S}(x)dx \tag{3.1}$$

certainly satisfies these requirements [15], but by no means represents the general form of a local metric. In particular, locality allows also derivatives to occur.

To find the maximal generalization allowed by the symmetry requirements, we perform a derivative expansion and include the magnetic field, as we did for the Lagrangian. This leads to the form:

$$\begin{aligned} ds_g^2 \ = \ & \frac{1}{V}\int \Big\{ (d\vec{S})^2 + \frac{a_1}{F^{4/(d-2)}}(\partial_\mu d\vec{S})^2 + \frac{a_2}{F^{4/(d-2)}}(d\vec{S})^2(\partial_\mu \vec{S})^2 \\ & + \frac{a_3}{F^{8/(d-2)}}(\partial^2 d\vec{S})^2 + \dots \\ & + b_1\frac{\Sigma}{F^{2d/(d-2)}}(d\vec{S})^2(\vec{H}\vec{S}) + b_2\frac{\Sigma}{F^{2(d+2)/(d-2)}}(\partial_\mu d\vec{S})^2(\vec{H}\vec{S})\dots \Big\} dx \end{aligned} \tag{3.2}$$

where we introduce new coupling constants for the non-leading contributions. The explicit discussion of this general measure is given in appendix B. It turns out that all its non-leading terms only yield physically irrelevant power divergences, i.e. they do not contribute to the dimensionally regularized – more generally: not to the renormalized – free energy.

In the same appendix we also give the transformation rules of the couplings in the measure under the field transformations (2.11) - (2.13). They show in particular that the form

[15] The factor $1/V$ is unimportant here, i.e. a question of normalization. Polyakov did not include such a factor; he considers a *curved* space where it would have violated locality.

(3.1) does not permit such substitutions: starting from the measure (3.1), they cause non-trivial changes of all the coupling constants in (3.2) (which are, however, not relevant in dimensional regularization, since they only affect power divergences). At last we demonstrate the invariance of the entire partition function to 3 loops – including all possible couplings in the Lagrangian and in the measure as well as the leading power divergences – under those transformations with arbitrary coefficients.

It looks as if the quantization of theories including derivative couplings contain a considerable number of degrees of freedom, here represented by the unspecified constants a_i, b_i. The essential point of appendix B is, however, that the freedom of choice for the coupling constants in the measure does not mean an increased number of degrees of freedom of the system.

So it is justified to consider in the main part of this work the simplified measure (3.1), which yields relevant contributions, as we will see. In this section we give its evaluation in terms of the collective variables introduced in section 2.

Without consideration of a magnetic field, the set of functions $\{\vec{S}(x)\}$ is parame-trized by the direction of the mean magnetization

$$\vec{M} = \frac{1}{V}\int_V \vec{S}(x)dx$$

– which plays the role of the collective variable introduced in section 2 – and by a set of coordinates associated with the remaining degrees of freedom, which has to be regularized.

The collective variable may be identified with a subset of the rotation group, namely the rotations in the planes containing a fixed vector, say $\vec{e} = (1, \underline{0})$. There is a rotation Ω of this subset that takes the direction of $\vec{M}$ into $\vec{e}$,

$$\vec{m} \doteq \frac{\vec{M}}{|\vec{M}|} = \Omega^T \vec{e} \tag{3.3}$$

This is the rotation needed to represent the general element $\vec{S}(x)$ of our space of functions by eq. (2.8).

The non-zero modes we parametrize by a decomposition into a complete orthonormal system of periodic functions $u_n^i(x)$:

$$\pi^i(x) = \Sigma_n{}' u_n^i(x) q^n \tag{3.4}$$

where $n = (k, \bar{n})$ runs over the flavours $k = 1 \dots N-1$ and over the modes

$\bar{n} = (\bar{n}_1, \dots, \bar{n}_d), \quad \bar{n}_\mu \in \mathbb{Z}.$

The form $u_n^i(x) = \delta_k^i u_{\bar{n}}(x)$ insures the remaining $O(N-1)$ symmetry and the prime indicates that the zero mode is excluded, in accordance with eq. (2.9). The mode functions obey

$$\frac{1}{V}\int u_n^i{}^*(x) u_m^i(x) dx \quad = \quad \delta_{nm} \tag{3.5}$$

$$\frac{1}{V}\Sigma'_{nm}u^i_n(x)u^k_m(y) \;=\; \delta^{ik}\left(\delta(x-y)-\frac{1}{V}\right) \tag{3.6}$$

where $1/V$ has to be subtracted in eq. (3.6) because of the missing zero mode. From relation (2.10) we see that the coefficients q^n are of order $O(L^{1-d/2})$.

In terms of the variables Ω, $\vec{\pi}(x)$ the metric takes the form:

$$ds^2 = \frac{1}{V}\int_V [d\vec{\pi}(x) - \omega\vec{\pi}(x)]^2 dx \qquad (\omega = d\Omega\cdot\Omega^T = -\omega^T = \text{ inf. rotation matrix}).$$

Inserting the decomposition (3.4) this becomes:

$$\begin{aligned} ds^2 \;&=\; g_{mn}\delta q^m\delta q^n + 2g_{mi}\delta q^m\omega^{0i} + g_{ik}\omega^{0i}\omega^{0k} \\ \text{where} \qquad & \delta q^n = dq^n - \frac{1}{V}\omega^{ik}\int u^i_n(x)\pi^k(x)dx \\ & g_{mn} = \delta_{mn} + \frac{1}{V}\int \frac{u^i_m(x)u^k_n(x)\pi^i(x)\pi^k(x)}{[\pi^0(x)]^2}dx \\ & g_{ni} = \frac{1}{V}\int u^k_n(x)\left(\delta_{ik}\pi^0(x) + \frac{\pi^i(x)\pi^k(x)}{\pi^0(x)}\right)dx \\ & g_{ik} = \delta_{ik} + \frac{1}{V}\int [\pi^i(x)\pi^k(x) - \delta_{ik}\underline{\pi}^2(x)]dx \end{aligned}$$

(with summation over double indices). Accordingly, the regularized volume element is given by $[d\vec{S}] = \sqrt{\mathrm{g}}\ [d\pi]'\Pi_{i=1}^{N-1}\omega^{0i}$, where $[d\pi]' \doteq \prod'_n dq^n$ and

$$\mathrm{g} = det\begin{pmatrix} g_{mn} & g_{mi} \\ g_{kn} & g_{ki} \end{pmatrix} .$$

Note that the volume element only involves the components ω^{0i} of ω. From definition (3.3) we get $d\vec{m}\cdot d\vec{m} = \Sigma_{i=1}^{N-1}(\omega^{0i})^2$. This shows that their product is the volume element on the unit sphere: $\Pi_{i=1}^{N-1}\omega^{0i} = d\mu(\vec{m})$.

The determinant g can be evaluated perturbatively. $\sqrt{\mathrm{g}}$ is an H-independent factor, so we need it only to the 2^{nd} order, i.e. to $(q^n)^4$. [16]

We first write it in the form

$$\mathrm{g} = det(1+\varepsilon) \qquad \varepsilon = \begin{pmatrix} \varepsilon_{mn} & \varepsilon_{mi} \\ \varepsilon_{kn} & \varepsilon_{ki} \end{pmatrix} \doteq (\varepsilon_{ab})$$

where all the matrix elements $\varepsilon_{ab} = g_{ab} - \delta_{ab}$ have the magnitude L^{2-d}. Hence:

$$\sqrt{\mathrm{g}} = (e^{tr\ ln(1+\varepsilon)})^{1/2} = e^{\frac{1}{2}tr\varepsilon - \frac{1}{4}tr\varepsilon^2} + O((L^{2-d})^3)$$

[16] In order to be very precise we had to say that actually in 0^{th} order the integral over $[d\pi]'$ does not even occur, so g=1 is already the first order and the numeration of the orders is shifted by one. But such a notation would cause considerable confusion.

This is the form of $\sqrt{g}$ we need in order to add $-ln\sqrt{g}$ to the action.

Gaussian integration yields the contraction rule

$$< \pi^i(x)\pi^k(y) >= \delta^{ik}\frac{1}{F^2}G(x-y) , \tag{3.7}$$

where $G(x-y)$ is the propagator, which satisfies

$$-\partial^2 G(x) = \delta(x) - \frac{1}{V} . \tag{3.8}$$

The measure can be expressed in terms of G. Hence *all* the contributions to the partition function Z are determined by the propagator G and its derivatives.

The most obvious choice for the mode functions $u^i_n(x)$ is plane waves (Fourier decomposition). We are going to refer to this decomposition in the forthcoming when we deal with momenta p_n : $p_n u^k_n = -i\nabla u^k_n$. This implies the form

$$G(x) = \frac{1}{V}\sum_{\bar{n}}{}' \frac{e^{ip_{\bar{n}}x}}{p^2_{\bar{n}}} \tag{3.9}$$

(again the prime at the summation sigma indicates that we omit the zero mode). The choice of the complete orthonormal system is unimportant and we will often refer to plane waves only for convenience.

The traces in $ln\sqrt{g}$, however, contain UV divergences – e.g. the undefined term $\delta(0)$ – so the system needs regularization.

Many regularization schemes are known. They all violate important physical properties – if this could be avoided, regularization would not be needed. Since the violated properties vary from one regularization scheme to an other, the question of their equivalence in the final limit is highly non-trivial.

Our model is very sensitive for the type of regularization; e.g. a sharp cutoff in momentum space or the Pauli-Villars regularization turn out to be unsuitable, see subappendix A.1.

A better possibility is *dimensional regularization* [1]. It has been used in [9] and [14], and also in this work we apply it.

Its essential peculiarity can be demonstrated if we decompose the two-point Green function into the limit at infinite volume and a volume-dependent correction:

$$G(x-y) = G(x-y)|_{V\to\infty} + g(x-y) . \tag{3.10}$$

The first term is a distribution that takes in Fourier decomposition the form $\frac{1}{(2\pi)^d}\int d^dp \frac{e^{ip(x-y)}}{p^2}$. Regularization reduces it to a function $G^\Lambda(x-y)$, whose Laplacian is a regularized δ-function :

$$\delta^{\Lambda}(x-y) \doteq -\partial^2 G^{\Lambda}(x-y) \tag{3.11}$$

At $x = y$ this is in general a function of the regularization parameter that diverges when the parameter is removed, whereas in the particular case of dimensional regularization it simply vanishes. This happens to all the singularities, which – in case of a momentum cutoff – contain powers of the cutoff (such as $G^{\Lambda}(0)$ for $d > 2$); there remain, however, logarithmic singularities (like $G^{\Lambda}(0)$ for $d = 2$). For a discussion of dimensional regularization on compact manifolds, see [17].

Whenever we deal with the singularity structurerefpropzerleg) of the propagator and refer to dimensional regularization. For technical reasons, however, we are going to consider a more general construction which keeps track also of the leading power divergence of each term occurring in other regularization schemes. We will see that the essential properties of the result still hold for this generalized singularity structure: we show in appendix A that perturbative renormalization still works and in appendix B we verify the invariance of the entire partition function under the field transformations (2.11) ... (2.13). As far as the power divergences are concerned, we consider only those with the highest possible power of the UV cutoff (that we denote by Λ, see appendix A). In divergences of the second order, e.g., we only include the term $\propto \Lambda^{2(d-2)}$ and drop the one $\propto \Lambda^{d-2}$ which is sensitive to the used regularization. Of course the logarithmic divergences that also occur in dimensional regularization are not discarded.

As a consequence, throughout the main part of this work we assume $\delta^{\Lambda}(x)$ to act under the integral like an exact δ-function. This means an interchange of limits, which is justified for dimensional regularization, but which would be dangerous in other regularizations (the results without of this assumption are discussed in subappendix A.1).

So we will permit ourselves the luxury of including power divergences such as $\delta^{\Lambda}(0)$; they will reveal some aspects hidden by dimensional regularization, in particular it provides us with very significant consistency tests.

Using the relation,

$$\frac{1}{V}\sum_{n}{}' u_n^i(x) u_n^k(y)|_{reg.} = \delta^{ik}\left(\delta^{\Lambda}(x-y) - \frac{1}{V}\right) , \tag{3.12}$$

a lengthy calculation leads to:

$$\begin{aligned} ln\sqrt{\mathbf{g}} \simeq & \frac{V\delta^{\Lambda}(0) - N + 1}{2V}\int \underline{\pi}^2(x)dx + \frac{2V\delta^{\Lambda}(0) - N + 1}{8V}\int (\underline{\pi}^2(x))^2 dx \\ & -\frac{N-1}{8}\left(\frac{1}{V}\int \underline{\pi}^2(x)dx\right)^2 \end{aligned} \tag{3.13}$$

As a consistency test we compare our result (3.13) to the one of [14] (section 6), where –

also in the framework of dimensional regularization – the Faddeev-Popov method yields [17] :

$$Z = \mathrm{N_0} \int d\mu(\vec{m}) \int [d\pi]' |\vec{M}|^{N-1} e^{-S} \quad .$$

So $|\vec{M}|^{N-1}$ should be the measure for $\delta^\Lambda(0) = 0$. With $|\vec{M}| = \frac{1}{V}\int \pi^0 dx$ we get:

$$\begin{aligned} |\vec{M}|^{N-1} &= e^{(N-1)\, ln(\frac{1}{V}\int \sqrt{1-\underline{\pi}^2 dx})} \\ &\simeq exp\Big((N-1)\Big[-\frac{1}{2V}\int \underline{\pi}^2 dx - \frac{1}{8}\{\frac{1}{V}\int \underline{\pi}^2 dx\}^2 - \frac{1}{8V}\int \{\underline{\pi}^2\}^2 dx\Big]\Big) \end{aligned} \tag{3.14}$$

This is indeed the $\delta^\Lambda(0)$-independent part of $\sqrt{\mathrm{g}}$, so we can affirm the consistency and write:

$$\sqrt{\mathrm{g}}\ [d\pi]' = [dq]|\vec{M}|^{N-1} \ ; \quad [dq] \simeq e^{\frac{1}{4}\delta^\Lambda(0)[2\int \underline{\pi}^2 dx + \int (\underline{\pi}^2)^2 dx]}[d\pi]'.$$

In contrast to the Faddeev-Popov method, the integration over the collective variable Ω here only extends over a subset of the rotation group. This difference is inessential, however, for the following reason. Perform the change of variables $\pi^i(x) \to R^{ik}\pi^k(x)$, where R is an element of the little group belonging to $\vec{e} = (1, \underline{0})$, and replace Z by an average over the little group. Both, the magnetization and the measure $[dq]$ are invariant under this transformation. Since every $\Omega \in O(N)$ can be decomposed as $\Omega = R\Omega_{\vec{m}}$ (where $\Omega_{\vec{m}}$ belongs to the subset specified above), an integral over all elements of the little group followed by an integral over all directions $\vec{m}$ amounts to an integral of the full group. Finally, exploiting also $O(N)$ - invariance of the action: $S(\Omega^T\vec{\pi}, \vec{H}) = S(\vec{\pi}, \Omega\vec{H})$ we arrive at

$$Z = \mathrm{N} \int d\Omega \int [d\pi]' \sqrt{\mathrm{g}}\ e^{-\int \mathcal{L}(\vec{\pi}, \Omega\vec{H})dx} \tag{3.15}$$

where $d\Omega$ is the Haar measure on $O(N)$.

The procedure of [14] was first applied in [11] where Hasenfratz finds for lattice regularization the measure

$$[d\pi]' \prod_{i=1}^{N-1} \delta\Big(\sum_x \pi_x^i\Big)\ exp\Big(-\sum_x ln\pi_x^0\Big)\ exp\Big((N-1)ln\sum_x \pi_x^0\Big)$$

($\sum_x$: sum over lattice points).

In the continuum limit, the last factor becomes $const.\cdot|\vec{M}|^{N-1}$. The factor $exp(-\sum_x ln\pi_x^0)$ has been omitted in [14] because for dimensional regularization the exponent vanishes. In the present case, the leading power divergence corresponds to

$$\delta^\Lambda(0) = \frac{1}{a^d} \qquad (a \doteq \text{lattice constant}) \tag{3.16}$$

[17] We discuss here the analogue to the Polyakov measure (3.1). Of course a generalization with non-leading terms is possible for the Faddeev-Popov measure too.

so this factor becomes:

$$exp\left(-\delta^{\Lambda}(0)\int ln\pi^0 dx\right) = exp\left(\frac{1}{2}\delta^{\Lambda}(0)\ \left[\int \underline{\pi}^2 + \frac{1}{2}\int(\underline{\pi}^2)^2 dx \ldots\right]\right) = \frac{[dq]}{[d\pi]'}$$

The only difference between the lattice regularization in [11] and the regularized expansion in a complete orthonormal system of periodic functions used here is, as far as the functional measure is concerned, the representation for the regularized volume element associated with the non-zero modes.

We have seen explicitly to the second order that the measures of Polyakov and Faddeev-Popov coincide.

An important motivation for testing the Polyakov method is its applicability to a larger class of symmetry groups than it is the case for the FP measure. In particular it can be applied to every symmetry breaking $SU(N) \times SU(N) \to SU(N)$, in contrast to the FP method; for those symmetry breakings it is not known how one could apply the FP method if $N > 2$ (as we mentioned in section 1, the case $N = 3$ is of interest for chiral QCD with three flavors). [18]

4 1-Loop calculation of the partition function

For the 1-loop calculation of the partition function we have to consider:
$\sqrt{g}$ to the 0^{th} order ($= 1$), S_0, $\tilde{S}_0(H)$ and $\tilde{S}_1(H)$. Inserting this in (3.15) we get: [19]

$$Z \approx N\int d\Omega \int [d\pi]' e^{-\frac{F^2}{2}\int \partial_\mu \underline{\pi}\partial_\mu \underline{\pi} dx + \gamma\Omega^{00} - \frac{\gamma\Omega^{00}}{2V}\int \underline{\pi}^2 dx}$$

where we defined:

$$\gamma \doteq \Sigma H V \propto O(1) \quad .$$

The evaluation of the generating functional is more convenient than the study of individual Green functions [5]. $d = 3$ and $d = 4$ needn't be distinguished yet.

With the substitution $q_1^n \doteq \sqrt{\frac{F^2}{2}Vp_n^2 + \frac{\gamma\Omega^{00}}{2}}\ q^n \propto O(1)$, the second integral only contributes to the normalization constant, hence

$$Z \approx N\int d\Omega e^{\gamma\Omega^{00}} \int [d\pi]' e^{-\underline{\pi}_1^2} = N_1 \int d\Omega e^{\gamma\Omega^{00}} \left(\prod_n{}' \left[\frac{F^2}{2}Vp_n^2 + \frac{\gamma\Omega^{00}}{2}\right]\right)^{-\frac{1}{2}}$$

[18]That the transition to the unitary groups is not straightforward can be seen from the fact that for $SU(N) \times SU(N)$ there is not such a simple invariant as $|\vec{S}|$ in the case of $O(N)$, but there is a set of invariants, which is not easy to handle.

[19]Here, we clearly recognize the GB mass $m^2 = \frac{\Sigma H}{F^2}$.

Now we have to expand the Jacobian: $\frac{F^2}{2}V(p_n)^2 + \frac{\gamma\Omega^{00}}{2} \doteq \frac{1}{2}L^{d-2}(\alpha_n + \frac{\beta}{L^{d-2}})$, where $\alpha_n, \beta \propto O(1)$, α_n being independent of H.

$$\prod_n{}' \left\{\frac{1}{2}L^{d-2}(\alpha_n + \frac{\beta}{L^{d-2}})\right\} = \mathrm{N}'\left[1 + \frac{\beta}{L^{d-2}}\Sigma_n' \frac{1}{\alpha_n} + O((L^{2-d})^2)\right]$$

Inserting this, we obtain:

$$\mathrm{Z} \approx \mathrm{N_2} \int d\Omega e^{\gamma\Omega^{00}(1-\frac{1}{2F^2}\frac{1}{V}\sum_n'[1/p_n^2])} = \mathrm{N_3}Y_N\left(\gamma\left[1 - \frac{N-1}{2F^2}G_1\right]\right) \tag{4.1}$$

The last step is an identity of the 'modified Bessel function' [20], obeying

$$Y_N''(x) + \frac{N-1}{x}Y_N'(x) - Y_N(x) \equiv 0 \quad , \tag{4.2}$$

and $G_1 \doteq G(0) = \frac{1}{V}\sum_{\bar{n}}' \frac{1}{(p_{\bar{n}})^2}$. We expand this definition to $G_k \doteq \frac{\Gamma(k)}{V}\sum_{\bar{n}}' \frac{1}{(p_{\bar{n}}^2)^k}$, see figure 1.

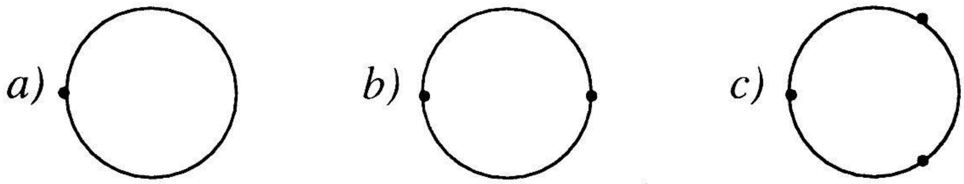

Figure 1: Diagrams representing the terms: a) G_1 b) G_2 c) G_3 etc.

Eq. (4.1) is the result Hasenfratz and Leutwyler found with a different method [14]. But the 'Jacobian method' demonstrated here is not applicable beyond the 1-loop level. So for the 3-loop calculation we will follow the straightforward procedure (expansion of the exponential, Wick contractions).

5 The partition function to 3 loops in 3 dimensions

We first consider $d = 3$. If we insert the results of section 2 and 3, the partition function takes the form :

$$\begin{aligned} \mathrm{Z} &\cong \mathrm{N} \int d\Omega \int \sqrt{\mathrm{g}}\, [d\pi]' e^{-S_0 - S_1 - S_2 - S_3(H)} \\ &\cong \mathrm{N} e^{\frac{\gamma^2}{F^6 V}k_3} \int d\Omega e^{\gamma\Omega^{00} + \frac{\gamma^2}{F^6 V}k_2(\Omega^{00})^2} \int [d\underline{\pi}]' e^{-\frac{1}{2}F^2\int(\partial_\mu \underline{\pi})^2 dx} \quad . \end{aligned}$$

[20]The identity is $\int d\Omega e^{x\Omega^{00}} \equiv \Gamma(\frac{N}{2})Y_N(x)$ for $\Omega \in O(N)$. The modified Bessel function has the expansion $Y_N(x) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(k+\frac{N}{2})}\left(\frac{x}{2}\right)^{2k}$, which is not oscillating, in contrast to the common Bessel function.

$$
\begin{aligned}
&\{1-\frac{F^2}{2}\int(\underline{\pi}\partial_\mu\underline{\pi})^2dx-\frac{\beta}{2V}\int\underline{\pi}^2dx\\
&+\frac{F^4}{8}(\int(\underline{\pi}\partial_\mu\underline{\pi})^2dx)^2+\frac{\beta^2-N+1}{8V^2}(\int\underline{\pi}^2dx)^2+\frac{F^2\beta}{4V}(\int(\underline{\pi}\partial_\mu\underline{\pi})^2dx)(\int\underline{\pi}^2dy)\\
&-\frac{F^2}{2}\int\underline{\pi}^2(\underline{\pi}\partial_\mu\underline{\pi})^2dx-\frac{\rho}{8V}\int(\underline{\pi}^2)^2dx\\
&-\frac{F^4\gamma\Omega^{00}}{16V}(\int(\underline{\pi}\partial_\mu\underline{\pi})^2dx)^2(\int\underline{\pi}^2dy)-\frac{F^2\beta^2}{16V^2}(\int(\underline{\pi}\partial_\mu\underline{\pi})^2dx)(\int\underline{\pi}^2dy)^2\\
&-\frac{\beta(\beta^2-3(N-1))}{48V^3}(\int\underline{\pi}^2dx)^3+\frac{F^2\gamma\Omega^{00}}{16V}(\int(\underline{\pi}\partial_\mu\underline{\pi})^2dx)(\int(\underline{\pi}^2)^2dy)\\
&+\frac{F^2\gamma\Omega^{00}}{4V}(\int\underline{\pi}^2dx)(\int\underline{\pi}^2(\underline{\pi}\partial_\mu\underline{\pi})^2dy)+\frac{\beta\rho}{16V^2}(\int\underline{\pi}^2dx)(\int(\underline{\pi}^2)^2dy)\\
&-\frac{\gamma\Omega^{00}}{16V}\int(\underline{\pi}^2)^3dx-\frac{\gamma\Omega^{00}k_1}{F^4V}\int\partial_\mu\underline{\pi}\partial_\mu\underline{\pi}dx\}
\end{aligned}
\tag{5.1}
$$

where

$$\beta\doteq\gamma\Omega^{00}+N-1-V\delta^\Lambda(0)\qquad\text{and}\qquad\rho\doteq\beta-V\delta^\Lambda(0)\quad . \tag{5.2}$$

In the expansion $\{\dots\}$ only the $\gamma-$ (i.e. $H-$) dependent contributions to the 3^{rd} order have to be included. The rest has been omitted, rsp. absorbed by the normalization constant. Accordingly for the coefficients β^2, $\beta(\beta^2-3(N-1))$ and $\beta\rho$ in the 3^{rd} order – that is: in the last four lines – only the γ-dependent part needs to be included.

We denote: $<\dots>\doteq\frac{\int[d\pi]'e^{-\frac{1}{2}F^2\int(\partial_\mu\underline{\pi})^2dx}(\;\dots\;)}{\int[d\pi]'e^{-\frac{1}{2}F^2\int(\partial_\mu\underline{\pi})^2dx}}$.

With $<1>=1$ there remain 15 terms to be evaluated. To this end we use:

$$
\begin{aligned}
<\frac{1}{V}\int\underline{\pi}\cdot\underline{\pi}\;dx> &= \frac{N-1}{F^2}G_1\\
<\int\partial_\mu\underline{\pi}\partial_\mu\underline{\pi}\;dx> &= -\frac{N-1}{F^2}V\partial^2G(0)=\frac{N-1}{F^2}\left(V\delta^\Lambda(0)-1\right)
\end{aligned}
$$

Together with the contraction rules this enables us to calculate the terms in eq. (5.1). We repeat that throughout these calculations (i.e. throughout section 5 and also section 6) $\delta^\Lambda(x)$ is treated like an exact δ-function under the integrals. For the moment the difference between the dimensionally regularized system we actually refer to and the more general singularity structure we also want to consider manifests itself only in the presence of the term $\delta^\Lambda(0)$.

From the contraction rules we obtain:

1) $$<-\frac{F^2}{2}\int(\pi^i\partial_\mu\pi^i)(\pi^k\partial_\mu\pi^k)dx>=-\frac{N-1}{2F^2}(V\delta^\Lambda(0)-1)G_1 \tag{5.3}$$

2) $$<-\frac{\beta}{2V}\int\underline{\pi}^2dx>=-\frac{\beta}{2F^2}(N-1)G_1 \tag{5.4}$$

Here we recognize the 1-loop result again.

$$3) \quad < \frac{F^4}{8}\Big(\int(\underline{\pi}\partial_\mu\underline{\pi})(\underline{\pi}\partial_\mu\underline{\pi})dx\Big)\Big(\int(\underline{\pi}\partial_\nu\underline{\pi})(\underline{\pi}\partial_\nu\underline{\pi})dy\Big) > \doteq (N-1)\cdot\mathcal{I} \tag{5.5}$$

We leave it like this because – as we will see – we don't need to know $\mathcal{I}$ explicitly. We just mention that it contains the term

$$J_2 \doteq \int(\partial_\mu G\partial_\mu G)^2 dx \quad , \tag{5.6}$$

which can not be expressed in terms of G-functions at $x=0$. [21]

$$4) \quad < \frac{\beta^2-N+1}{8V^2}\Big(\int\underline{\pi}^2 dx\Big)^2 > = \frac{\beta^2-N+1}{8F^4}(N-1)\Big[(N-1)G_1^2+\frac{2}{V}G_2\Big] \tag{5.7}$$

$$5) \quad \frac{F^2\beta}{4V} < \Big[\int(\pi^i\partial_\mu\pi^i)(\pi^j\partial_\mu\pi^j)dx\Big]\Big[\int\pi^k\pi^k dy\Big] >$$

There are two types of pairing that contribute: we can either contract the two derivated fields – this situation is similar to 4) – or contract each of them with one $\pi^k(y)$. The sum of these two contributions is :

$$= \frac{\beta}{4F^4}(V\delta^\Lambda(0)-1)(N-1)\Big[(N-1)G_1^2+\frac{2}{V}G_2\Big]+\frac{\beta}{2F^4}(N-1)G_1^2 \tag{5.8}$$

$$\begin{aligned} 6) \quad & -\frac{F^2}{2} < \int(\pi^i\pi^i)(\pi^j\partial_\mu\pi^j)(\pi^k\partial_\mu\pi^k)dx > \\ = \quad & -\frac{V\delta^\Lambda(0)-1}{2} < (\delta_{ii}\delta_{jj}+2\delta_{ij})\int(\pi^i\pi^i)(\pi^j\pi^j)dx > \\ = \quad & -\frac{V\delta^\Lambda(0)-1}{2F^4}(N^2-1)G_1^2 \end{aligned} \tag{5.9}$$

$$7) \quad -\frac{\rho}{8V} < \int(\pi^i\pi^i)(\pi^k\pi^k)dx > = -\frac{\rho}{8F^4}(N^2-1)G_1^2 \tag{5.10}$$

Now we have finished the first and second order. (A term of order ℓ can easily be recognized by the factor $F^{-2\ell}$.) Except for G_1, G_2 that were already represented by diagrams in fig. 1, there occurs G_1^2 , see fig. 2. J_2, which is included in $\mathcal{I}$ is represented in fig. 3.

Figure 2: Diagrams to: a) G_1^2 b) G_1^3 etc.

[21] For completeness we give the result nevertheless. The calculation yields:
$\mathcal{I} = \frac{1}{8F^4}\Big[\{(N-1)(V\delta^\Lambda(0))^2+8V\delta^\Lambda(0)-N-9\}G_1^2+\{2(V\delta^\Lambda(0))^2-4V\delta^\Lambda(0)+4\}\frac{1}{V}G_2+2(N-2)VJ_2\Big]$

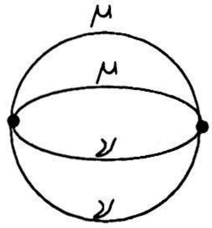

Figure 3: Graph for the term J_2. Here and in the following diagrams, greek letters denote partial derivatives.

The remaining 8 terms are of third order and require more effort. We show the treatment of the most difficult one that includes all the steps we used for the third order. Then we satisfy ourselves for the rest by quoting the result of each term.

$$8) \qquad -\frac{F^4\gamma\Omega^{00}}{16V} < \int(\pi^i\partial_\mu\pi^i)(\pi^j\partial_\mu\pi^j)dx\int(\pi^k\partial_\nu\pi^k)(\pi^\ell\partial_\nu\pi^\ell)dy\int(\pi^w\pi^w)dz >$$

We can reduce the effort remarkably by inserting the $\mathcal{I}$ of definition (5.5).

$$= -\mathcal{I}\cdot\frac{(N-1)^2}{2F^2}G_1 - \frac{F^4\gamma\Omega^{00}}{16V} < [(a\mathbf{b})(c\mathbf{d})][(ef)(gh)][(r|s)] > \qquad (5.11)$$

in an obvious notation: $a\dots s$: momenta, [ ]: same space-point, ( ) same flavor, bold/roman: $\partial_\mu, \partial_\nu$, $\;|$: pairing $r=s$ is excluded .
Thus r,s have to be paired with $a\dots h$. For this there are 7 equivalence classes:

i)	$a=r,\ c=s$	(class with 4 variants)	*v*)	$a=r,\ e=s$	(8)
ii)	$a=r,\ b=s$	(8)	*vi*)	$a=r,\ f=s$	(16)
iii)	$a=r,\ d=s$	(8)	*vii*)	$b=r,\ f=s$	(8)
iv)	$b=r,\ d=s$	(4)			

i) We further distinguish 3 subclasses and get:

$$-\frac{\gamma\Omega^{00}(N-1)}{4F^6V}\Big[(N-1)(V\delta^\Lambda(0))^2+2(V\delta^\Lambda(0)-1)+2\int\partial_{\mu\nu}G\partial_{\mu\nu}Gdu\Big]G_1G_2 =$$
$$-\frac{\gamma\Omega^{00}(N-1)}{4F^6V}\Big[(N-1)(V\delta^\Lambda(0))^2-2(N-3)V\delta^\Lambda(0)+N-5\Big]G_1G_2 \qquad (5.12)$$

ii) and *iii*) include the factor $\partial_\mu G_2 = 0$.
iv) Again there are 3 subclasses. Using $\int\partial_\mu G\partial_\mu G dx = G_1$ we find:

$$-\frac{\gamma\Omega^{00}(N-1)}{4F^6}\Big((N-1)(V\delta^\Lambda(0)-1)G_1^3+2(V\delta^\Lambda(0)-1)\frac{G_1G_2}{V}+2G_1^3\Big) \qquad (5.13)$$

In the remaining classes, r and s are connected in a mixed way to the $x-$ and $y-$block, i.e. the factor $\int G(x-z)G(y-z)dz$ occurs:

$$\frac{1}{V^2}\int\Big(\Sigma'_{\bar n}\frac{e^{ip_{\bar n}(x-z)}}{p^2_{\bar n}}\Big)\Big(\Sigma'_{\bar m}\frac{e^{ip_{\bar m}(y-z)}}{p^2_{\bar m}}\Big)dz = \frac{1}{V}\Sigma'_{\bar n}\frac{e^{ip_{\bar n}(x-y)}}{(p^2_{\bar n})^2} \doteq \dot G(x-y)$$

Figure 4: Diagrams for : a) J_3 b) Γ_3

Explication of the notation: $G(x) = \sum_{\vec{n}}' \frac{e^{ip_{\vec{n}}x}}{m^2+p_{\vec{n}}^2}|_{m=0}$; $\dot{G}(x) \doteq -\frac{d}{dm^2}G(x) \Rightarrow -\partial^2\dot{G} = G$, $\dot{G}(0) = G_2$, $\ddot{G}(0) = G_3$ etc. (see appendix C).
As in $i)$ we denote $u = x - y$. Then in $v) \dots vii)$ there occur integrals over 4 G-functions of u, whereby one dot and four partial derivatives (two by μ and two by ν) are distributed in all possible ways. So we have to deal with the terms:

$$
\begin{array}{ll}
\Gamma_0 \doteq \int G(\partial_{\mu\nu}G)^2\dot{G}du & \Gamma_1 \doteq \int \partial_\mu G\partial_\nu G\partial_{\mu\nu}G\dot{G}du \\
\Gamma_2 \doteq \int G\partial_\nu G\partial_{\mu\nu}G\partial_\mu\dot{G}du & \Gamma_3 \doteq \int \partial_\mu G\partial_\nu G\partial_\nu G\partial_\mu\dot{G}du \\
\Gamma_4 \doteq \int G^2\partial_{\mu\nu}G\partial_{\mu\nu}\dot{G}du & \Gamma_5 \doteq \int G\partial_\mu G\partial_\nu G\partial_{\mu\nu}\dot{G}du
\end{array}
\tag{5.14}
$$

With partial integrations all these quantities can be expressed in terms of one of them; we choose Γ_3 $(= \dot{J}_2/4)$.

$$
\begin{array}{lcl}
\Gamma_0 & = & -\frac{1}{6}G_1^3 + (\delta^\Lambda(0) - \frac{1}{2V})G_1G_2 - \frac{1}{4V^2}G_3 - \frac{1}{12V}J_3 + \Gamma_3 \\
\Gamma_1 & = & \frac{1}{2}(-\frac{1}{V}G_1G_2 + \frac{1}{2V^2}G_3 + \frac{1}{2V}J_3 + \Gamma_3) \\
\Gamma_2 & = & \frac{1}{6}(G_1^3 - \frac{1}{V}J_3) - \frac{1}{2}\Gamma_3 \\
\Gamma_4 & = & \frac{1}{3}(2G_1^3 + \frac{1}{V}J_3) + \Gamma_3 \\
\Gamma_5 & = & -\frac{1}{6}(G_1^3 + \frac{2}{V}J_3) - \frac{1}{2}\Gamma_3
\end{array}
\tag{5.15}
$$

where $J_3 \doteq \int G^3(u)du$.

Thus we can express everything through G_1, G_2, G_3, J_3 and Γ_3, represented in figures 1, 2 and 4. The occurrence of precisely these terms in the 3^{rd} order is consistent with the massive expansion in [9], as we show in appendix C.

We continue to decompose the equivalence classes in subclasses. Each of them contributes a summand to:

$$
\begin{array}{rl}
v) & -\frac{\gamma\Omega^{00}}{2F^6}(N-1)[(V\delta^\Lambda(0)-1)^2\frac{G_3}{2V^2} + N\Gamma_0 + (N+2)\Gamma_1] \\
vi) & -\frac{\gamma\Omega^{00}}{F^6}(N-1)[(V\delta^\Lambda(0)-1)\frac{1}{V}G_1G_2 + \frac{N+2}{6}(G_1^3 - \frac{1}{V}J_3) + (\frac{N}{2}-1)\Gamma_3)] \\
vii) & -\frac{\gamma\Omega^{00}}{2F^6}(N-1)[G_1^3 + N\Gamma_4 + (N+2)\Gamma_5]
\end{array}
$$

If we insert the identities of the Γ's and add up we arrive at

$$
\begin{array}{l}
-\frac{F^4\gamma\Omega^{00}}{16} < [(ab)(cd)][(ef)(gh)][(r|s)] >= -\frac{\gamma\Omega^{00}}{4F^6}(N-1) \cdot \\
\Big\{\Big[(N-1)V\delta^\Lambda(0) + \frac{N}{3} + \frac{17}{3}\Big]G_1^3 + \Big[(N-1)(V\delta^\Lambda(0))^2 + 12V\delta^\Lambda(0) - N - 13\Big]\frac{1}{V}G_1G_2 \\
+\Big[(V\delta^\Lambda(0))^2 - 2V\delta^\Lambda(0) + 2\Big]\frac{1}{V^2}G_3 - \frac{N+5}{3V}J_3 + 4(N-2)\Gamma_3\Big\}
\end{array}
\tag{5.16}
$$

9)...15) can be treated in the same way following Wick's theorem: determination of the classes that avoid summation over linear momenta, decomposition of each class in subclasses corresponding to the possibilities to pair the remaining fields. No further term is as lengthy as 8), and exept for 11) the results include only propagators at 0. We write them down such that each line stems from one equivalence class.

9) $$-\frac{\beta^2}{16F^6}(V\delta^\Lambda(0)-1)(N-1)\Big[(N-1)^2G_1^3+6(N-1)\frac{1}{V}G_1G_2+\frac{4}{V^2}G_3\Big]$$
$$-\frac{\beta^2}{4F^2}(N-1)\Big[(N-1)G_1^3+\frac{2}{V}G_1G_2\Big]$$
$$-\frac{\beta^2(N-1)}{2F^6V}G_1G_2$$

(There is a fourth class but its contribution vanishes.)

10) $$-\frac{\beta(\beta^2-3(N-1))}{48V^3}(N-1)\Big[(N-1)^2G_1^3+(N-1)\frac{6}{V}G_1G_2+\frac{4}{V^2}G_3\Big]$$

11) $$\frac{\gamma\Omega^{00}}{16F^6}(V\delta^\Lambda(0)-1)(N^2-1)\Big[(N-1)G_1^3+\frac{4}{V}G_1G_2\Big]$$
$$+\frac{\gamma\Omega^{00}}{12F^6}(N^2-1)\Big[5G_1^3-\frac{2}{V}J_3\Big]$$

For the latter we used : $\int G^2\partial_\mu G\partial_\mu G du=\frac{1}{3}(G_1^3-\frac{1}{V}J_3)$

12) $$\frac{\gamma\Omega^{00}}{4F^6}(V\delta^\Lambda(0)-1)(N^2-1)\Big[(N-1)G_1^3+\frac{4}{V}G_1G_2\Big]$$
$$+\frac{\gamma\Omega^{00}}{2F^6}(N^2-1)G_1^3$$

13) $$\frac{\beta\rho}{16F^6}(N^2-1)\Big[(N-1)G_1^3+\frac{4}{V}G_1G_2\Big]$$

11 *i*), 12 *i*), and 13) are essentially based on the same calculation.

14) $$-\frac{\gamma\Omega^{00}}{16F^6}(N^2-1)(N+3)G_1^3$$

15) $$-\frac{\gamma\Omega^{00}k_1}{F^6V}(N-1)(V\delta^\Lambda(0)-1)$$

Having completed the evaluation, we add everything up and the factor $<\{\dots\}>$ in eq. (5.1) takes the form:

$$<\{\dots\}>=1+(N-1)\Big[\mu_1+\mu_2+\gamma\Omega^{00}(\nu_1+\nu_2+\nu_3)+(\gamma\Omega^{00})^2(\rho_2+\rho_3)+(\gamma\Omega^{00})^3\sigma_3\Big]$$

where the indices correspond to the order of magnitude. If we lift this up to the exponent we get

$$<\{\dots\}>\;\cong\; const.\cdot exp\{(N-1)[(\alpha_1+\alpha_2+\alpha_3)\gamma\Omega^{00}+(\beta_2+\beta_3)(\gamma\Omega^{00})^2+\gamma_3(\gamma\Omega^{00})^3]\}$$

where the *const.* is $H-$independent and

$$\alpha_1=\nu_1,\qquad \alpha_2=\nu_2-(N-1)\mu_1\nu_1,\quad \alpha_3=\nu_3-(N-1)(\nu_1\mu_2+\mu_1\nu_2)+(N-1)^2\mu_1^2\nu_1$$

$$\beta_2 = \rho_2 - \frac{N-1}{2}\nu_1^2, \quad \beta_3 = \rho_3 - (N-1)(\mu_1\rho_2 + \nu_1\nu_2) + (N-1)^2\mu_1\nu_1^2$$
$$\gamma_3 = \sigma_3 - (N-1)\nu_1\rho_2 + \frac{(N-1)^2}{3}\nu_1^3$$

The most complicated contributions, μ_2 and ν_3 only occur in α_3. There the ugly term $\mathcal{I}$ cancels, so it was justified not to insert its evaluation.

Now we actually have a representation of Z, but we prefer to have the same form as in section 4, i.e. we want to transform the integral on the unit sphere in the iso-space, $\int d\Omega e^{\gamma\Omega^{00} + \frac{\gamma^2}{F^6 V}k_2(\Omega^{00})^2} exp\{\ldots\}$, to the form $e^{\delta_1'+\delta_2'+\delta_3'} \int d\Omega e^{\gamma\Omega^{00}(1+\epsilon_1'+\epsilon_2'+\epsilon_3')}$.

This can be realized making use of the differential equation (4.2), as we outline in appendix D. The result is

$$\delta_1' = 0, \quad \delta_2' = (N-1)\beta_2, \quad \delta_3' = (N-1)(\beta_3 - (N-1)\gamma_3) + \tfrac{k_2}{F^6}$$
$$\epsilon_1' = (N-1)\alpha_1, \quad \epsilon_2' = (N-1)(\alpha_2 - (N-1)\beta_2)$$
$$\epsilon_3' = (N-1)[\alpha_3 - (N-1)\beta_3 + (\gamma^2 + N(N-1))\gamma_3 + (N-1)^2\alpha_1\beta_2 - \tfrac{k_2}{F^6}]$$

Inserting everything we arrive at the final result:

$$\mathrm{Z} \cong \mathrm{N}\, e^{\frac{\gamma^2}{F^4}(\delta_2 + \frac{1}{F^2}\delta_3)} \; Y_N\left(\gamma\Big[1 + \frac{\epsilon_1}{F^2} + \frac{\epsilon_2}{F^4} + \frac{\epsilon_3}{F^6}\Big]\right) \tag{5.17}$$

$$\epsilon_1 = -\frac{N-1}{2}G_1 \tag{5.18}$$

$$\delta_2 = \frac{N-1}{4V}G_2 \tag{5.19}$$

$$\epsilon_2 = \frac{(N-1)(N-3)}{8}\left(-G_1^2 + \frac{2}{V}G_2\right) \tag{5.20}$$

$$\delta_3 = \frac{N-1}{4}\left(\frac{N-3}{V}G_1G_2 - \frac{2N-5}{3V^2}G_3\right) + \frac{k_2+k_3}{V} \tag{5.21}$$

$$\begin{aligned}\epsilon_3 = {} & (N-1)\Big\{\frac{(N-3)(3N-7)}{48}(-G_1^3 + \frac{6}{V}G_1G_2) - ((N-3)(N-4)+\gamma^2)\frac{G_3}{12V^2} \\ & -\frac{N-3}{12V}J_3 - (N-2)\Gamma_3 - \frac{(V\delta^{\Lambda}(0)-1)k_1 + k_2}{V}\Big\}\end{aligned} \tag{5.22}$$

It is not surprising that all the contributions are proportional to the number of flavors, $N-1$, except for the terms with k_2 and k_3, which do not stem from a coupling of $\underline{\pi}$-fields. On the other hand we notice a repeated appearance of the factors $(N-3)$ and $(3N-7)$ that can only be interpreted in the context of renormalization. Even more striking is that there are many terms occurring in the course of the calculation that cancel at the end. Already in the first order the (H-independent) $\delta^{\Lambda}(0)$-contributions of the measure and the action just compensate each other. Further examples for such terms are in $\delta_2 : NG_1^2$ and G_1^2. We write $\delta_2 : \quad (N,1)G_1^2 \quad$ etc.

$$\begin{array}{ll} \delta_2 : & (N,1)G_1^2 \qquad \varepsilon_2 : \quad (N,1)V\delta^\Lambda(0)G_1^2, \quad (N,1)\delta^\Lambda(0)G_2, \quad (N^2,N,1)VJ_2 \\ \delta_3 : & (N^2,N,1)V\delta^\Lambda(0)G_1^3, \quad (N^3,N^2,N,1)G_1^3, \quad (N,1)\delta^\Lambda(0)G_1G_2, \quad N^3G_1G_2/V \\ \varepsilon_3 : & (N^2,N,1)(V\delta^\Lambda(0))^2G_1^3, \quad (N^2,N,1)V\delta^\Lambda(0)G_1^3, \quad (N,1)(\delta^\Lambda(0))^2VG_1G_2 \\ & (N^2,N,1)\delta^\Lambda(0)G_1G_2, \quad (N,1)\delta^\Lambda(0)G_3/V, \quad (N,1)VJ_2G_1 \end{array}$$

This long list of canceled terms exhibits a remarkable property of the system. The vast part of it would have been ignored if we would have restricted ourselves to the terms that really appear in dimensional regularization. We will see that all these cancellations of measure and Lagrangian are strictly required by the perturbative renormalizability of the structure that includes the leading power divergences.

6 3-Loop expansion of the partition function for d=4

The difference in the expansion of $d = 3$ and $d = 4$ is due to the magnitude of the terms with the coupling constants $k_1 \dots k_6$. Generally, the terms with $k_1 \dots k_5$ are all $\propto \frac{1}{V}$, which means of $3^{rd}, 2^{nd}$ order for $d = 3,4$ respectively, and the k_6-term is $\propto L^{-4}$, i.e. of $4^{th}, 2^{nd}$ order for $d = 3,4$. Thus for $d = 3$ only the H-dependent terms with k_1, k_2, k_3 had to be taken into account. For $d = 4$, however, those terms contribute to the 2^{nd} and 3^{rd} order and additionally contribute mixed terms with the first order in the exponential expansion. The latter is also true for the $k_4 \dots k_6$-terms, so we have to include three coupling constants more than in section 5.

We write down the expansion of the partition function as in (5.1):

$$\begin{aligned} Z \cong\ & \mathrm{N}e^{\frac{\gamma^2 k_3}{F^4 V}} \int d\Omega e^{\gamma\Omega^{00} + \frac{(\gamma\Omega^{00})^2 k_2}{F^4 V}} \int [d\pi]' e^{-\frac{F^2}{2}\int (\partial_\mu \underline{\pi})^2 dx} . \\ & \{1 - \frac{\beta}{2V}\int \underline{\pi}^2 dx - \frac{F^2}{2}\int (\underline{\pi}\partial_\mu\underline{\pi})^2(1+\underline{\pi}^2)dx - \frac{\rho}{8V}\int (\underline{\pi}^2)^2 dx - \frac{N-1-\beta^2}{8V^2}\left(\int \underline{\pi}^2 dx\right)^2 \\ & + \frac{F^4}{8}\left(\int (\underline{\pi}\partial_\mu\underline{\pi})^2 dx\right)^2 - \frac{\gamma\Omega^{00}k_1}{F^2 V}\int \partial_\mu\underline{\pi}\partial_\mu\underline{\pi}dx + \frac{\beta F^2}{4V}\int \underline{\pi}^2 dx \int (\underline{\pi}\partial_\mu\underline{\pi})^2 dy \\ & - \frac{1}{4}\int [k_4(\partial_\mu\underline{\pi}\partial_\mu\underline{\pi})^2 + k_5(\partial_\mu\underline{\pi}\partial_\nu\underline{\pi})^2 + k_6\frac{2}{F^2}(\partial_\mu\partial^2\underline{\pi})^2]dx \\ & - \frac{\beta^3}{48V^3}\left(\int \underline{\pi}^2 dx\right)^3 - \frac{\beta^2 F^2}{16V^2}\left(\int \underline{\pi}^2 dx\right)^2 \int (\underline{\pi}\partial_\mu\underline{\pi})^2 dy - \frac{\beta F^4}{16V}\left(\int \underline{\pi}^2 dx\right)\left(\int (\underline{\pi}\partial_\mu\underline{\pi})^2 dy\right)^2 \\ & + \frac{\beta F^2}{4V}\int \underline{\pi}^2 dx \int (\underline{\pi}\partial_\mu\underline{\pi})^2\underline{\pi}^2 dy + \frac{\beta\rho}{16V^2}\int \underline{\pi}^2 dx \int (\underline{\pi}^2)^2 dy + \frac{\beta(N-1)}{16V^3}\left(\int \underline{\pi}^2 dx\right)^3 \\ & + \frac{\beta\gamma\Omega^{00}k_1}{2F^2V^2}\int \underline{\pi}^2 dx \int \partial_\mu\underline{\pi}\partial_\mu\underline{\pi}dy + \frac{\gamma\Omega^{00}}{8V}\int \underline{\pi}^2 dx \int [k_4(\partial_\mu\underline{\pi}\partial_\mu\underline{\pi})^2 + k_5(\partial_\mu\underline{\pi}\partial_\nu\underline{\pi})^2]dy \\ & + \frac{F^2\rho}{16V}\int (\underline{\pi}\partial_\mu\underline{\pi})^2 dx \int (\underline{\pi}^2)^2 dy + \frac{\gamma\Omega^{00}k_1}{2V}\int (\underline{\pi}\partial_\mu\underline{\pi})^2 dx \int (\partial_\nu\underline{\pi})^2 dy \end{aligned}$$

$$+\frac{\gamma\Omega^{00}k_1}{2F^4V^2}\int \underline{\pi}^2(\partial_\mu\underline{\pi})^2dx - \frac{\gamma\Omega^{00}}{16V}\int(\underline{\pi}^2)^3dx - \frac{\gamma\Omega^{00}k_1}{F^2V}\int(\underline{\pi}\partial_\mu\underline{\pi})^2dx - \frac{(\gamma\Omega^{00})^2k_2}{F^4V^2}\int\underline{\pi}^2dx$$
$$+\frac{\gamma^2k_2}{F^4V^2}\Omega^{0i}\Omega^{0k}\int\pi^i\pi^kdx + \frac{\gamma\Omega^{00}k_6}{4VF^2}\int\underline{\pi}^2dx\int(\partial_\mu\partial^2\underline{\pi})^2dy\} \tag{6.1}$$

where β and ρ keep the meaning given in definition (5.2).

Most of these terms have already been evaluated in section 5 (identically or only with different coefficients). We just discuss the new ones.

First we define two new distributions :

$$D^\Lambda(x) \doteq -\partial^2\delta^\Lambda(x) = \partial^2\partial^2G^\Lambda(x) = \partial^2\partial^2G(x) \quad \text{and} \quad \Delta^\Lambda(x) \doteq -\partial^2D^\Lambda(x) \quad . \tag{6.2}$$

In the framework of our decomposition $G = G^\Lambda + g$, where $-\partial^2G = \delta^\Lambda - 1/V$, we find $\partial^2g(x) \equiv 1/V$ and $D^\Lambda(x)$, $\Delta^\Lambda(x)$ are volume independent.

At $x = 0$ they are pure power divergences, stronger than those introduced before:

$$D^\Lambda(0) = \frac{1}{V}\sum_{\bar{n}} p_{\bar{n}}^2 \quad , \qquad \Delta^\Lambda(0) = \frac{1}{V}\sum_{\bar{n}}(p_{\bar{n}}^2)^2$$

They vanish in dimensional regularization but as in the case of $\delta^\Lambda(0)$ we do not omit them.

In addition we define:

$$G_{\mu\nu} \doteq \partial_\mu\partial_\nu G(x)|_{x=0}$$

Then we find to the

2^{nd} order

$-\frac{k_4}{4} < \int(\partial_\mu\underline{\pi}\partial_\mu\underline{\pi})(\partial_\nu\underline{\pi}\partial_\nu\underline{\pi})dx >= -\frac{(N-1)k_4}{4F^4V}\Big[(N-1)(V\delta^\Lambda(0)-1)^2 + 2V^2(G_{\mu\nu})^2\Big]$

$-\frac{k_5}{4} < \int\partial_\mu\pi^i\partial_\nu\pi^i\partial_\mu\pi^k\partial_\nu\pi^kdx >= -\frac{(N-1)k_5}{4F^4V}\Big[(V\delta^\Lambda(0)-1)^2 + NV^2(G_{\mu\nu})^2\Big]$

$-\frac{k_6}{2F^2} < \int\partial_\mu\partial^2\underline{\pi}\partial_\mu\partial^2\underline{\pi}dx >= k_6\frac{N-1}{2F^2}V\Delta^\Lambda(0)$

3^{rd} order

$< \frac{\beta\gamma\Omega^{00}k_1}{2F^2V}\int\underline{\pi}^2dx\int\partial_\mu\underline{\pi}\partial_\mu\underline{\pi}dy >= \frac{\beta\gamma\Omega^{00}k_1}{2F^6V}(N-1)\Big[(N-1)(V\delta^\Lambda(0)-1)+2\Big]G_1$

$\frac{\gamma\Omega^{00}k_4}{8V} < \int\int\pi^i(x)\pi^i(x)\partial_\mu\pi^j(y)\partial_\mu\pi^j(y)\partial_\nu\pi^k(y)\partial_\nu\pi^k(y)dxdy >$
$= \frac{\gamma\Omega^{00}k_4}{8F^6V}(N-1)\Big[\{(N-1)(V\delta^\Lambda(0)-1)^2 + 4(V\delta^\Lambda(0)-1) + 2V^2(G_{\mu\nu})^2\}(N-1)G_1$
$+8VG_{\mu\nu}\dot{G}_{\mu\nu}\Big]$

where we have used : $\int\partial_\mu G\partial_\nu Gdu = \dot{G}_{\mu\nu}$.

$\frac{\gamma\Omega^{00}k_5}{8V} < \int\int\pi^i(x)\pi^i(x)\partial_\mu\pi^j(y)\partial_\nu\pi^j(y)\partial_\mu\pi^k(y)\partial_\nu\pi^k(y)dxdy >$
$= \frac{\gamma\Omega^{00}k_5}{8F^6V}(N-1)\Big[\{(N-1)(V\delta^\Lambda(0)-1)^2 + 4(V\delta^\Lambda(0)-1) + N(N-1)V^2(G_{\mu\nu})^2\}G_1$
$+4NG_{\mu\nu}\dot{G}_{\mu\nu}\Big]$

$\frac{\gamma\Omega^{00}k_1}{2V} < \int(\underline{\pi}\partial_\mu\underline{\pi})(\underline{\pi}\partial_\mu\underline{\pi})dx\int\partial_\nu\underline{\pi}\partial_\nu\underline{\pi}dy >$
$= \frac{\gamma\Omega^{00}k_1}{2F^6V}(N-1)\Big[(N-1)(V\delta^\Lambda(0)-1)^2 + 4(V\delta^\Lambda(0)-1)\Big]G_1$

where we inserted $\int \partial_{\mu\nu}G\partial_{\mu\nu}G du = \delta^\Lambda(0) - 1/V$.

$$\frac{\gamma\Omega^{00}k_1}{2F^2V} < \int \underline{\pi}^2(\partial_\mu\underline{\pi}\partial_\mu\underline{\pi})dx >= \frac{\gamma\Omega^{00}k_1}{2F^6V}(N-1)^2(V\delta^\Lambda(0)-1)\; G_1$$

$$\frac{\gamma^2k_2}{F^4V^2} < \int[\Omega^{0i}\Omega^{ok}\pi^i\pi^k - (\Omega^{00})^2\underline{\pi}^2]dx >= \frac{\gamma^2k_2}{F^6V}[1-N(\Omega^{00})^2]\; G_1$$

$$\frac{\gamma\Omega^{00}k_6}{4VF^2} < \underline{\pi}^2dx \int \partial_\mu\partial^2\underline{\pi}\partial_\mu\partial^2\underline{\pi}dy >= \frac{\gamma\Omega^{00}k_6}{4F^6}(N-1)\Big[-(N-1)G_1V\Delta^\Lambda(0)+2D^\Lambda(0)\Big]$$

So we found in connection with the new coupling constants $k_4 \dots k_6$ also new kinds of terms: $(G_{\mu\nu})^2$ and $G_{\mu\nu}\dot{G}_{\mu\nu}$ - see fig. 5 - are associated with k_4, k_5 and include a regular contribution. Their occurrence is again consistent with [9], see appendix C.

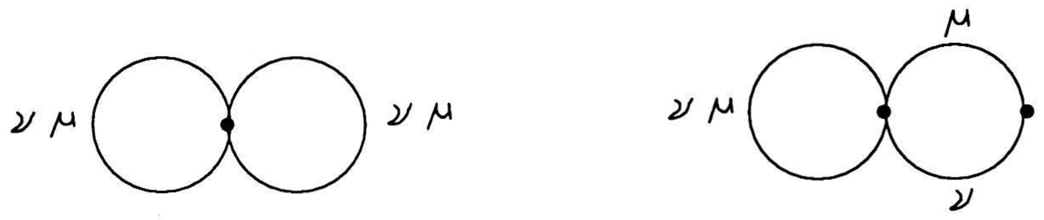

Figure 5: Diagrams for $G_{\mu\nu}G_{\mu\nu}$ and $G_{\mu\nu}\dot{G}_{\mu\nu}$

k_6 contributes the remaining new terms $V\Delta^\Lambda(0)$, $G_1V\Delta^\Lambda(0)$ and $D^\Lambda(0)$ that vanish in dimensional regularization (i.e. they can not be found in [9]); so the coupling constant k_6 is physically irrelevant. (It can easily be seen that this is true to all orders of magnitude.)

But the new terms don't prevent us from following exactly the same procedure as in section 5. We add all the summands of the integrand of Z, heave it in the exponent and remove the parts with $(\Omega^{00})^2$ and $(\Omega^{00})^3$ according to appendix D. Thus we arrive at the result:

$$Z \;\cong\; \mathrm{N}e^{\frac{\gamma^2}{F^4}(\delta_2+\frac{1}{F^2}\delta_3)}\quad Y_N\left(\gamma[1+\frac{\varepsilon_1}{F^2}+\frac{\varepsilon_2}{F^4}+\frac{\varepsilon_3}{F^6}]\right) \tag{6.3}$$

$$\varepsilon_1 \;=\; -\frac{N-1}{2F^2}G_1 \tag{6.4}$$

$$\delta_2 \;=\; \frac{N-1}{4V}G_2 + \frac{k_2+k_3}{V} \tag{6.5}$$

$$\varepsilon_2 \;=\; -\frac{(N-1)(N-3)}{8}\left(G_1^2 - \frac{2}{V}G_2\right) - \frac{N-1}{V}\Big[(V\delta^\Lambda(0)-1)k_1 + k_2\Big] \tag{6.6}$$

$$\delta_3 \;=\; (N-1)\left(\frac{N-3}{4V}G_1G_2 - \frac{2N-5}{12V^2}G_3 + \frac{k_1-k_2}{V}G_1\right) \tag{6.7}$$

$$\begin{aligned}\varepsilon_3 \;=\; & (N-1)\Big\{-\frac{(N-3)(3N-7)}{48}(G_1^3 - \frac{6}{V}G_1G_2) \\ & -\frac{(N-3)(N-4)+\gamma^2}{12V^2}G_3 - \frac{N-3}{12V}J_3 - (N-2)\Gamma_3\end{aligned}$$

$$+\frac{1}{2V}\Big[(N-1)V\delta^{\Lambda}(0)k_1-(N+1)(k_1-k_2)+(V\delta^{\Lambda}(0)-1)\{(N-1)k_4+k_5\}\Big]G_1$$
$$+(k_4+\frac{N}{2}k_5)G_{\mu\nu}\dot{G}_{\mu\nu}+\frac{1}{2}k_6D^{\Lambda}(0)\Big\} \tag{6.8}$$

Now we have completed the evaluation of the partition functions for $d=3$ and $d=4$ to the 3^{rd} order. We repeat that for dimensional regularization we can omit $\delta^{\Lambda}(0)$ and $D^{\Lambda}(0)$, and as a consequence the coupling constant k_6.

If we want the 2-loop result we simply neglect δ_3 and ε_3. Then the difference of $d=3$ and $d=4$ are only the additional terms of the latter with k_1, k_2, k_3. The 2-loop result has been given already in [14]. There one finds in the appendices also a description of methods and results for the numerical determination of g_1 and g_2, the regular parts of G_1, G_2.

As *new* cancellations for $d=4$ we can report:

$$\begin{array}{ll}
\delta_3: & (N^2,N,1)\delta^{\Lambda}(0)k_1G_1 \\
\varepsilon_3: & (N^2,N,1)\delta^{\Lambda}(0)k_1G_1, \quad (N^2,N)k_1G_1/V \\
 & (N^2,N,1)V(\delta^{\Lambda}(0))^2G_1(k_4,k_5), \quad N^2(\delta^{\Lambda}(0),\frac{1}{V})k_4G_1, \quad N(\delta^{\Lambda}(0),\frac{1}{V})k_5G_1 \\
 & (N,1)k_4V(G_{\mu\nu})^2G_1, \quad (N^2,N,1)k_5(G_{\mu\nu})^2G_1, \quad (N^2,N,1)G_1k_6V\Delta^{\Lambda}(0)
\end{array}$$

Again the power singularities are strongly represented in this list, in particular there are very few $\delta^{\Lambda}(0)$ in the final result – even though we met whole polynomials of $V\delta^{\Lambda}(0)$ in the calculation – and $\Delta^{\Lambda}(0)$ does not show up at all.

All this is part of a sensitive consistency check, because only due to this cancellations even the generalized singularity structure we consider is perturbatively renormalizable, see appendix A.

7 Perturbative renormalization

Now we are going to verify that the results obtained in sections 5 and 6 fulfill the constraints imposed by renormalizability [22]. We repeat that for the singularity structure we refer to the decomposition (3.10). Most singularities of our system of leading divergences are power divergent (i.e. physically irrelevant). In this section we restrict the consideration to the relevant singularities that really occur in the dimensional regularized system. The remarkable property that even the generalized structure we dealt with so far is perturbatively renormalizable is discussed in appendix A.

We apply the "mass independent" renormalization prescription, that sets all the finite parts of the counter terms zero [20]. This prescription is preferable especially in view of the

[22] For brevity here we write sometimes "renormalizability" where we actually mean "perturbative renormalizability", as specified in section 1.

β-functions [19]. In the path integral formalism, the rescaling of the fields to be integrated over can be absorbed by rescaling the source accordingly [19]. But since H only occurs in the product ΣH, it suffices to renormalize all the coupling constants, where $\Sigma_r H$ actually means $(\Sigma H)_r$ (the subscript r denotes the renormalized quantities). On each of the three levels, the couplings should be able to absorb the divergences without picking up a V-dependence (see section 2). The volume independence of the counter terms is the constraint that provides us with the non-trivial check of our results.

For the subsequent discussion we introduce a measure for the degree of divergence. Let Λ be a characteristic regularization parameter with the dimension of momentum, so the non-regularized system corresponds to $\lim_{\Lambda\to\infty}$. (In the naive momentum cutoff "regularization", Λ would be the cutoff, or more generally a characteristic length of the support in momentum space, e.g. for a smooth cutoff. In a lattice regularization Λ would be proportional to the inverse lattice constant, etc.) Then we can express the degree of divergence in powers of Λ. We recall that for dimensional regularization only the singularities $\propto ln\Lambda$ remain and that we only consider them in this section.

It is required by the concept of low energy expansion and included in our Bessel representation that after renormalization the leading coupling constants Σ and F coincide with the bare couplings, so we don't need to renormalize them here.

This is not true, however, for the non-leading coupling constants k_j. There we make the ansatz:

$$k_{jr} = k_j + \kappa_{0,j} \qquad\qquad \kappa_{0,j} \propto ln\Lambda \ , \tag{7.1}$$

where in the counter terms $\kappa_{0,j}$ the "mass independent" renormalization prescription excludes additional (finite) terms.

The partition function has the form $\mathrm{Z} = \mathrm{N}e^{\frac{(\Sigma HV)^2}{F^4}\rho_2}\, Y_N(\Sigma HV\rho_1)$. Hence the renormalization has to provide:

$$\Sigma_r \cdot \rho_{1r} \;=\; \Sigma \cdot \rho_1 \tag{7.2}$$

$$\frac{\Sigma_r^2}{F_r^4} \cdot \rho_{2r} \;=\; \frac{\Sigma^2}{F^4} \cdot \rho_2 \tag{7.3}$$

where in ρ_{1r}, ρ_{2r} all the singularities are removed. We are going the evaluate these two equations order by order.

In the first order there are no counter terms available since only the leading coupling constants are involved. This is in accordance with the fact that the 1-loop result does not contain singularities; $G_1^\Lambda \propto \Lambda^{d-2}$ vanishes in dimensional regularization (both, for $d = 3$ and $d = 4$).

For the second and third order, $d = 3$ and $d = 4$ have to be discussed separately. We start with $\mathbf{d = 3}$:

$G_n|_{V\to\infty}$ diverges if $2n \le d$. Then we regularize it to [23] $G_n^\Lambda \propto \Lambda^{d-2n}$ and $G_n = G_n^\Lambda + g_n(V)$ This concerns for $d = 3$ just G_1, whereas $G_2, G_3 \dots$ are regular V-dependent functions.

As a consequence the partition function has no divergences even to the second order, in agreement with the observation that there are still no counter terms available.

For the remaining terms that contain integrations over V, a corresponding decomposition is more complicated. Let J be such a term: $J = \int_V T(x) d^d x$, which shall be brought to the form $J = j(V) + \sum_\ell J_\ell^\Lambda j_\ell(V)$, J_ℓ^Λ being V-independent divergences (that can be absorbed by the renormalized coupling constants), and j, j_ℓ being regular. T is some combination of G-functions. Their decomposition into $G^\Lambda(x) + g(x,V)$ yields the form $T = \sum_k T_k^\Lambda(x) \cdot t_k(x,V)$, where the t_k are regular functions, whereas the T_k^Λ depend on Λ but not V. If $T_k^\Lambda \propto \Lambda^{a_k}$ and $t_k \propto L^{-b_k}$, then $(a_k + b_k)$ will be fixed for all k, since Λ has the dimension of a momentum. This also means that close to the origin $T_k^\Lambda(x) \propto x^{-a_k}$. Thus only the summands with $a_k \ge d$ are really singular for $\Lambda \to \infty$ (if $t_k(0,V) \ne 0$) [24] , the rest contributes to j.

For the treatment of those singularities, let's call them $\int_V T_\ell^\Lambda(x) t_\ell(x,V) d^d x$, we apply a technique that was similarly used in [9]. Let S be a sphere around the origin inside the box V. If we decompose $\int_V \dots$ into $\int_S \dots + \int_{V-S} \dots$, only the first integral is singular (at $x = 0$); the second one can be added to j. Let $t_\ell^{(0)}$ be the Taylor expansion of $t_\ell(x)$ around $x = 0$ to the order $a_\ell - d$ (its coefficients depend on V); then we write the singular term as:

$$\int_S T_\ell^\Lambda(x)[t_\ell(x,V) - t_\ell^{(0)}(x,V)] d^d x + \int_S T_\ell^\Lambda(x) t_\ell^{(0)}(x,V) d^d x \quad .$$

The first term is regular and contributes to j. Finally:
$\int_S T_\ell^\Lambda(x) t_\ell^{(0)}(x,V) d^d x = \int_{\mathbb{R}^d} T_\ell^\Lambda(x) t_\ell^{(0)}(x,V) d^d x - \int_{\mathbb{R}^d - S} \dots$. We include the last term in j again, and the integral over the entire Euclidean space is the desired $J_\ell^\Lambda j_\ell(V)$; J_ℓ^Λ is independent of V, with a leading divergence $\propto \Lambda^{a-d}$. [25]

That this procedure corresponds to the decomposition of the G_k described above can be confirmed if we apply it on $\int_V (G(x))^2 d^d x = G_2$; we arrive at the same G_2^Λ (for $d \ge 4$).

With this concept, we investigate the structure of

$$\frac{1}{V} J_3 = \frac{1}{V} \int_V [G^{\Lambda 3} + 3 G^{\Lambda 2} g + 3 G^\Lambda g^2 + g^3] d^3 x$$

We look for singularities close to the origin where $G^\Lambda(x) \propto 1/x$. Only the first term is singular. We find:

$$J_3 = J^\Lambda + j_3 \; , \quad \text{where } J^\Lambda \propto ln\Lambda \; .$$

[23] For $d = 2n$ G_n^Λ becomes $\propto ln\Lambda$, i.e. relevant for our discussion. This is also the meaning of Λ with vanishing power in the following.

[24] The generalization for the case $t_k(0,V) = 0$ is straightforward.

[25] We introduce the sphere S instead of just writing $\int_V T_\ell^\Lambda t_\ell^{(0)} dx = \int_{\mathbb{R}^d} T_\ell^\Lambda t_\ell^{(0)} dx - \int_{\mathbb{R}^d - V} T_\ell^\Lambda t_\ell^{(0)} dx$ because it permits an additional selection of the singularities, as we will see.

The singularity J^Λ survives dimensional regularization, so it must be renormalized, see below. Our most complicated term is $\Gamma_3 = \int \partial_\mu(G^\Lambda + g)\partial_\mu(G^\Lambda + g)\partial_\nu(G^\Lambda + g)\partial_\nu(\dot{G}^\Lambda + \dot{g})\, d^3x$. $g(x)$ is an even function, so $\partial_\mu g = cx_\mu + c_\mu^{\alpha\beta\gamma} x_\alpha x_\beta x_\gamma + \cdots$ (from section 3 we know: $\partial_{\mu\nu} g|_{x=0} = \delta_{\mu\nu}/V \;\rightarrow\; c = \frac{1}{V}$).
About $\dot{g}(x)$ we know: $\partial_\mu \dot{g}(x)|_{x=0} = 0$, $\partial_{\mu\nu}\dot{g}(x)|_{x=0} = -\delta_{\mu\nu} g_1$, so $\partial_\nu \dot{g}$ has the expansion: $\partial_\nu \dot{g} = -g_1 x_\nu + c_\nu'^{\alpha\beta\gamma} x_\alpha x_\beta x_\gamma + \cdots$. Applying this we find:

$$\Gamma_3 = \Gamma_a^\Lambda + \Gamma_b^\Lambda g_1 + \Gamma_c^\Lambda \frac{1}{V} + \gamma_3 \quad \text{where} : \Gamma_a^\Lambda \propto \Lambda^3,\ \Gamma_b^\Lambda \propto \Lambda^2,\ \Gamma_c^\Lambda \propto ln\Lambda,\ \gamma_3 \text{ regular}$$

It might seem that there is also a singularity $\propto \Lambda$ associated with c', but the corresponding volume-dependent factor vanishes as we see when we integrate over S. This happens to all the non-covariant terms. In addition only Γ_c^Λ is relevant for the present discussion.

The 3^{rd} order of ρ_1, ρ_2 is denoted by $\frac{1}{F^6}\varepsilon_3$, $\frac{1}{F^2}\delta_3$. Inserting the result of section 5 we see that in δ_3 there are no logarithmic singularities, so we find for the counter term the constraint:

$$\kappa_{0,2} + \kappa_{0,3} = 0 \quad . \tag{7.4}$$

Exploiting in the same way the 3^{rd} order of (7.2), we arrive at:

$$\kappa_{0,2} - \kappa_{0,1} = \frac{N-3}{12} J^\Lambda + (N-2)\Gamma_c^\Lambda \tag{7.5}$$

We conclude that the set of counter terms $\{\kappa_{0,1},\ \kappa_{0,2},\ \kappa_{0,3}\}$ is submitted to the two (independent) constraints (7.4) and (7.5), so the counter terms keep one degree of freedom. But as we mentioned in section 2 we could have used the transformation (2.12) to eliminate either k_2 or k_3: if we do so, i.e. if we exploit maximally the freedom of choice of the fields to reduce the number of coupling constants in the Lagrangian (for $\vec{H} = const.$), then the two remaining counter terms are uniquely determined.

Now we carry out the procedure for **d** $= 4$.

In the final result of section 6 there are three terms that contain a singularity in dimensional regularization:

$$\begin{aligned} G_2 &= G_2^\Lambda + g_2 \\ J_3 &= J_b^\Lambda g_1 + j_3 \\ \Gamma_3 &= \Gamma_d^\Lambda \frac{1}{V} g_1 + \gamma_3 \end{aligned}$$

and the renormalization involves already the second order of eqs. (7.2) and (7.3). We insert the results for δ_2 and ε_2 :

$$\kappa_{0,2} + \kappa_{0,3} = \frac{N-1}{4} G_2^\Lambda \tag{7.6}$$

$$\kappa_{0,1} - \kappa_{0,2} = \frac{N-3}{4} G_2^\Lambda \tag{7.7}$$

The 2-loop results of $d = 3$ and 4 are alike, only the k_j-terms are one order higher for $d = 3$. It is remarkable that on the other hand the mechanism of renormalization is much different. Here G_2 is divergent, but the counter terms $\kappa_{0,j}$ allow it to occur in δ_2 and ε_2 .

Third order

If we insert δ_3 in (7.3) we make the interesting observation that the constraint (7.7) is identically repeated.

In the third order of eq. (7.2) also the counter terms associated with k_4, k_5 occur [26]. If we insert ε_3 and apply eq. (7.7), only the new counter terms remain and and can be determined:

$$\begin{pmatrix} \kappa_{0,4} \\ \kappa_{0,5} \end{pmatrix} = \frac{2}{N(N-1)-2} \begin{pmatrix} -N/2 \\ 1 \end{pmatrix} \left\{ \frac{(N-3)(2N-3)}{2} G_2^\Lambda - \frac{N-3}{6} J_b^\Lambda - 2(N-2)\Gamma_d^\Lambda \right\} \tag{7.8}$$

Actually there are three constraints imposed on $\{\kappa_{0,1},\ \kappa_{0,2},\ \kappa_{0,3}\}$, but only two of them are independent, so the set $\{\kappa_{0,1} \dots \kappa_{0,5}\}$ keeps one degrees of freedom. However, if we eliminate k_2 or k_3, then all the counter terms are uniquely determined, as for $d = 3$.

In summary we repeat that perturbative renormalizability can be affirmed on all the three levels of magnitude, for $d = 3$ and $d = 4$. This we could demonstrate in the framework of dimensional regularization without determining the singularities (nor the regular terms) explicitly.

If we reduce the number of coupling constants in the Lagrangian by means of field transformations to its minimum, then renormalization can only be realized due to the coincidence of various constraints imposed on the counter terms, which are associated with the remaining coupling constants. In this case, all the counter terms are determined uniquely, both for $d = 3$ and $d = 4$.

8 Conclusions

We have investigated the non-linear σ-model in 3 and 4 dimensions, describing a system of Goldstone bosons in a large but finite volume and in presence of a weak magnetic field of the order of the inverse volume (such that the Goldstone bosons feel the finite size strongly). The corresponding partition function is perturbatively renormalizable as we have shown explicitly to 3 loops.

[26] Here and to all order no counter terms of k_6 are involved in the renormalization.

We can also confirm the applicability of Polyakov's functional measure that contains relevant contributions in terms of the finite size. Referring to dimensional regularization, an arbitrary linear combination of further invariant terms can be added to this measure without yielding any contribution to the action.

The explicit three loop results for the large volume expansion of the source dependent part of the free energy are given for spatial dimension $d = 3, 4$ in section 5, 6, respectively, without specification of the isospin space dimension N. They take a particularly simple form for $N = 3$ (Heisenberg model). They provide a basis for the interpretation of Monte Carlo results, in particular for their extrapolation to infinite volume.

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A Perturbative renormalization with power divergences

In this appendix we are first going to show that the results of sections 5 and 6 are perturbatively renormalizable, even if we include the leading power divergences of all the singular terms, as specified in section 3. In the second part we add some remarks about other attempts to regularize our model.

The property of perturbative renormalizability with leading power divergences is much more general than the one shown in section 7 where we restricted the discussion to the terms that occur in dimensional regularization. The latter could be deduced from this appendix as a corollary. The generalization, however, suffers from the problem that we do not know a regularization scheme that leaves us exactly with the divergences we include here. What we consider are the terms that all regularizations displaying power divergences have in common. So we are not very surprised that this structure is meaningful. On the other hand the generalizability is remarkable, since there are many divergences involved with many different volume dependent prefactors. Hence the constraints imposed on the counter terms become very narrow; in particular various counter terms are actually overdetermined by the number of constraints and renormalizability only holds due to the coincidence of several of them. This coincidence requires very special relations among the coefficients of the terms occurring in our results, which provide us with the highly non-trivial check announced in the introduction.

Like in section 7 we use the "mass independent renormalization prescription" which does, however, not affect the additional counter terms we include here. They will be expressed in positive powers of Λ^{d-2}/F^2, where Λ is the characteristic regularization parameter (of dimension of a momentum) introduced in section 7.

We will proceed in a manner that does not only show the renormalizability by determining the counter terms explicitly, but also the significance of this property. Starting from the ℓ loop result ($\ell = 0,1,2$) we make a quite general ansatz for the $(\ell+1)$ level – including all the terms occurring in the calculation to this level – and examine, which constraints renormalizability imposes on its coefficients. Although some coefficients are arbitrary from this point of view, a lot of them are determined exactly: from the beginning many of them must be zero (in accordance with the long list of canceled terms given at the end of section 5 and 6) and for $\ell = 2$ we can even "predict" non-vanishing coefficients. [27] This clarifies the meaning of the result and gives a sound basis for the conclusions about the multiloop terms ($\ell > 3$).

As a general ansatz for the renormalized leading coupling constants we write:

$$\begin{aligned} \Sigma_r &= \Sigma(1+\sigma_1+\sigma_2+\sigma_3\ldots) \\ F_r &= F(1+f_1+f_2+f_3\ldots) \qquad \sigma_n,\ f_n \propto \frac{\Lambda^{(d-2)n}}{F^{2n}} \end{aligned} \tag{A.1}$$

where Σ, F are the bare couplings and σ_n, f_n their associated counter terms. The renormalized non-leading coupling constants can also contain logarithmic counter terms (known from

[27] As we will see, the renormalizability does not depend on any special ratio between different singularities of the same power. This justifies the predictions mentioned above. If we could assume this independence of singularity ratios also to higher orders, we could immediately predict the coefficients of various multiloop terms, such as G_1^k, $G_1^{k-2}G_2/V$, $(k = 4,5\ldots)$ etc. However, we have no prove for this assumption.

Hence for exact predictions we would need a general prove of this assumption as well as the assumption that the singularity structure we consider is really renormalizable to all orders. So we write "predictions" in inverted commas.

section 7), so there we write:

$$k_{jr} = k_j + \kappa_{0,j} + \kappa_{1,j} + \kappa_{2,j} + \dots \qquad \kappa_{0,j} \propto ln\Lambda \ , \ \kappa_{n,j} \propto \frac{\Lambda^{(d-2)n}}{F^{2n}} \ . \tag{A.2}$$

Again we discuss order by order eqs. (7.2) and (7.3).

First we consider the 1-loop result of section 4; here only (7.2) is involved. It takes the form:

$$\Sigma(1+\sigma_1+\dots)\left(1-\frac{N-1}{2(F[1+f_1+\dots])^2}\, g_1\right) = \Sigma\left(1-\frac{N-1}{2F^2}(G_1^\Lambda+g_1)\right)$$

This may be exploited to the first order ($\propto F^{-2}$) :

$$\sigma_1 = -\frac{N-1}{2F^2}G_1^\Lambda \tag{A.3}$$

So the 1-loop result *is* renormalizable and fixes σ_1. Now we wonder, how significant this is, i.e. which subset of the possible 1-loop results can be renormalized.
A general ansatz is:

$$\rho_1 = 1+\frac{a_1+P_{\alpha,1}}{F^2}\, G_1 \ , \text{ where } \ P_{\alpha,1} \doteq \sum_{k\geq 1} \alpha_{k,1}(V\delta^\Lambda(0))^k$$

a_1 and $\alpha_{k,1}$ being arbitrary constants. The V-independent part yields:
$\sigma_1 = \frac{a_1}{F^2}G_1^\Lambda$, and the V- dependent part: $\alpha_{k,1}=0 \quad (\forall\ k)$. Thus we already have non-trivial constraints, although the renormalizable class is quite large up to now.

We consider $\mathbf{d=3}$ and list the singularity structures of the divergent terms occurring in the 3-loop calculation:

$$\begin{aligned}
G_1 &= G_1^\Lambda + g_1 \ ; \qquad G_1^\Lambda \propto \Lambda \ , \quad g_1 \text{ regular}\\
J_2 &= I_a^\Lambda + I_b^\Lambda \frac{1}{V} + j_2 \ ; \qquad I_a^\Lambda \propto \Lambda^5, \ I_b^\Lambda \propto \Lambda^2, \quad j_2 \text{ regular}\\
J_3 &= J^\Lambda + j_3 \ ; \qquad J^\Lambda \propto ln\Lambda, \quad j_3 \text{ regular}\\
\Gamma_3 &= \Gamma_a^\Lambda + \Gamma_b^\Lambda g_1 + \Gamma_c^\Lambda \frac{1}{V} + \gamma_3 \ ; \qquad \Gamma_a^\Lambda \propto \Lambda^3, \ \Gamma_b^\Lambda \propto \Lambda^2, \ \Gamma_c^\Lambda \propto \ln\Lambda, \quad \gamma_3 \text{ regular}
\end{aligned}$$

Presuming the already exploited 1-loop result, a general ansatz for the 2-loop result is:

$$\begin{aligned}
\rho_1 &= 1-\tfrac{N-1}{2F^2}G_1 + \tfrac{1}{F^4}\{(a_2+P_{\alpha,2})G_1^2 + \tfrac{b+P_\beta}{V}G_2 + (t+P_\tau)VJ_2\}\\
\rho_2 &= (\tilde{a}_2+P_{\tilde{\alpha},2})G_1^2 + \tfrac{\tilde{b}+P_{\tilde{\beta}}}{V}G_2 + (\tilde{t}+P_{\tilde{\tau}})VJ_2
\end{aligned}$$

where the P's are again polynomials in $V\delta^\Lambda(0)$. For ρ_1 we have to consider the second order of (7.2) :

$$\sigma_2 - \frac{N-1}{2F^2}g_1(\sigma_1-2f_1) =$$

$$\frac{1}{F^4}\Big\{(a_2+P_{\alpha,2})(G_1^{\Lambda 2}+2G_1^\Lambda g_1)+P_{\alpha,2}g_1^2+\frac{P_\beta}{V}G_2+(t+P_\tau)(VI_a^\Lambda+I_b^\Lambda)+P_\tau j_2\Big\}$$

We conclude: $P_{\alpha,2}=P_\beta=t=P_\tau=0$, $\sigma_2=a_2G_1^{\Lambda 2}/F^4$, $f_1=(\frac{a_1}{2}-\frac{a_2}{a_1})G_1^\Lambda$. a_2 and b are arbitrary. The required cancellations are achieved, thanks to the compensation of the $\delta^\Lambda(0)$-contributions of the measure and the Lagrangian. Inserting a_1, a_2 we get:

$$\sigma_2=-\frac{(N-1)(N-3)}{8F^4}\,G_1^{\Lambda 2}\ ,\quad f_1=-\frac{N-2}{2F^2}\,G_1^\Lambda \tag{A.4}$$

Now also (7.3) must hold to the 2^{nd} order ($\propto F^{-4}$). The renormalized side can not compensate any divergent terms, so the general *renormalizable* ansatz is just: $\rho_2=\tilde{b}\,G_2/V$. This is indeed what we found, with $\tilde{b}=(N-1)/4$.

Third order

Here the constraints imposed by renormalizability develop for the first time their full power. They no longer only exclude certain terms and cut off the polynomials in $(V\delta^\Lambda(0))$ in the coefficients, but also "predict" the exact values of non-vanishing coefficients.

The 3^{rd} order of ρ_1, ρ_2 is denoted by $\frac{1}{F^6}\varepsilon_3$, $\frac{1}{F^2}\delta_3$. As a general ansatz for ε_3, δ_3 we take an arbitrary linear combination of the terms :
G_1^3, $\frac{1}{V}G_1G_2$, $\frac{1}{V^2}G_3$, $\frac{1}{V}k_j$ $(j=1,2,3)$, $\frac{1}{V}J_3$, Γ_3, and VG_1J_2 ,
where again the coefficients include polynomials in $(V\delta^\Lambda(0))$. Presupposing the 2-loop result, the 3^{rd} order of (7.3) requires that in δ_3 a lot of contributions vanish and fixes the form:

$$\delta_3=\frac{(N-1)(N-3)}{4V}G_1G_2+\frac{\tilde{c}}{V^2}G_3+\frac{1}{V}(\tilde{r}J_3+\sum_j\tilde{d}_jk_j)$$

which is in accordance with section 5. $\tilde{c}$ is arbitrary and if we insert the result in the remaining constraint: $\sum_j d_j\kappa_{0,j}=\tilde{r}J^\Lambda$ we arrive at eq. (7.4).

Exploiting in the same way the 3^{rd} order of eq. (7.2), we find for ε_3 the form :

$$\begin{aligned}\varepsilon_3 \;=\;& \frac{(N-1)(N-3)(3N-7)}{48}\Big(-G_1^3+\frac{6}{V}G_1G_2\Big)+\frac{c}{V^2}G_3\\ &+\frac{1}{V}\sum_j\Big(d_j+V\delta^\Lambda(0)d_{1,j}\Big)k_j+\frac{r}{V}J_3+s\Gamma_3\end{aligned}$$

where c, d_j, $d_{1,j}$, r and s are almost arbitrary. Again the specified coefficients are in accordance with the result of section 5.

So to the 34 terms that had to vanish non-trivially for $d=3$ we can add 3 more coefficients that precisely take the only value that permits renormalization of the structure we consider.

The counter terms are :

$$\sigma_3 = -\frac{N-1}{F^6}\left(\frac{(N-3)(3N-7)}{48}G_1^{\Lambda 3} + \delta^\Lambda(0)k_1 + (N-2)\Gamma_a^\Lambda\right) \quad (A.5)$$

$$f_2 = -\frac{N-2}{F^4}\left(\frac{N-2}{4}G_1^{\Lambda 2} + \Gamma_b^\Lambda\right) \quad (A.6)$$

$$\sum_{j=1}^{3} d_j \kappa_{0,j} = rJ^\Lambda + s\Gamma_c^\Lambda \quad (A.7)$$

Inserting the explicit result in eq. (A.7) yields constraint (7.5). We note that for $N = 2$ and for $N = 3$ the counter terms take a particularly simple form.

Now we carry out the procedure for **d = 4**.

Since no confusion is possible, we denote the parameters equally as for $d = 3$, as we already started to in section 6 and 7. So a_k, b, c etc. have a local meaning for the dimension we are discussing at present.

The singularity structures of the terms occurring up to the 3^{rd} order take the following form:

$$\begin{aligned}
G_1 &= G_1^\Lambda + g_1\ , \qquad G_1^\Lambda \propto \Lambda^2 \qquad \text{(first degree of divergence)}\\
G_2 &= G_2^\Lambda + g_2\ , \qquad G_2^\Lambda \propto ln\Lambda\\
J_2 &= I_a^\Lambda + I_b^\Lambda \tfrac{1}{V} + I_c^\Lambda \tfrac{1}{V^2} + j_2\ , \qquad I_a^\Lambda \propto \Lambda^8, I_b^\Lambda \propto \Lambda^4, I_c^\Lambda \propto ln\Lambda\\
J_3 &= J_a^\Lambda + J_b^\Lambda g_1 + j_3\ , \qquad J_a^\Lambda \propto \Lambda^2, J_b^\Lambda \propto ln\Lambda\\
\Gamma_3 &= \Gamma_a^\Lambda + \Gamma_b^\Lambda g_1 + \Gamma_c^\Lambda \tfrac{1}{V} + \Gamma_d^\Lambda \tfrac{1}{V} g_1 + \gamma_3\\
& \qquad \Gamma_a^\Lambda, \Gamma_b^\Lambda, \Gamma_c^\Lambda, \Gamma_d^\Lambda \propto \Lambda^6, \Lambda^4, \Lambda^2, ln\Lambda \quad \text{(respectively)}\\
G_{\mu\nu} &= -\delta_{\mu\nu}\, \delta^\Lambda(0) + g_{\mu\nu}\ , \qquad \delta^\Lambda(0) \propto \Lambda^4\\
\dot{G}_{\mu\nu} &= -\delta_{\mu\nu}\, G_1^\Lambda + \dot{g}_{\mu\nu}\ , \qquad G_1^\Lambda \equiv \dot{\delta}^\Lambda(0) \propto \Lambda^2\\
D^\Lambda(0) &\propto \Lambda^6\ ; \quad \Delta^\Lambda(0) \propto \Lambda^8
\end{aligned}$$

where the last term is regular everywhere.

The general ansatz for ε_2 has to be extended to:

$$\varepsilon_2 = (a_2 + P_{\alpha,2})G_1^2 + \frac{b + P_\beta}{V}G_2 + \frac{1}{V}\sum_{j=1}^{3}(d_j + P_{d_j})k_j + (t + P_\tau)VJ_2 + (r + P_\rho)V\Delta^\Lambda(0)$$

and again the same for δ_2, with $\tilde{a}_2$ etc.

From the 2^{nd} order of (7.3) we see:
$\tilde{a}_2 = P_{\tilde{\alpha}_2} = P_{\tilde{\beta}} = P_{\tilde{d}_j} = \tilde{t} = P_{\tilde{\tau}} = \tilde{r} = P_{\tilde{\rho}} = 0, \quad \sum_j \tilde{d}_j \kappa_{0,j} = \tilde{b}G_2^\Lambda$
(except for this, $\tilde{d}_j$ and $\tilde{b}$ are free). Inserting what we found in section 6 we arrive at eq. (7.6).

From eq. (7.2) we can only conclude: $P_{\alpha_2} = P_\beta = d_{k>1,j} = t = P_\tau = r = p_\rho = 0$. This time already in the 2^{nd} order $\delta^\Lambda(0)$ can occur, and indeed it does. The rest is almost free, and:

$$\begin{aligned}
\sigma_2 &= \frac{1}{F^4}\Big\{a_2 G_1^{\Lambda\,2} + \delta^\Lambda(0)\sum_j d_{1,j}k_j\Big\} \\
&= -\frac{(N-1)(N-3)}{8F^4}G_1^{\Lambda\,2} - \frac{N-1}{F^4}\delta^\Lambda(0)k_1 && \text{(A.8)} \\
f_1 &= \frac{1}{F^2}\Big\{\frac{\sigma_1}{2} + \frac{2a_2}{N-1}G_1^\Lambda\Big\} = -\frac{N-2}{2F^2}G_1^\Lambda && \text{(A.9)} \\
\sum_{j=1}^{3} d_j\kappa_{0,j} &= bG_2^\Lambda && \text{(A.10)}
\end{aligned}$$

Eq. (A.10) contains the slight restriction of the freedom announced above and leads to constraint (7.7).

Third order

This time the ansatz for ε_3, δ_3 is a linear combination of the following terms:

G_1^3, $\frac{1}{V}G_1G_2$, $\frac{1}{V^2}G_3$, $\frac{G_1}{V}k_j$ $(j = 1\dots 5)$, $\frac{1}{V}J_3$, Γ_3, VG_1J_2, $G_1V(G_{\mu\nu})^2k_j$ $(j = 4,5)$, $G_{\mu\nu}\dot{G}_{\mu\nu}k_j$ $(j = 4,5)$, $k_6D^\Lambda(0)$, $k_6G_1V\Delta^\Lambda(0)$.

In δ_3 once more most terms have to vanish. Omitting them it just remains:

$$F^2(2\sigma_1 - 4f_1)(\tfrac{N-1}{4}g_2 + k_2 + \kappa_{2,0} + k_3 + \kappa_{3,0}) + \kappa_{1,2} + \kappa_{1,3} + \sum \tilde{e}_j\kappa_{0,j} =$$
$$\tilde{b}_3(G_1^\Lambda G_2^\Lambda + G_1^\Lambda g_2 + g_1G_2^\Lambda) + G_1^\Lambda\sum\tilde{e}_jk_j$$

Inserting the counter terms known from the 2^{nd} order, we find :

$$\begin{aligned}
\tilde{b}_3 &= \frac{(N-1)(N-3)}{4} \\
\sum\tilde{e}_j\kappa_{0,j} &= \tilde{b}_3G_2^\Lambda && \text{(A.11)} \\
\kappa_{1,2} + \kappa_{1,3} &= \frac{1}{F^2}(N-1)(k_1 - k_2)G_1^\Lambda && \text{(A.12)}
\end{aligned}$$

As we mentioned in section 7, eq. (A.11) reproduces identically the constraint (7.7). In addition we see now that also constraint (7.6) has been repeated.

Eq. (7.2) causes more work; generally there we can exclude less quantities because it starts from the tree-level. A lengthy book-keeping yields in the pure G_n sector what we got for $d = 3$ before, but beyond that some novelties:

$$\varepsilon_3 = \tfrac{(N-1)(N-3)(3N-7)}{48}(-G_1^3 + \tfrac{6}{V}G_1G_2) + \tfrac{c}{V^2}G_3 + \tfrac{G_1}{V}\textstyle\sum_j(e_j + e_{1,j}V\delta^\Lambda(0))k_j$$
$$+\tfrac{r}{V}J_3 + s\Gamma_3 + G_{\mu\nu}\dot{G}_{\mu\nu}\textstyle\sum_j w_jk_j + pk_6D^\Lambda(0)$$

$c, e_j, e_{1,j}, r, s, w_j$ and p are almost arbitrary, as we see if we compute the counter terms:

$$\sigma_3 = \tfrac{N-1}{F^6}\Big[-\tfrac{(N-3)(3N-7)}{48} G_1^{\Lambda\, 3} + \tfrac{1}{2}([N-1]k_1 + [N+1][k_4+k_5]) G_1^\Lambda \delta^\Lambda(0) \\ -(N-2)\Gamma_a^\Lambda + \tfrac{1}{2}k_6 D^\Lambda(0)\Big]$$

$$f_2 = \tfrac{1}{F^4}\left[-\tfrac{(N-2)^2}{8} G_1^{\Lambda\, 2} + \tfrac{1}{2}([N+3]k_1 + [N+1][k_4+k_5])\delta^\Lambda(0) - (N-2)\Gamma_b^\Lambda \right] \tag{A.13}$$

$$\kappa_{1,2} - \kappa_{1,1} = \tfrac{1}{2F^2}\Big[\{2(N-1)k_1 - 2(N-3)k_2 + (N+1)(k_4+k_5)\} G_1^\Lambda \\ + \tfrac{N-3}{6} J_a^\Lambda + 2(N-2)\Gamma_c^\Lambda \Big]$$

where on the logarithmic order eq. (7.8) has to be added.

In summary we repeat that perturbative renormalizability can be affirmed even for the singularity structure with leading power divergences on all the three levels of magnitude, for $d = 3$ and $d = 4$. This we could demonstrate without determining the power singularities explicitly. This property imposes very narrow constraints on the coefficients of the 3 loop result.

The treatment of power divergences becomes in part applicable when we discuss conclusions about the renormalizable case $d = 2$ in appendix E. There, e.g. the singularity G_1^Λ is logarithmic, so it must be included in the renormalization.

A.1 Constraints on the regularization

In this subappendix we add some remarks about the problems that occur if we try to regularize our model in a way different from dimensional regularization.

In the main part of this work we have treated the regularized δ-function $\delta^\Lambda(x)$ like an exact δ-function under the spatial integral without worrying. If we don't choose dimensional regularization, this is risky since we anticipate a limit which we ought to take only at the very end, after renormalization. First we are going to give generalized results for the measure and the partition function without any assumptions about $\delta^\Lambda(x)$ (rsp. $G^\Lambda(x)$). Then we observe which properties have been used in the first part of this appendix and how far they are necessary for perturbative renormalizability. As an example, the Pauli-Villars regularization fails to fulfil the required properties. Concerning the physical properties, it maintains covariance but violates unitarity (the opposite is the case for lattice regularization). At the end we outline why also a sharp momentum cutoff is unsuitable for this model. We are not much surprised about this when we consider that it violates both, unitarity as well as covariance.

First we consider the measure and only use relation (3.12), which is now understood as a *definition* of $\delta^\Lambda(x)$. No further properties of this function are presupposed. Then the measure takes the generalized form:

$$\begin{aligned} ln\sqrt{\mathbf{g}} &= \frac{V\delta^{\Lambda}(0)-N+1}{2V}\int \underline{\pi}^2(x)dx + \frac{V\delta^{\Lambda}(0)-1}{2V}\int (\underline{\pi}^2(x))^2 dx \\ &-\frac{N-1}{8}\Big(\frac{1}{V}\int \underline{\pi}^2(x)dx\Big)^2 - \frac{N-5}{8V}\int\int \delta^{\Lambda}(x-y)\underline{\pi}^2(x)\underline{\pi}^2(y)dxdy \\ &-\frac{1}{4}\sum_{i,k}\int\int (\delta^{\Lambda}(x-y))^2\pi^i(x)\pi^k(x)\pi^i(y)\pi^k(y)dxdy \end{aligned} \tag{A.14}$$

Here we already observe modifications in the second order.

Let us consider the partition function for $\mathbf{d}=\mathbf{3}$ and generalize the result of section 5. We insert eq. (A.14) and – in accordance with eq. (3.12) – we use eq. (3.11) when performing the Wick contractions. In the evaluation of the contracted terms, we always maintain the generality of $\delta^{\Lambda}(x)$, except for the following three assumptions about the regularized system:
a) The regularized propagator is translation invariant : $G^{\Lambda}(x,y)=G^{\Lambda}(x-y)$.
b) Partial integrations are permitted everywhere without causing extraordinary terms.
c) The regularization does not require additional terms in the Lagrangian.

For example on the lattice all the three assumptions are not fulfilled: e.g. the non-covariance requires to include terms like $g_4^{(4)}(\partial_\mu\vec{S}\partial_\mu\vec{S})(\partial_\mu\vec{S}\partial_\mu\vec{S})$ in the Lagrangian.

As a consequence of assumption a) an odd number of derivatives of the propagator at the origin has to vanish. This property has been used very extensively:
Already to the first order, the evaluation (5.3) only holds with the (non-trivial) constraint

$$\partial_\mu G^{\Lambda}(x)|_{x=0}=0 \quad .$$

This would be violated for "regularizations", which are not symmetric around the origin in momentum space, e.g. a sharp cutoff $|p-p_0|\leq\Lambda$. We see easily that it is required from the 2-loop renormalization; else it would cause there a non-vanishing contribution $(\partial_\mu G^{\Lambda}(x)|_{x=0})^2 G^{\Lambda}$, which can not be absorbed.

On the 2-loop level we find modifications for the following terms:

$$\begin{aligned} I_1 &\doteq < \frac{F^4}{8}\Big(\int(\underline{\pi}\partial_\mu\underline{\pi})^2dx\Big)^2 > \\ -I_2 &\doteq < -\frac{N-5}{8V}\int\int \delta^{\Lambda}(x-y)\underline{\pi}^2(x)\underline{\pi}^2(y)dxdy > \\ -I_3 &\doteq < -\frac{1}{4}\sum_{i,k}\int\int(\delta^{\Lambda}(x-y))^2\pi^i(x)\pi^k(x)\pi^i(y)\pi^k(y)dxdy > \end{aligned}$$

$$< \frac{F^2\beta}{4V}\Big(\int(\underline{\pi}\partial_\mu\underline{\pi})^2dx\Big)\Big(\int\underline{\pi}^2dy\Big) > \;=\; \frac{\beta(N-1)}{4F^4}\Big[(V\delta^{\Lambda}(0)-1)\Big((N-1)G_1^2+\frac{2}{V}G_2\Big)$$

$$+2G_1 \int \delta^\Lambda(x)G(x)dx\Big]$$

The source independent terms I_1, I_2, I_3 do not enter the final result because their product with $S_1(H)$ is cancelled on the 3-loop level. So we don't need to evaluate them.

On the 3-loop level we focus our attention again on the troublesome term

$$\begin{aligned} < -\frac{F^4\gamma\Omega^{00}}{16V}\Big(\int(\underline{\pi}\partial_\mu\underline{\pi})^2dx\Big)^2\Big(\int\underline{\pi}^2dz\Big) > \; &= \; -\gamma\Omega^{00}\frac{N-1}{2F^2}G_1 I_1 - \\ &\quad \frac{\gamma\Omega^{00}}{16F^6} < [(\mathbf{ab})(\mathbf{cd})][(\mathbf{ef})(\mathbf{gh})][(\mathbf{r}|\mathbf{s})] > \end{aligned}$$

with the notation introduced in eq. (5.11). In particular the functions $\Gamma_0 \ldots \Gamma_5$ defined in eqs. (5.14) can not be expressed in terms of Γ_3 as easily as in section 5. Instead of eqs. (5.15) we have:

$$\begin{array}{ll} \Gamma_0 = \Gamma_3 + \gamma_2 + \frac{1}{2}(\gamma_1 - \gamma_4) & \text{where :} \\ \Gamma_1 = -\frac{1}{2}(\Gamma_3 + \gamma_1) & \gamma_1 \doteq \int(-\delta^\Lambda + \frac{1}{V})\partial_\mu G\partial_\mu G\dot{G}dx \\ \Gamma_2 = \frac{1}{2}(\gamma_4 - \Gamma_3) & \gamma_2 \doteq \int \partial_\mu G\partial_\mu\delta^\Lambda G\dot{G}dx \\ \Gamma_4 = \Gamma_3 + \gamma_3 - \gamma_4 & \gamma_3 \doteq \int \partial_\mu\delta^\Lambda G^2\partial_\mu\dot{G}dx \\ \Gamma_5 = \gamma_4 - \frac{1}{2}(\Gamma_3 + \gamma_3) & \gamma_4 \doteq \frac{1}{3}(\int \delta^\Lambda G^3dx - \frac{1}{V}J_3) \end{array}$$

Also in the remaining nine H-dependent 3-loop terms there are numerous modifications. Some of them include again the quantities $\gamma_1 \ldots \gamma_4$. Following all the steps of section 5 with these generalized terms we arrive at a partition function of the form (5.17) with:

$$\begin{aligned} \varepsilon_1 &= -\frac{N-1}{2}\,G_1 \\ \delta_2 &= \frac{N-1}{4V}\,G_2 \\ \varepsilon_2 &= \frac{N-1}{8}\Big[-(N+1)G_1^2 + \frac{2(N-3)}{V}G_2 + 4G_1\int\delta^\Lambda(x)G(x)dx\Big] \\ \delta_3 &= \frac{N-1}{2V}\Big[\frac{-2N+5}{3V}G_3 + \frac{N-1}{2}G_1G_2 - G_2\int\delta^\Lambda(x)G(x)dx\Big] + \frac{k_2+k_3}{V} \\ \varepsilon_3 &= (N-1)\Big\{-\frac{1}{16}(N+1)(N+3)G_1^3 - \frac{1}{12V^2}\Big[(N-3)(N-4)+\gamma^2\Big]G_3 \\ &\quad +\frac{1}{8V}\Big[N^2-8N+11-4V\int(\delta^\Lambda(x))^2dx + 2(N^2-6N+7)\int\delta^\Lambda(x)dx\Big]G_1G_2 \\ &\quad +\frac{3N+1}{4}G_1^2\int\delta^\Lambda(x)G(x)dx + -\frac{N-3}{2V}G_2\int\delta^\Lambda(x)G(x)dx \\ &\quad +\frac{1}{2}G_1\int(\delta^\Lambda(x))^2\dot{G}(x)dx + \frac{N}{2}\int(\delta^\Lambda(x))^2G(x)\dot{G}(x)dx \\ &\quad +\frac{N-5}{2V}\int\delta^\Lambda(x)G(x)\dot{G}(x)dx - \frac{1}{2}G_1\Big(\int\delta^\Lambda(x)G(x)dx\Big)^2 \\ &\quad -\frac{1}{4}\Big[4(N-2)\Gamma_3 - 2\gamma_1 + 2N\gamma_2 + (N-2)\gamma_3 + (-N+6)\gamma_4\Big] \end{aligned}$$

$$-\frac{1}{V}\Big[(V\delta^{\Lambda}(0)-1)k_1-k_2\Big]\Big\}$$

Comparison to the corresponding result of section 5 (eqs (5.18) ... (5.22)) shows that:

> ε_1 and δ_2 are unchanged.

> ε_2 and δ_3 coincide with the corresponding quantities of section 5 only if

$$\int \delta^{\Lambda}(x)G(x)dx = G_1 \quad . \tag{A.15}$$

> For ε_3 the analogous transition requires eq. (A.15) and 8 further constraints:

$$\int \delta^{\Lambda}dx = 1 \tag{A.16}$$
$$\int (\delta^{\Lambda})^2 dx = \delta^{\Lambda}(0) \tag{A.17}$$
$$\int (\delta^{\Lambda})^2 \dot{G} dx = \delta^{\Lambda}(0)G_2 \tag{A.18}$$
$$\int (\delta^{\Lambda})^2 G\dot{G} dx = \delta^{\Lambda}(0)G_1G_2 \tag{A.19}$$
$$\int \delta^{\Lambda} G\dot{G} dx = G_1G_2 \tag{A.20}$$
$$\int \delta^{\Lambda}\partial_\mu G\partial_\mu G\dot{G} dx = 0 \tag{A.21}$$
$$\int \delta^{\Lambda} G\partial_\mu G\partial_\mu \dot{G} dx = 0 \tag{A.22}$$
$$\int \delta^{\Lambda} G^3 dx = G_1^3 \tag{A.23}$$

So the nine properties (A.15) ... (A.23) of δ^{Λ} rsp. G^{Λ} are required for the transformation of this 3 loop partition function to the former, simplified form.

Note that in the evaluations of section 5 we have additionally used two more relations of that kind, but since they affect only I_2 and I_3 (which cancel separately), they are not needed for the final result.

We have, however, not answered the crucial question, which among those constraints are really necessary for the perturbative renormalizability of the 3 loop result.

To handle this question we procceed as follows: we assume that we are dealing with a regularization in the proper sense, i.e. if we remove the regularization parameters the propagator and all its derivatives return to the original form. We consider now the constraints (A.15) ... (A.23) for such a proper, but general regularization and list the additional singular terms that might occur. Using in particular the symmetry of the regularized propagator in momentum space, we find that

1. A discrepancy on the level of the leading divergence is possible in eqs (A.15), (A.17) ... (A.20) and (A.23).
2. Additional (non-leading) divergences can occur in

$$\int (\delta^\Lambda)^2 dx \ , \quad \int (\delta^\Lambda)^2 \dot{G} dx \ , \quad \int (\delta^\Lambda)^2 G\dot{G} dx \quad \text{and} \quad \int \delta^\Lambda G^3 dx \quad .$$

Their possible, non-leading contributions are of order Λ, $ln\Lambda$, Λ, Λ, respectively. Thus ε_3 can receive an extra term of the form

$$\alpha_1 \Lambda L^{-2} G_1 G_2 + \alpha_2 ln\Lambda \ L^{-2} G_1 + N\alpha_3 \Lambda L^{-2} + (N-6)\alpha_4 \Lambda L^{-2}$$

($\alpha_i = const.$). Obviously all these terms contain a singularity with the volume dependent prefactor $L^{-2} \propto g_1^2$, which can (in general) *not* be renormalized: the only counter terms that can absorb singularities with the same prefactor in ε_3 are σ_1 and f_1. But they are uniquely determined from the 2 loop level and therefore not able to absorb further singularities.

For the analogous reasons all the constraints about the leading divergences are necessary.

If in a regularization such additional divergences exist, they have to fulfill very special relations to preserve perturbative renormalizability.

For **d = 4** the generalized partition function has again the form (5.17), where

$$\begin{aligned}
\varepsilon_1 &= -\frac{N-1}{2} G_1 \\
\delta_2 &= \frac{N-1}{4V} G_2 + \frac{k_2+k_3}{V} \\
\varepsilon_2 &= (N-1)\Big[-\frac{N+1}{8}G_1^2 + \frac{N-3}{4V}G_2 + \frac{1}{2}G_1 \int \delta^\Lambda G dx - \frac{(V\delta^\Lambda(0)-1)k_1 - k_2}{V}\Big] \\
\delta_3 &= \frac{N-1}{V}\Big[\frac{N-1}{4}G_1 G_2 - \frac{1}{2}G_2 \int \delta^\Lambda G dx - \frac{2N-5}{12V}G_3 + k_1 \int \delta^\Lambda G dx - k_2 G_1\Big] \\
\varepsilon_3 &= (N-1)\Big\{-\frac{1}{16}(N+1)(N+3)G_1^3 - \frac{1}{12V^2}\Big[(N-3)(N-4)+\gamma^2\Big]G_3 \\
&\quad + \frac{1}{8V}\Big[N^2 - 8N + 11 - 4V\int (\delta^\Lambda)^2 dx + 2(N^2-6N+7)\int \delta^\Lambda dx\Big] G_1 G_2 \\
&\quad + \frac{3N+1}{4}G_1^2 \int \delta^\Lambda G dx - \frac{N-3}{2V} G_2 \int \delta^\Lambda G dx \\
&\quad + \frac{1}{2}G_1 \int (\delta^\Lambda)^2 \dot{G} dx + \frac{N}{2}\int (\delta^\Lambda)^2 G\dot{G} dx \\
&\quad + \frac{N-5}{2V}\int \delta^\Lambda G\dot{G} dx - \frac{1}{2}G_1 \Big(\int \delta^\Lambda G dx\Big)^2 \\
&\quad - \frac{1}{4}\Big[4(N-2)\Gamma_3 - 2\gamma_1 + 2N\gamma_2 + (N-2)\gamma_3 - (N-6)\gamma_4\Big] \\
&\quad + \frac{1}{V}k_1\Big[\frac{N-3}{2}(V\delta^\Lambda(0)-1)G_1 - \int \delta^\Lambda G dx + G_1 \int \delta^\Lambda (V\delta^\Lambda - 1) dx\Big] \\
&\quad + k_2 \frac{N+1}{2V} G_1 + \Big[(N-1)k_4 + k_5\Big]\frac{V\delta^\Lambda(0)-1}{2V}\int \delta^\Lambda G dx
\end{aligned}$$

$$+(k_4 + \frac{N}{2}k_5)G_{\mu\nu}\dot{G}_{\mu\nu} + \frac{1}{2}k_6 \int \delta^\Lambda D^\Lambda dx\}$$

In its evaluation, the assumptions a) ... c) – and the comments about them – remain unchanged. For the simplification to the result of section 6 we still need eqs (A.15) ... (A.23) and in addition:

$$\int \delta^\Lambda D^\Lambda dx = D^\Lambda(0) \tag{A.24}$$

Concerning the eqs. that have to hold non-trivially on the level of leading divergences, eq. (A.24) has to be added to the list given for $d = 3$.

The variety of possible non-leading divergences is much larger here than in the 3 dimensional case. [28] The following terms can cause additional contributions, the form of which is given in the right column:

$$\left.\int (\delta^\Lambda)^2 G\dot{G}dx\ ,\quad \int \delta^\Lambda G^3 dx\ ,\quad \int \delta^\Lambda D^\Lambda dx\ \right\}\ \alpha_1\Lambda^4 L^{-2} + \alpha_2\Lambda^2 L^{-4} + \alpha_3 ln\Lambda\ L^{-6}$$

$$\left.\int (\delta^\Lambda)^2 dx\ ,\quad \int (\delta^\Lambda)^2 \dot{G}dx\ \right\}\ \alpha_4\Lambda^2 L^{-2} + \alpha_5 ln\Lambda\ L^{-4}$$

$$\left.\int \delta^\Lambda G dx\ ,\quad \int \delta^\Lambda G\dot{G}dx\ ,\quad V\int \delta^\Lambda G\partial_\mu G\partial_\mu \dot{G}dx\ \right\}\ \alpha_7 ln\Lambda\ L^{-2}$$

($\alpha_i = const.$). To see that we can not permit all these terms to occur with arbitrary coefficients, it suffices to look at the contributions $\propto L^{-4}$ i.e. $\propto g_1^2$: like in the 3 dimensional case, the only counter terms with the same volume dependent prefactor that enter ε_3 are uniquely determined from the 2 loop level.

A possibility to regularized the power singularities without eliminating all of them is provided by the *Pauli-Villars regularization.* Referring to the Fourier decomposition the propagator is manipulated to take the form:

$$G_{PV}(x) = \frac{1}{V}\sum_n{}'\Big[\frac{1}{p_n^2} + \sum_i \frac{c_i}{p_n^2 + M_i^2}\Big]e^{ip_n x} \tag{A.25}$$

(the c_i are constants and the M_i are heavy regularization masses that go to infinity in the final limit). We only regularize $G(x)$ and do not try to construct a correspondingly extended Lagrangian. [29]

This is a proper regularization in the above sense. To regularize a singularity of power Λ^{2n} we have to introduce at least $n+1$ different regularization masses. If we choose this

[28] This can be understood from the fact that in the Laplacian expansion of G^Λ every step corres-ponds to $2/(d-2)$ loop orders.

[29] Such a construction is not to be feasible: the only way to introduce masses M_i is to break again the $O(N)$ symmetry. After angular integration we do not end up with the desired form.

minimal number, the appropriate coefficients are given by:

$$\begin{pmatrix} c_1 \\ . \\ . \\ c_k \end{pmatrix} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ M_1^2 & M_2^2 & \dots & M_k^2 \\ M_1^4 & M_2^4 & \dots & M_k^4 \\ . & . & . & . \\ . & . & . & . \\ M_1^{2k-2} & M_2^{2k-2} & \dots & M_k^{2k-2} \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ 0 \\ 0 \\ . \\ . \\ 0 \end{pmatrix}$$

($\sum_i c_i = -1$ is the minimal condition to have a regularization at all). Since the strongest divergences (to 3 loops) are of order Λ^3, Λ^6 for $d = 3,\ 4$, respectively, we have to introduce at least 2 rsp. 4 different masses.

It turns out, however, that the Pauli-Villars regularization does not obey the constraints listed above. To illustrate this we consider eq. (A.15) for $d = 4$: we find the difference

$$G_{1PV} - \int \delta^{\Lambda}_{PV}(x) G_{PV}(x) dx = \frac{1}{V} \sum_i \frac{c_i}{M_i^2} + \frac{1}{16\pi^2} \sum_{i,k} c_i c_k M_k^2 \frac{M_i^2 ln M_i^2 - M_k^2 ln M_k^2}{M_k^2 - M_i^2}$$

which diverges quadratically. For renormalizability it is not sufficient that this difference is volume independent: as we saw the counter terms are overdetermined and have solutions only because their constraints are not independent. In the presence of such discrepancies, the additional terms would have to match in a very special manner to keep the regularization applicable.

At last we give a brief illustration, why a *sharp cutoff in momentum space* turns out not to be a suitable regularization. More precisely: the sharp momentum cutoff does not even deserve the name "regularization" because its limit $\Lambda \to \infty$ does not always reproduce the non-regularized quantities.

This can be illustrated by considering the contribution to a scattering amplitude at low energy, illustrated in figure 6.

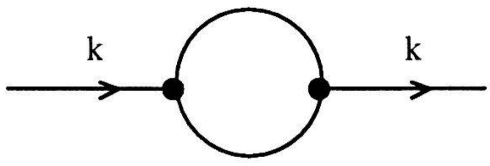

Figure 6: 2 loop contribution to a scattering amplitude at low energy

If we apply a sharp momentum cutoff $\Lambda >> |k|$ and require that the incoming momentum at every vertex vanishes, this contribution will include

$$J = \int dx e^{ikx} G^{\Lambda}(x) \partial^2 G^{\Lambda}(x) \propto \int_B \frac{1}{p^2} d^d p$$

where $B \doteq \{p|(|p| \leq \Lambda) \wedge (|p-k| \leq \Lambda)\}$ is the intersection of the balls with radius Λ and the centers 0 and k. In the 0^{th} approximation ($k = 0$) we get

$$J_0 = \frac{2\pi^{d/2}}{\Gamma(d/2)(d-2)}\Lambda^{d-2}$$

The "moon" to be subtracted from this is to first order in $|k|$ proportional to the sphere of the ball $B|_{k=0}$, hence

$$J = J_0 - \alpha\Lambda^{d-3}|k| + O(|k|^2) \qquad (\alpha > 0)$$

In particular we find for $d = 4$:

$$J = \pi^2\Lambda^2 - \frac{4\pi}{3}\Lambda|k| - \frac{1}{4}\pi^2|k|^2 + O(|k|/\Lambda) \quad .$$

Remembering that the available counter terms have magnitudes $\Lambda^{(d-2)n}$ ($n \in \mathbb{N}$), we see that already the second term is not renormalizable for $d > 3$, i.e. this term is an artifact of the sharp cutoff ("echo effect"), which destroys locality.

The troublesome difference $J - J_0$ stems from the difference $\delta^\Lambda(x) - \delta(x)$; the occurrence of odd power differences in the expansion of J shows the violation of the basic symmetry properties that the correctly regularized $G^\Lambda(x)$ must have.

For the free energy the situation is a little different since the diagrams have no external legs. The role of the perturbation $|k|$ has to be played by $1/L$. Note that e.g. eq. (A.15) holds in this case; however we run into trouble of the kind illustrated above on the 3 loop level when dealing with the integrands that contain more than 2 momenta: the cutoff acts on each of them, hence also on the sums of them but one.

B Generalization of the Polyakov measure

In section 3 we have calculated the measure according to the definition (3.1) containing only the simplest invariant term. The application of this measure in the following sections has been successful, particularly in view of the the perturbative renormalization (section 7, appendix A), which could only be performed due to cancellations of singularities stemming from the measure and from the Lagrangian.

We repeat that from a physical point of view, it is reasonable to require the following properties for terms entering the measure:
a) locality
b) Euclidean translation invariance
c) rotational invariance in isospin space
– just like the terms in the Lagrangian. Now we are going to generalize the measure by

adding an arbitrary linear combination of all additional terms fulfilling these conditions. The generalized measure takes the form:

$$d=3: \quad ds_g^2 = \frac{1}{V}\int\Big\{(d\vec{S})^2[1+b\frac{\Sigma}{F^6}(\vec{H}\vec{S})]+\frac{a}{F^4}(\partial_\mu d\vec{S})^2+\dots\Big\}dx \tag{B.1}$$

$$\begin{aligned} d=4: \quad ds_g^2 = & \frac{1}{V}\int\Big\{(d\vec{S})^2+\frac{a_1}{F^2}(\partial_\mu d\vec{S})^2+\frac{a_2}{F^2}(d\vec{S})^2(\partial_\mu\vec{S})^2 \\ & +\frac{a_3}{F^4}(\partial^2 d\vec{S})^2+\dots \\ & +b_1\frac{\Sigma}{F^4}(d\vec{S})^2(\vec{H}\vec{S})+b_2\frac{\Sigma}{F^6}(\partial_\mu d\vec{S})^2(\vec{H}\vec{S})\dots\Big\}dx \end{aligned} \tag{B.2}$$

where we wrote down the terms that might contribute to our three loop result, with dimensionless coefficients. The selection of these terms has to be performed with some care, paying attention to the relations

$$(\vec{S}d\vec{S})=\frac{1}{2}d(\vec{S}^2)=0=(\vec{S}\partial_\mu\vec{S}) \quad \text{and} \quad \partial_\mu\vec{S}=\Omega\partial_\mu\vec{\pi}\propto L^{-d/2}\ .$$

It appears quite natural to include the further invariants that can be built purely from $\vec{S}$ (with a coefficients), whereas the source dependent terms (with b coefficients) might surprise a little. To us it seems useful to observe the consequences of such terms too. We consider it physically plausible that an external field might influence the metric in configuration space.

Moreover the reduction to the first term is a priori not acceptable since we have exploited the freedom of choice of the fields already when simplifying the Lagrangian in section 2. We recall that we got rid of three coupling constants by suitable redefinitions of the fields. Two of the redefinitions included the magnetic field, which also supports the consideration of the source dependent terms in the measure.

To be explicit, let us start from the generalized measures (B.1), (B.2) and perform the substitutions (2.11), (2.12) and (2.13) of section 2. Then the coefficients of ds_g^2 change in terms of the dimensionless parameters α, β, λ as follows:

$$\{a,b\} \rightarrow \{a-2\alpha, b-2\beta\} \tag{B.3}$$

$$\begin{aligned} \{a_1,a_2,a_3,b_1,b_2\} \rightarrow & \{a_1-2\alpha, a_2+2\alpha, a_3+\alpha^2-2a_1\alpha, \\ & b_1-2\beta, b_2-2\lambda-2(a_1-2\alpha)\beta\} \end{aligned} \tag{B.4}$$

We see that for $d=3$ the full generalization (B.1) has to be considered. Also for $d=4$ all the parameters of ds_g^2 get activated, but the changes in the three $\vec{H}$-independent parameters depend only on α, thus they are not independent. [30] Nevertheless we consider the completely generalized form (B.2).

[30] If we want to eliminate non-leading couplings in the maximally generalized measure instead of the Lagrangian, we can reduce it with these transformations to Polyakov's form for $d=3$, whereas for $d=4$ there remain two of the a_i coefficients.

In his original paper [18], Polyakov did not consider such a generalization. He had renormalizable models in mind and there it is not motivated to include additional invariants, not in the Lagrangian nor in the measure. The two dimensional non-linear σ-model corresponds to the renormalizable case Polyakov refers to. There he mentions that his measure (for bosonic strings composed of two terms) is unique, where he requires locality in a stricter sense than it is done here, i.e. in the sense that also derivative couplings are excluded.

Here the situation is different: non-leading couplings are needed and instead of adding them only in $\mathcal{L}$ (as it is usually done), we can do so in $[d\vec{S}]$ as well. In which way those extensions are related to each other is not evident and will be discussed explicitly.

Of interest is, how far this generalization is physically permissible and to which extent the non-leading coupling constants k_i in the Lagrangian can be replaced by new parameters from the measure. At last, for $d > 2$ those terms were included in order to enable a perturbative renormalization. It will be observed if we need less of them for this purpose after generalizing the measure.

Now we are going to consider the maximal generalization mentioned above. If we insert the collective variables introduced in section 2, expand everything in terms of the transversal fields $\underline{\pi}(x)$ and consider that non-diagonal elements of the metric tensor only enter the determinant quadratically, we arrive at the form:

$$
\begin{aligned}
d=3: \quad ds_g^2 &\cong \frac{1}{V}\int dx\Big\{(d\vec{\pi}-\omega\vec{\pi})^2[1+b\frac{\Sigma}{F^6}H\Omega^{00}]+\frac{a}{F^4}(\partial_\mu d\underline{\pi})^2\Big\} && \text{(B.5)}\\
d=4: \quad ds_g^2 &\cong \frac{1}{V}\int dx\Big\{(d\vec{\pi}-\omega\vec{\pi})^2\Big[1+\frac{a_2}{F^2}\partial_\mu\underline{\pi}\partial_\mu\underline{\pi} && \text{(B.6)}\\
&+b_1\frac{\Sigma}{F^4}H(\Omega^{00}[1-\frac{1}{2}\underline{\pi}^2]+\Omega^{0i}\pi^i)\Big]+\frac{a_1}{F^2}\Big[(\partial_\mu d\underline{\pi})^2+(\partial_\mu(\underline{\pi}d\underline{\pi}))^2\Big]\\
&+b_2\frac{\Sigma}{F^6}H\Omega^{00}(\partial_\mu d\underline{\pi})^2+\frac{a_3}{F^4}(\partial^2 d\underline{\pi})^2\Big\}
\end{aligned}
$$

We begin with $\mathbf{d}=\mathbf{3}$. For the generalized determinant $\mathbf{g}_g$ we get :

$$
\sqrt{\mathbf{g}_g}\cong\sqrt{\mathbf{g}}\; exp\{b\frac{\Sigma}{2F^6}H\Omega^{00}(N-1)V\delta^\Lambda(0)+a\frac{N-1}{2F^4}VD^\Lambda(0)\} \qquad \text{(B.7)}
$$

where $\sqrt{\mathbf{g}}$ is given in (3.13) and we have defined $D^\Lambda(x)$ in (6.2).

The most important property displayed by eq. (B.7) is that the non-leading couplings in the measure do not yield any physically relevant contribution. To be explicit: in dimensional regularization the determinants simply obey the relation:

$$
\mathbf{g}_g=\mathbf{g} \quad . \qquad \text{(B.8)}
$$

We recall that there are relevant measure terms, but they are all included in the leading term that defines Polyakov's measure (3.1).

If we apply our ansatz for the renormalized non-leading coupling constants also on the parameters of the measure, the new counter terms do not enter the renormalizability conditions because they do not multiply any regular term. Thus the non-leading coupling constants of the measure do not contribute new degrees of freedom to the set of counter terms, constrained by renormalization.

For completeness we also show the renormalizability of the leading power divergences involved in this generalization:

$D^{\Lambda}(0) \propto \Lambda^{d+2}$ can be absorbed by the normalization constant $\mathbf{N}$ of the partition function.

The b-contribution shifts in the final result (5.17) ε_3 in the following way:

$$\varepsilon_3 \to \varepsilon_3 + \frac{b}{2}(N-1)\delta^{\Lambda}(0) \tag{B.9}$$

Qualitatively this term is not new: a term $\propto \delta^{\Lambda}(0)$ was already found with k_1. So we can interpret b as a shift of k_1, where k_2 has to perform the same shift in order to keep the regular term $(k_1 - k_2)/V$ unchanged. We conclude that indeed a and b are completely arbitrary, even if we include the leading power divergences.

To be explicit, we just have to replace in eq. (A.5)

$$\sigma_3 \to \sigma_3 + \frac{b}{2F^6}(N-1)\delta^{\Lambda}(0) \quad .$$

Let us consider the more complicated case $\mathbf{d} = \mathbf{4}$. We find:

$$\begin{aligned} ln\sqrt{\mathbf{g}_g} &= \frac{1}{2}tr\varepsilon_g - \frac{1}{4}tr\varepsilon_g^2 \\ &= \frac{1}{2}tr\varepsilon - \frac{1}{4}tr\varepsilon^2 \\ &\quad + a_1\frac{N-1}{2F^2}VD^{\Lambda}(0) + \frac{a_1 + (N-1)a_2}{2F^2}\delta^{\Lambda}(0)\int \partial_\mu\underline{\pi}\partial_\mu\underline{\pi}dx \\ &\quad + (2a_3 - a_1^2)\frac{N-1}{4F^4}V\Delta^{\Lambda}(0) + b_1\frac{\gamma\Omega^{00}}{2F^4}(N-1)\delta^{\Lambda}(0)\left(1 - \frac{1}{2V}\int \underline{\pi}^2 dx\right) \\ &\quad + (b_2 - a_1 b_1)\frac{\gamma\Omega^{00}}{2F^6}D^{\Lambda}(0) \end{aligned} \tag{B.10}$$

Δ^{Λ} is also defined eq. (6.2), and ε_g is the generalization of the matrix $\varepsilon = g - 1$ introduced in section 3. We note that there occur three cancellations in the considered order from the trace of the linear and the quadratic matrix ε_g. They concern the terms

$$b_1\frac{\gamma\Omega^{00}}{2F^4V}tr\varepsilon \quad , \quad \frac{a_1}{2F^2V}\int(\partial_\mu\underline{\pi})^2 dx \qquad \text{and} \quad \frac{a_1}{2F^2}D^{\Lambda}(0)\int \underline{\pi}^2 dx \tag{B.11}$$

Eq. (B.10) reveals that our central observation of $d = 3$ – that the non-leading measure couplings do not contribute any physically relevant term – still holds for $d = 4$. Here eq. (B.8) for the dimensionally regularized system is confirmed on a highly non-trivial level. In particular the first two terms of the list (B.11) would have destroyed this property.

Hence it was justified to evaluate only the leading measure term in section 3.

Since none of the non-leading coupling constants in the measure multiplies any regular term, its meaning is already exhausted with their contributions to the (powerful) counter terms. In particular the measure can not provide any counter terms that enter the renormalization equations (they behave like k_6), so the number of non-leading coupling constants in the Lagrangian required for the renormalization of the three loop result remains unchanged (as we observed for $d = 3$ before).

Concerning the power divergences, we note that the last term of the cancellation list (B.11) would have been forbidden by perturbative renormalizability.

The factor

$$exp\left(\frac{N-1}{2F^2}\left[a_1 D^\Lambda(0) + \frac{2a_3 - a_1^2}{2F^2} V\Delta^\Lambda(0)\right]\right)$$

of $\sqrt{g_g}$ can be absorbed by N. The rest changes ε_2, ε_3 in the following way (referring to (6.6), (6.8))

$$\varepsilon_2 \quad\longrightarrow\quad \varepsilon_2 + b_1\frac{N-1}{2}\delta^\Lambda(0) \tag{B.12}$$

$$\varepsilon_3 \quad\longrightarrow\quad \varepsilon_3 - \frac{N-1}{2}\Big[\{a_1 + (N-1)(a_2 + b_1/2)\}\delta^\Lambda(0)G_1 + (a_1 b_1 - b_2)D^\Lambda(0)\Big] \tag{B.13}$$

where again a forbidden term ($\propto V(\delta^\Lambda(0))^2 G_1$) cancels in ε_3.

The singularities in the additional terms can be renormalized by generalizing the formulas (7.8). The counter terms of the leading coupling constants receive the following additional summands (with respect to eqs (A.8) and (A.13)) :

$$\begin{aligned}
\sigma_2 &\quad\longrightarrow\quad \sigma_2 + \frac{N-1}{2F^4}b_1\delta^\Lambda(0) \\
\sigma_3 &\quad\longrightarrow\quad \sigma_3 - \frac{N-1}{2F^6}\Big[\{a_1 + (N-1)(a_2 + b_1/2)\}\delta^\Lambda(0)G_1^\Lambda + (a_1 b_1 - b_2)D^\Lambda(0)\Big] \\
f_2 &\quad\longrightarrow\quad f_2 - \frac{N-1}{2F^4}\{a_1 + (N-1)(a_2 + b_1/2)\}\delta^\Lambda(0)
\end{aligned}$$

Since an important motivation for considering $d^2 s_g$ was given by the transformations (2.11)...(2.13), let us at last take a look at their actual effect, i.e. we want to observe the outcome if the three terms

$$g_4^{(1)}(\partial^2\vec{S})^2 \quad ; \quad h_{1,2}^{(2)}(\vec{H}\partial^2\vec{S}) \quad ; \quad h_{1,4}^{(3)}(\vec{H}\vec{S})(\partial^2\vec{S})^2$$

are included in the Lagrangian. From power counting we see that for $d = 3$ all the three terms can affect our result only to the third order, unlike $d = 4$ where $g_4^{(1)}$ and $h_{1,2}^{(2)}$ could appear already to the second order.

Next we recall that the elimination of the $h_{1,2}^{(2)}$-term is only motivated by the possibility of a space-dependent magnetic field $\vec{H}(x)$. In the case of a constant external field considered here, this term does not contribute to the action.

Let us discuss the influence of the remaining two terms, provided with dimensionless coupling constants K_1, K_2 (we recall that we choose F to be the only dimension-carrying coupling). We modify the Lagrangian of section 2 as follows:

$$\mathcal{L} \;\rightarrow\; \mathcal{L} + K_1\frac{1}{2F^2}(\partial^2\vec{S})^2 \qquad (d=3)$$

$$\mathcal{L} \;\rightarrow\; \mathcal{L} + \frac{1}{2}K_1(\partial^2\vec{S})^2 - K_2\frac{\Sigma}{F^6}(\vec{H}\vec{S})(\partial^2\vec{S})^2 \qquad (d=4)$$

For $d = 3$ only the K_1-term is relevant to 3 loops. In the final result, this alters the argument of the Bessel function such that

$$\varepsilon_3 \rightarrow \varepsilon_3 + \frac{1}{2}K_1(N-1)\left(\delta^\Lambda(0) - \frac{1}{V}\right)$$

The same kinds of terms were also found with the k_i couplings. In dimensional regularization the singularity structure does not change due to K_1. However, in contrast to the non-leading couplings in the measure, K_1 creates a regular contribution.

As a consequence, the new counter term introduced by K_1 rises the degree of freedom of the set of logarithmic counter terms from 1 to 2.

Referring to the renormalization of the leading power divergences we first note that a forbidden term $\propto (N-1)VG_1D^\Lambda(0)$ cancels in ε_3. Hence it suffices to replace $\sigma_3 \rightarrow \sigma_3 + \frac{1}{2}K_1(N-1)\delta^\Lambda(0)$.

For $d = 4$ the action to three loops changes as follows:

$$S \rightarrow S + \frac{1}{2}K_1\int\left[(\partial^2\underline{\pi})^2 + (\partial_\mu\underline{\pi}\partial_\mu\underline{\pi} + \underline{\pi}\partial^2\underline{\pi})^2\right]dx - K_2\frac{\gamma\Omega^{00}}{F^4V}\int\partial^2\underline{\pi}\partial^2\underline{\pi}dx$$

Here the modifications are much more complicated, mainly because the K_1-term is of first order. A lengthy calculation – along the lines of section 6 – yields:

$$\varepsilon_2 \;\rightarrow\; \varepsilon_2 + \frac{1}{2}K_1(N-1)\left(\delta^\Lambda(0) - \frac{1}{V}\right)$$

$$
\begin{aligned}
\delta_3 &\rightarrow \delta_3 - K_1 \frac{N-1}{2V} G_1 \\
\varepsilon_3 &\rightarrow \varepsilon_3 + (N-1)\big\{ K_1([N+1]V\delta^\Lambda(0) - 3N + 5)\frac{1}{4V} G_1 + 2K_1 G_{\mu\nu}\dot{G}_{\mu\nu} \\
&\quad + (K_1[k_1 - \frac{1}{2}K_1] + K_2)\ D^\Lambda(0)\big\}
\end{aligned}
$$

The result for ε_3 in $d = 3$ is shifted down to ε_2 here, since $1/V$ is classified in the second order now. Also for $d = 4$ only K_1 is relevant in dimensional regularization. K_2 does not contribute to the considered order, i.e. it behaves like k_6 and all the non-leading couplings in the measure. Also the observation still holds that K_1 does not change the singularity structure but it does change the regular part. Its associated (logarithmic) counter term rises again the degree of freedom of the logarithmic counter terms from 1 to 2. (Note that this counter term enters constraint (7.7) – which was identically imposed by ε_2 and δ_3 – both times in such a way that the constraints due to ε_2 and δ_3 remain identical.)

In the notation used at the end of section 5 and 6, the terms occurring in the intermediary results and cancelling at the end are:

$$
\begin{aligned}
\varepsilon_2 &: (N^2,N,1)K_1G_1VD^\Lambda(0) \\
\delta_3 &: (N^3,N^2,N,1)VD^\Lambda(0)G_1^2,\ (N^2,N,1)D^\Lambda(0)G_2,\ (N^2,N,1)\delta^\Lambda(0)G_1,\ N^2G_1/V \\
\varepsilon_3 &: K_1 : (N^3,N^2,N,1)\{G_1V(\delta^\Lambda(0))^2, V^2\delta^\Lambda(0)D^\Lambda(0)G_1^2, VD^\Lambda(0)G_1^2, G_2D^\Lambda(0)\}, \\
&\qquad (N^2,N,1)VD^\Lambda(0)\delta^\Lambda(0)G_2,\ N^3\{G_1/V, \delta^\Lambda(0)G_1\} \\
&\quad K_1^2 : (N^3,N^2,N,1)G_1(VD^\Lambda(0))^2,\ (N^2,N,1)\{G_1V\Delta^\Lambda(0), V\delta^\Lambda(0)D^\Lambda(0)\}, \\
&\qquad N^2D^\Lambda(0) \\
&\quad K_1k_1 : (N^2,N,1)V\delta^\Lambda(0)D^\Lambda(0),\ N^2D^\Lambda(0)
\end{aligned}
$$

Most of these cancellations are required by renormalizability of the structure with power divergences (in particular there are no counter terms available with volume-dependent prefactors, whose powers of L are larger than zero; in the δ-sector even pure singularities are forbidden.)
The exceptions are N^2G_1/V in δ_3 and $N^3\{G_1/V, \delta^\Lambda(0)G_1\}$, $N^2D^\Lambda(0)\{K_1^2, K_1k_1\}$ in ε_3: here the singularities could be absorbed, but each of these terms would be a strange novelty to compare with section 6, [31] whereas existing terms just shift the regular and the power divergent contributions of the coupling constants k_i. About the renormalization we remark that since the singular contributions associated with K_1 and K_2 consist of terms we found before in section 6, their renormalization – along the lines of appendix A – works without any problems.

The transformations (2.11)... (2.13) alter the non-leading coupling constants of the Lagrangian like this:

$$
k_1 \rightarrow k_1 - \alpha - \beta\ , \quad k_2 \rightarrow k_2 - \beta\ , \quad k_3 \rightarrow k_3 + \beta
$$

[31] Such terms would not permit the field transformation invariance of Z, pointed out below.

$$k_4 \to k_4 + 4\alpha\ , \quad k_6 \to k_6 + \alpha^2 - 2\alpha K_1 \tag{B.14}$$
$$K_1 \to K_1 - 2\alpha\ , \quad K_2 \to K_2 - \frac{1}{2}\alpha^2 + 2\alpha k_1 + \beta K_1 - 2\alpha\beta + \lambda$$

and we can confirm our claim of section 2 that K_1 and K_2 can be chosen to be zero, as well as k_2 or k_3.

If we include in the partition function (for $d = 3$ and $d = 4$) *all* possible couplings in the Lagrangian and in the measure and apply the transformation rules for both sets of couplings, we observe that $Z(\vec{S}, \vec{H}, k_i, K_i, a_i, b_i)$ – including the leading power divergences – is invariant under the transformations (2.11) ... (2.13) for arbitrary α, β and λ. [32] This is a very sensitive consistency test for our results.

Thus we have found an other remarkable aspect that supports that the structure with all the leading divergences still fulfills the important properties. This invariance is due to an exchange of terms among all the non-leading coupling constants, except for k_5.

As a corollary we can conclude that the overall invariance of Z under the discussed field transformations also holds for the dimensionally regularized system. There, however, the consistency test is less sensitive; only $k_1 \dots k_4$ and K_1 participate in an exchange of regular terms and transformation (2.13) is irrelevant.

The conclusion of this appendix is that the maximal generalization of the measure – including all the terms fulfilling the three physical properties listed in the beginning of this appendix – is permitted by perturbative renormalizability and only affects the power divergences in the Lagrangian, i.e. in dimensional regularization they do not yield any contribution at all. Hence they do not reduce the number of required non-leading coupling constants in the Lagrangian.

Since the coefficients for the non-leading terms are completely arbitrary, there is an infinite set of equivalent measures of the path integral for our model – each measure corresponding to a particular type of quantization – that all yield the same physically relevant contributions. The Polyakov measure belongs to this set and has the advantages of its simplicity and its compatibility to renormalizable theories.

Without the transformations eliminating some terms in the Lagrangian, in the regular part the coupling constants $k_1 \dots k_5$ would have been shifted. The singularities, which are present in dimensional regularization, however, are not affected by these transformations. The leading power singularity structure would have been altered without destroying its perturbative renormalizability.

[32] To the considered order there occur no mixed products of non-leading coupling constants from the measure and from the Lagrangian. So to verify this invariance one just has to insert the transformation rules in the results given above.

C Identification of the terms by massive expansion

If we assume the GB to be massive, their Green functions take the form

$$G_m(x) = \frac{1}{V}\sum_{\bar{n}} \frac{e^{ip_{\bar{n}}x}}{m^2 + p_{\bar{n}}^2} \quad .$$

For small masses we can expand this in terms of the massless $G(x)$:

$$G_m(x) = \frac{1}{Vm^2} + G(x) - m^2\dot{G}(x) + \frac{m^4}{2}\ddot{G}(x) + \cdots$$

where $\dot{G} \doteq -\frac{d}{dm^2}G_m|_{m=0}$ etc.

In our case we have to insert $m^2 = \frac{\Sigma H}{F^2} = \frac{\gamma}{F^2 V}$, so: $-m^2\frac{d}{dm^2}|_{m=0} \propto \frac{\gamma}{F^2}L^{2-d}$ lowers the magnitude by one unit.
Thus we get e.g. the series $\frac{1}{F^2}G_1$, $\frac{1}{F^4 V}G_2$, $\frac{1}{F^6 V^2}G_3 \dots$ (index = order).

Gerber and Leutwyler carried out a 3-loop calculation with G_m for d=4 [9]. There only the temperature was finite, not the spatial box, but still the type of terms ought to coincide with ours. They found, except for G_1, G_2, G_3 the terms:

$$\frac{m^4}{F^4}\int G_m^4 dx \ , \quad \frac{1}{F^4}\int (\partial_\mu G_m \partial_\mu G_m)^2 dx \ , \quad \frac{1}{F^4}(\partial_{\mu\nu}G_m(x)|_{x=0})^2$$

We want to show that this corresponds to the type of terms we found up to the 3^{rd} order.

$$i) \qquad m^4 \int \left(\frac{1}{Vm^2} + G - m^2\dot{G} + \frac{m^4}{2}\ddot{G}\dots \right)^4 dx$$
$$= \frac{1}{V^3 m^4} + \frac{6}{V}G_2 + \frac{4m^2}{V}(J_3 - \frac{3}{2V}G_3) + O(m^4)$$

If we proceed one order by $-m^2\frac{d}{dm^2}$ and put $m = 0$, we can identify J_3 as a 3^{rd} order term, which is relevant for us since m^2 introduces an H-dependence. (The singularity at $m = 0$ corresponds to the 0-mode.)

$$ii) \quad \int \partial_\mu G_m \partial_\mu G_m \partial_\nu G_m \partial_\nu G_m dx = J_2 + 4m^2\Gamma_3 + 0(m^4)$$

Here we can confirm in exactly the same way the term Γ_3.

iii) For $d = 4$: $\frac{1}{F^4}(G_{\mu\nu})^2 \propto L^{-8}$, so $\frac{1}{F^4}G_{\mu\nu}\dot{G}_{\mu\nu}$ can enter the 3^{rd} order, but not with a factor m^2. As we know from section 6, it does occur, with factors $\frac{\gamma}{F^2}k_j$ instead of m^2. For $d = 3$ this term is only of 4^{th} order, in contrast to the results of *i*) and *ii*) that are classified equally for $d = 3$ and $d = 4$.

To look at *iii*) one might wonder if one couldn't include terms of even lower orders that one gets by further dot-derivatives.

But introducing such terms – or also the corresponding terms that could be produced in *i*) and *ii*) – would contradict our low energy expansion based on two leading coupling constants as well as the perturbative renormalizability.

D Transformation to the modified Bessel function

The differential equation (4.2) and its first derivative state:

$$\int d\Omega e^{z\Omega^{00}}\left[z^2\Omega^{00\ 2}+(N-1)z\Omega^{00}-z^2\right] = 0 \qquad \text{(D.1)}$$

$$\int d\Omega e^{z\Omega^{00}}\left[z^3\Omega^{00\ 3}+Nz^2\Omega^{00\ 2}-z^3\Omega^{00}-z^2\right] = 0 \qquad \text{(D.2)}$$

We apply on [33]

$$\begin{aligned}&\int d\Omega e^{\gamma\Omega^{00}(1+\alpha_1+\alpha_2+\alpha_3)+(\gamma\Omega^{00})^2(\beta_2+\beta_3)+(\gamma\Omega^{00})^3\gamma_3}\\ &\cong \int d\Omega e^{\gamma\Omega^{00}}\quad \Big[1+\gamma\Omega^{00}(\alpha_1+\alpha_2+\alpha_3)+(\gamma\Omega^{00})^2(\tfrac{\alpha_1^2}{2}\alpha_1\alpha_2+\beta_2+\beta_3)\\ &\qquad +(\gamma\Omega^{00})^3(\tfrac{1}{6}\alpha_1^3+\alpha_1\beta_2+\gamma_3)\Big]\end{aligned}$$

the transformations enabled by (D.1) and (D.2) with the factors $-a$, $-b$ respectively:

$$\begin{aligned}&= \int d\Omega e^{\gamma\Omega^{00}}\quad \Big[1+\gamma^2(a+b)+\gamma\Omega^{00}(\alpha_1+\alpha_2+\alpha_3-(N-1)a+\gamma^2 b)\\ &\qquad +(\gamma\Omega^{00})^2(\tfrac{\alpha_1^2}{2}+\alpha_1\alpha_2+\beta_2+\beta_3-a-Nb)\\ &\qquad +(\gamma\Omega^{00})^3(\tfrac{\alpha_1^3}{6}+\alpha_1\beta_2+\gamma_3-b)\Big]\end{aligned}$$

This shall take the form: $e^{\delta_1+\delta_2+\delta_3}\int d\Omega e^{\gamma\Omega^{00}(1+\varepsilon_1+\varepsilon_2+\varepsilon_3)}$. Expanding this and comparing the coefficients to the orders of $(\gamma\Omega^{00})$ we get four equations for the unknown $a, b, \delta, \varepsilon$. On the 3 levels these are 12 variables to be determined by the 3 levels of the 4 equations, which impose 12 constraints. a and b we don't need to know explicitly, so we first eliminate them. From the remaining 2 equations we can determine on the level ℓ : δ_ℓ and ε_ℓ.

$$\begin{aligned}\delta_1 &= 0 & \varepsilon_1 &= \alpha_1\\ \delta_2 &= \gamma^2\beta_2 & \varepsilon_2 &= \alpha_2-(N-1)\beta_2\\ \delta_3 &= \gamma^2(\beta_3-(N-1)\gamma_3) & \varepsilon_3 &= \alpha_3+(N-1)(\alpha_1\beta_2-\beta_3)+\gamma_3(\gamma^2+N(N-1))\end{aligned}$$

This has been inserted in sections 5 and 6 in order to transform the row result for the partition function into the desired form.

[33]As usual the index coincides with the order of magnitude.

Corresponding transformations into this form are possible for results to any order. For the order ℓ we derivate (4.2) $\ell - 2$ times, get a differential equation of degree ℓ that allows to eliminate the term with $(\gamma\Omega^{00})^\ell$, etc. Thus we arrive inductively at the desired from.

E Conclusions about the non-linear σ-model in lower dimensions

For the three and the four dimensional case that we discussed in the main part of this paper, all the $O(N)$-invariant terms had to be included in order to enable a perturbative renormalization. This is different for the lower dimensions; there it is sufficient to include the leading coupling constants Σ and F. Thus in one dimension all singularities disappear (in the source dependent part we consider); in two dimensions all the singularities become logarithmic and can be renormalized solely by these two couplings.

E.1 The one dimensional case

Since we expand generally in powers of L^{d-2}/F^2, here the quantity that has to be small to permit our expansion is L/F^2, i.e. in contrast to the higher dimensions L must be small (one could imagine it to be a time slice) or the energy must be high when we translate our considerations to a scattering process. With respect to the partition function we deal with a high temperature expansion. Thus physics is turned upside down (note also that there are no GB's any more) but the mathematical methods remain applicable. In particular the zero mode is not weighted strongly any more, but its contribution still diverges and its treatment with collective variables is still a solution of this problem.

The explicit, finite terms that remain if we include only Σ and F are easily obtained:

$$\begin{aligned} G_1 &= \frac{L}{2^2 3}\,, \qquad \frac{1}{V}G_2 = \frac{L^2}{2^4 3^2 5} \qquad \frac{1}{V^2}G_3 = \frac{L^3}{2^4 3^3 5\cdot 7} \\ \frac{1}{V}J_3 &= \frac{1}{2V^2}G_3\,, \qquad \Gamma_3 = \frac{9}{4V^2}G_3 \end{aligned} \tag{E.1}$$

Clearly, including non-leading derivative couplings – or higher powers of the magnetic field – does not make sense if we expand in positive powers of L. (This concerns also non-leading couplings in the measure.)

Hence we let $k_i \equiv 0$ and find the partition function to be of the form (5.17) with

$$\varepsilon_1 = -\frac{N-1}{2^3 3}L \tag{E.2}$$

$$\varepsilon_2 = -\frac{(N-1)(N-3)}{2^7 3 \cdot 5} L^2 \tag{E.3}$$

$$\varepsilon_3 = \frac{(N-1)(5N^2 - 440N + 843 - 16\gamma^2)}{2^{10}3^4 5 \cdot 7} L^3 \tag{E.4}$$

$$\delta_2 = \frac{N-1}{2^6 3^2 5} L^2 \tag{E.5}$$

$$\delta_3 = \frac{(N-1)(13N-43)}{2^8 3^4 5 \cdot 7} L^3 \tag{E.6}$$

This model describes a free quantum mechanical spin 0 particle moving on a N dimensional unit sphere where L is the inverse temperature (or the time for a short time transition amplitude). Turning on the external magnetic field means geometrically a shift of the center of the sphere away from the origin. (Higher powers of $\vec{H}$ in $\mathcal{L}^{(sb)}$ would additionally deform the sphere).

E.2 The two dimensional case

This case has attracted very much attention in the literature because it is renormalizable and it represents an interesting toy model for QCD, see below.

We consider again the simplification $k_i \equiv 0$. Then the three loop partition function can be taken from section 5 or 6. The only terms that remain singular in two dimensions are G_1 and Γ_3.

Here dimensional regularization looses its special meaning since all the divergences are logarithmic. In particular:

$$G_1^\Lambda = c \cdot ln(\Lambda/\mu)$$

where Λ refers to the regularization parameter introduced in section 7, c is a positive constant and μ determines the mass scale. G_1 is independent of this scale, so

$$\mu \frac{\partial}{\partial \mu} G_1^\Lambda = -\mu \frac{\partial}{\partial \mu} g_1 \quad . \tag{E.7}$$

Γ_3 contains a term quadratic and a term linear in $ln\Lambda$, so it can be written in the form:

$$\Gamma_3 = c_A G_1^{\Lambda\,2} + c_B g_1 G_1^\Lambda + \gamma_3$$

where c_A, c_B are constants and γ_3 is regular. Analogously to (E.7) we find

$$\mu \frac{\partial}{\partial \mu} \gamma_3 = -c \cdot c_B g_1 \tag{E.8}$$

$$2c_A = c_B \quad . \tag{E.9}$$

Now we come to the renormalized coupling constants Σ_r and Φ_r, where we rename $\Phi \doteq 1/F^2$. They can be taken from appendix A with slight modifications:

$$\begin{aligned}
\Sigma_r &= \Sigma\Big[1 - \frac{N-1}{2}\Phi G_1^\Lambda - \frac{(N-1)(N-3)}{8}\Phi^2 G_1^{\Lambda 2} \\
&\quad - \frac{(N-1)(N-3)(3N-7)}{48}\Phi^3 G_1^{\Lambda 3} - (N-1)(N-2)c_A \Phi^3 G_1^{\Lambda 2} + O(\Phi^4)\Big] \\
\Phi_r &= \Phi\Big[1 + (N-2)\Phi G_1^\Lambda + (N-2)^2\Phi^2 G_1^{\Lambda 2} + 2(N-2)c_B \Phi^2 G_1^\Lambda + O(\Phi^3)\Big]
\end{aligned}$$

Hence their β-functions are :

$$\beta_\Sigma \doteq \mu\frac{\partial \Sigma_r}{\partial \mu} = \frac{N-1}{2}\cdot c \cdot \Sigma_r \Phi_r + O(\Phi_r^4) \tag{E.10}$$

$$\beta_\Phi \doteq \mu\frac{\partial \Phi_r}{\partial \mu} = -(N-2)\cdot c \cdot \Phi_r^2 - 2(N-2)\cdot c \cdot c_B \Phi_r^3 + O(\Phi_r^4) \tag{E.11}$$

The cancellation of the second and third order in the β-function of Σ is a consequence of our choice of the renormalization prescription.

If we include only the leading order of β_Φ , we find the solution:

$$\Phi_r(\mu) = \frac{\Phi_{r,s}}{1 + (N-2)\cdot c \cdot \Phi_{r,s} ln(\mu/\mu_s)} \tag{E.12}$$

where μ_s is a particular scale (i.e. the choice of one among the trajectories that solve the differential equation) and $\Phi_{r,s} \doteq \Phi_r(\mu_s)$.

For $N > 2$ this solution has a "Landau pole" at

$$\mu_L = \mu_s exp(-[(N-2)\cdot c \cdot \Phi_{r,s}]^{-1})$$

but of course in this regime the perturbative expansion is not applicable.

For large μ, however, Φ_r goes asymptotically to zero and perturbation theory becomes reliable. We clearly recognize asymptotic freedom for high energies (large μ), *just as in QCD.*

For higher orders in Φ, the fact that the β-functions must be independent of Λ yields in Σ_r, Φ_r immediately the 2, 3 counter terms (respectively), which are leading in the power of $ln(\Lambda/\mu)$.

The renormalization group equation also provides us with some knowledge about the partition function itself. Z has to be independent of μ and a variation of the quantity $\gamma = \Sigma H V$ shows that this independence holds separately for the argument of the modified Bessel function and the exponent of the Ω-independent prefactor. Also there it determines the coefficients of the leading divergences to higher orders of Φ.

But in this way we do not get any information about the regular contributions of the multiloop terms, which are actually of physical interest. Work about the way to deduce such information and its limits is in progress.

References

[1] C.G. BOLLINI, J.J. GIAMBIAGI, *Nuov. Cim.* **12B** (1972) 20
G. T'HOOFT, M. VELTMAN, *Nucl. Phys. B* **44** (1972) 189

[2] R. DASHEN, H. NEUBERGER, *Phys. Rev. Lett.* **50** (1983) 1897

[3] R. DASHEN, *Phys. Rev.* **183** (1969) 1245 and 1291 (with M. Weinstein)
S. COLEMAN, J. WESS, B. ZUMINO, *Phys. Rev* **177** (1969) 2239 and 2247 (with C.G. Callan)
H. PAGELS, *Phys. Rep.* **C16** (1975) 219

[4] J. GASSER, H. LEUTWYLER, *Phys. Lett. B* **125** (1983) 321, 325

[5] J. GASSER, H. LEUTWYLER, *Ann. Phys. (N.Y.)* **158** (1984) 142

[6] J. GASSER, H. LEUTWYLER, *Nucl. Phys. B* **250** (1985) 465
For a review of numerical simulations, see
E. MARINARI, *Nucl. Phys. (Proc. Suppl.)* **B9** (1989) 209

[7] J. GASSER, H. LEUTWYLER, *Phys. Lett. B* **184** 83 and *B* **188** 477 (1987)

[8] J. GASSER, H. LEUTWYLER, *Nucl. Phys. B* **307** (1988) 763

[9] P. GERBER, H. LEUTWYLER, *Nucl. Phys. B* **321** (1989) 387

[10] A. HASENFRATZ, T. NEUHAUS, *Nucl. Phys. B* **297** (1988) 205

[11] P. HASENFRATZ, *Phys. Lett. B* **141** (1984) 385

[12] P. HASENFRATZ, J. NAGER, *Z. Phys. C, Part. Fields* **37** (1988) 477 and Acta Phys Hung. 64 (1988) 147
A. HASENFRATZ, K. JANSEN, C.B. LANG, T. NEUHAUS, H. YONEYAMA, *Phys. Lett.* **199** (1987) 531 and Nucl. Phys. B317 (1989) 81 (with J. Jersak)
K. JANSEN, *Nucl. Phys. (Proc. Suppl.)* **B4** (1988) 422
J. KUTI, L. LIN, Y. SHEN, *Nucl. Phys. (Proc. Suppl.)* **B4** (1988) 397 and Phys. Rev. Lett. 61 (1988) 678

[13] For recent reviews, see e.g.
P. HASENFRATZ *Nucl. Phys. B (Proc. Suppl.)* **9** (1989) 3
H. NEUBERGER, *Nucl. Phys. B, Proc Suppl.* **17** (1990) 17
Since N is arbitrary throughout this paper, let us mention that for $N \to \infty$ the problem

is exactly solvable, but sets the upper bound of the Higgs mass much higher, see L. LIN, J. KUTI, Y. SHEN , in Proc. of Lattice Higgs Workshop, Tallahasee, FL (World Scientific, Singapore 1988) p.186

[14] P.HASENFRATZ, H. LEUTWYLER, *Nucl. Phys. B* **343** (1990) 241

[15] H. LEUTWYLER, *Nucl. Phys. (Proc. Suppl.)* **B4** (1988) 248
H. LEUTWYLER, *Phys. Lett. B* **189** (1987) 197

[16] A. LINDE, *Pis'ma Zh. Eksp. Teor. Fiz.* **23** (1976) 73
M. LINDNER, *Z. Phys.* **C 31** (1986) 295

[17] M. LÜSCHER, *Nucl. Phys. B* **219** (1983) 233

[18] A.M. POLYAKOV, *Phys. Lett.* **103B** (1981) 207
In this context also of interest is
J. HONERKAMP, *Nucl. Phys. B* **36** (1972) 272

[19] P. RAMOND, Field Theory, (Addison-Wesley, Redwood City (USA) 1989)

[20] G. T'HOOFT, *Nucl. Phys. B* **61** (1973) 455
ST. WEINBERG, *Phys. Rev. D* **8** (1973) 3497

[21] A.A. TSEYTLIN, *Int. J. Mod. Phys. A* **vol. 4 no. 6** (1989) 1257
Y.X. CHEN, *Prog. Theor. Phys.* **83** (1990) 1213
S. RANDJBAR-DAEMI, A. SALAM, J.A. STRATHDEE *Int. J. Mod. Phys. A* **2** (1987) 667
E. ABDALLA, M.C.B. ABDALLA, A. LIMA-SANTOS, *Int. J. Mod. Phys. A* **2** (1987) 753
For a purely metric calculation, see
S.J. GRAHAM, *Phys. Lett. B* **197** (1987) 543
A. FOAKES, N. MOHAMMEDI, *Phys. Lett. B* **198** (1987) 359

[22] S. WEINBERG, *Physica* **96A** (1979) 327

[23] S. WEINBERG, *Phys. Rev.* **166** (1968) 1586