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Defining Relative Energies For The Projected Ising Measure

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Abstract

Schonmann's [SCH] projection on a one-dimensional layer of the pure phases of the two-dimensional Ising model is investigated via the low temperature expansion. The non-Gibbsian character is illustrated by the identification of bad configurations. One-dimensional Gibbs measures are constructed in a restricted ensemble which resemble the original projection.

Keywords: Gibbs measures, low temperature expansion, projection, restricted ensemble.

1 Introduction.

There has recently been some interest in discovering and investigating non-Gibbsian measures describing the state of certain physically interesting systems. While this problem is apparently relevant for understanding so called non-equilibrium behavior and the nature of invariant states for certain dynamical systems, it has wider applications in exploring the frontiers of the Gibbs formalism.
An important question thereby is whether the standard methods of equilibrium statistical mechanics can still play a role in describing and classifying these non-Gibbsian measures. This probably depends on how close the state of the system is to being Gibbsian and we must turn to specific examples. They include discretizations of massless Gaussians [LM], measures invariant for certain stochastic dynamics ([LS],[MS]), fixed point measures for

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certain renormalization-group transformations [DvE] and also certain once renormalised Gibbs measures [vEFS].
A particularly nice example was provided by Schonmann's projection of the two-dimensional Ising phases on a one-dimensional layer. In [SCH] he proved that, below the critical temperature, there is no translation invariant summable interaction for which this projection is a Gibbs measure. The techniques he used ingeniously combine large deviations results, percolation methods and ferromagnetic correlation inequalities. In the light of the preceding question we were however motivated to understand this result more directly. We want to see what goes wrong in the low temperature expansion if we try to construct this projection as a Gibbs measure. We thus express the conditional probability distribution of the spin at the origin, given the other spin values on the one-dimensional layer, in a low temperature series and investigate the uniform convergence. Our findings can roughly be summarized as follows:

1. If, starting from the + phase of the two-dimensional Ising model, there would correspond a continuous version of the conditional probabilities of the projection, then, the same thing would hold starting from the − phase and the associated interactions would be the same (see Proposition 2 in [SCH]). We see however in our construction that this is not the case.

2. The Gibbsian description of the projected Ising phases amounts to a low temperature analysis of a one-dimensional quasi-local Ising system and therefore fails.

3. Relative energies can still be defined for the projection by the appropriate restriction of the allowed configurations, cf. the basic definition of Gibbs measures for unbounded spins [GE],[LP]. We have contented ourselves here with keeping a set of configurations which, for any decent Gibbs measure on the total configuration space, has measure zero. We believe this is far from optimal but the set is still sufficiently large (and closed) to define infinite volume Gibbs measures on it which resemble, for certain global properties, the original projected measures. In particular, on finite regions, they converge to each other as the temperature goes to zero. This procedure can also be found in studies of thermodynamic metastability [VA], the droplet picture for coexistence of phases [PF], and certain extensions of Pirogov-Sinai Theory [BKL].

We come back to these three points in section 4. The rest of the paper is organized as follows. Section 2 gives the necessary definitions and notations. Section 3 contains the standard low temperature set up. Section 4 is devoted to the discussion of our main observations. Section 5 contains the proofs. In an Appendix we briefly sketch the (more trivial) situation which exists at high temperature.

2 Definitions and notations.

Consider the planar lattice $\mathbf{Z}^2$ for which each site $i \in \mathbf{Z}^2$ has coordinates (x, y). The integer lattice $\mathbf{Z}$ can be viewed as a one-dimensional sublattice containing the origin. That is, we take $i \in \mathbf{Z}$ iff $i = (x, 0)$.
An Ising spin configuration on $\mathbf{Z}^2$ is an element σ of $\Theta = \{-1, +1\}^{\mathbf{Z}^2}$ in which to every site $i \in \mathbf{Z}^2$ a spin value $\sigma_i = \pm 1$ is associated. Similarily, $\Omega = \{-1, +1\}^{\mathbf{Z}}$ contains all spin configurations $\xi = \{\xi_x, x \in \mathbf{Z}\}$ on $\mathbf{Z}$. For any set $A \subset \mathbf{Z}$ and configurations $\sigma \in \Theta$, $\xi \in \Omega$, we say that $\sigma = \xi$ on A iff $\sigma_{(x,0)} = \xi_x, \forall x \in A$.

For a sequence of side lenghts $L \uparrow \infty$, we consider the square boxes

$$V \equiv V_L = \{i = (x, y) \in \mathbf{Z}^2 \mid -L \leq x, y \leq L\}$$

and their one-dimensional segments

$$\Lambda = V \cap \mathbf{Z} = \{i = (x, y) \in V \mid y = 0\}$$

The subvolume W on top of Λ is defined by

$$W = \{i = (x, y) \in V \mid y > 0\}.$$

As $L \uparrow \infty$, W tends to the upper half plane. With $+$ boundary conditions, the energy of a configuration σ on V is given by

$$H_V^+(\sigma) = - \sum_{<ij> \cap V \neq \emptyset} (\sigma_i \sigma_j - 1), \tag{2.1}$$

where the sum is over all nearest neighbor pairs $< ij >, i \in V$ or $j \in V$. We put $\sigma_i = +1$ for all $i \notin V$.
Let μ_V^+ be the corresponding Gibbs measure on V at inverse temperature β, i.e.

$$\mu_V^+(\sigma) = Z_V^{-1}(\beta) \exp(-\beta H_V^+(\sigma)),$$

where $Z_V(\beta)$ is the usual normalizing partition function. We want to study the projection ν_Λ^+ of μ_V^+, i.e. the restriction of μ_V^+ to the segment Λ. For a configuration ξ on Λ, ν_Λ^+ has weights

$$\begin{aligned} \nu_\Lambda^+(\xi) &= \mu_V^+(\sigma = \xi \text{ on } \Lambda) \\ &= \sum_{\sigma_i = \pm 1, i \in V \setminus \Lambda} \mu_V^+(\sigma) \mid_{\sigma = \xi \text{ on } \Lambda} . \end{aligned} \tag{2.2}$$

Obviously, for any finite L and β, ν_Λ^+ is a Gibbs measure for some (one-dimensional) Hamiltonian

$$\mathcal{H}_\Lambda^+(\xi) = -\log \nu_\Lambda^+(\xi). \tag{2.3}$$

Its conditional probability distribution at the origin is

$$\nu_\Lambda^+(\sigma_o = \xi_o \mid \sigma = \xi \quad \text{on} \quad \Lambda\backslash o) = \frac{1}{1+\exp(h_\Lambda^+(\xi))}. \tag{2.4}$$

Here,

$$h_\Lambda^+(\xi) = -\log \frac{\nu_\Lambda^+(\xi)}{\nu_\Lambda^+(\xi^o)} \tag{2.5}$$

is the energy difference $\mathcal{H}_\Lambda^+(\xi) - \mathcal{H}_\Lambda^+(\xi^o)$ or relative energy for flipping the spin at the origin:

$$\xi_x^o = \begin{cases} \xi_x, & \text{if } x \neq 0 \\ -\xi_o, & \text{if } x = 0 \end{cases}.$$

Also note that

$$\exp(-h_\Lambda^+(\xi)) = \exp(2\beta\xi_o(\xi_1 + \xi_{-1}))[\frac{Z_W^{+,\xi}(\beta)}{Z_W^{+,\xi^o}(\beta)}]^2, \tag{2.6}$$

where

$$Z_W^{+,\xi}(\beta) = \sum_{\sigma_i = \pm 1, i \in W} \exp(-\beta H_W^{+,\xi}(\sigma)).$$
$$H_W^{+,\xi}(\sigma) = -\sum_{<ij>\cap W \neq \emptyset} (\sigma_i\sigma_j - 1)\mid_{\sigma=\xi \text{ on } \Lambda} \tag{2.7}$$

is the Hamiltonian for the volume W with $+$ boundary conditions on top and on the sides and ξ boundary conditions below (see figure). Similar definitions can be made starting from $-$ boundary conditions on V changing all superscripts $+$ into $-$.

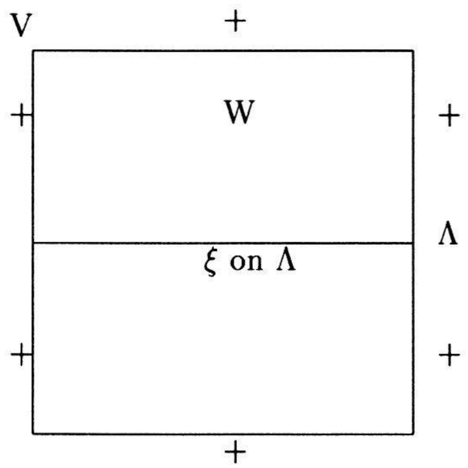

Figure: Volumes and boundary conditions.

3 The low temperature set up.

In this section we restrict ourselves to the case of + boundary conditions and the projection of the + phase. It will be straightforward to do the same thing starting from the − phase. The Hamiltonian (2.7) has the form

$$H_W^{+,\xi}(\sigma) = H_W^+(\sigma) + \sum_{x=-L}^{L}(1-\xi_x) - 2\sum_{x\in\Lambda^-(\sigma)}(1-\xi_x), \tag{3.1}$$

where

$$\Lambda^-(\sigma) = \{x \in \Lambda, \sigma_{(x,1)} = -1\}. \tag{3.2}$$

We will represent the configurations on W by sets of disjoint closed contours, as is usually done for + boundary conditions, [SI]. Let Γ_W denote the set of all closed contours on W. If a configuration σ is represented by a set of contours $\{\gamma_\alpha\}_{\alpha=1}^n \subset \Gamma_W$, then

$$H_W^+(\sigma) = 2\sum_{\alpha=1}^{n} \mid \gamma_\alpha \mid,$$

where $\mid \gamma_\alpha \mid$ is the lenght of γ_α.
For a given contour γ and $\xi \in W$, define

$$c(\gamma,\xi) = \text{card}\{x \in \mathbf{Z} : \xi_x = -1 \text{ and } (x,1) \text{ inside } \gamma\}, \tag{3.3}$$

so that in (3.1)

$$2\sum_{x\in\Lambda^-(\sigma)}(1-\xi_x) = 4\sum_{\alpha=1}^{n} c(\gamma_\alpha,\xi). \tag{3.4}$$

Combining (2.6) and (3.1)-(3.4) we have that

$$-\, h_\Lambda^+(\xi) = 2\beta\xi_o(\xi_1+\xi_{-1}) + 4\beta\xi_o + 2\log\frac{\tilde{Z}_W^{+,\xi}(\beta)}{\tilde{Z}_W^{+,\xi^o}(\beta)}, \tag{3.5}$$

where

$$\tilde{Z}_W^{+,\xi}(\beta) = 1 + \sum_{n=1}\frac{1}{n!}\sum_{\substack{\gamma_1\cdots\gamma_n\in\Gamma_W\\ \text{disjoint}}}\prod_{\alpha=1}^{n} z_\xi(\gamma_\alpha)$$

and

$$z_\xi(\gamma) = \exp(-2\beta \mid \gamma \mid + 4\beta c(\gamma,\xi)).$$

We use the technique of cluster expansion to calculate the ratio of the two partition functions (see e.g. [PF]). We obtain that

$$\begin{aligned} -h_\Lambda^+(\xi) = {} & 2\beta\xi_o(\xi_1+\xi_{-1}) + 4\beta\xi_o + \\ & 2\sum_{n=1}\frac{1}{n!}\sum_{\gamma_1\ldots\gamma_n\in\Gamma_W}\psi_n^T(\gamma_1\ldots\gamma_n)[\prod_{\alpha=1}^{n} z_\xi(\gamma_\alpha) - \prod_{\alpha=1}^{n} z_{\xi^o}(\gamma_\alpha)], \end{aligned} \tag{3.6}$$

where

$$\psi_n^T(\gamma_1 \dots \gamma_n) = \sum_{\substack{\text{C connected graphs} \\ \text{with n vertices}}} \prod_{\substack{(\alpha\alpha') \text{ is an} \\ \text{edge of C}}} \psi(\gamma_\alpha, \gamma_{\alpha'}),$$

$$\psi(\gamma_\alpha, \gamma_{\alpha'}) = \begin{cases} 0, & \text{if } \gamma_\alpha, \gamma_{\alpha'} \text{ disjoint} \\ -1, & \text{otherwise.} \end{cases} \tag{3.7}$$

It is clear that the difference in (3.6) will be zero, unless there is a contour which has the point $(0,1)$ in its inside. We rewrite (3.6) as

$$\begin{aligned} -h_\Lambda^+(\xi) &= 2\beta\xi_o(\xi_1 + \xi_{-1}) + 4\beta\xi_o \\ &+2\sum_{n=1} \frac{1}{n!} \sum_{\gamma_1 \dots \gamma_n \in \Gamma_W} \psi_n^T(\gamma_1 \dots \gamma_n) \prod_{\alpha=1}^n z_\xi(\gamma_\alpha)[1 - \exp(4\beta\xi_o b(\gamma_1 \dots \gamma_n))], \end{aligned} \tag{3.8}$$

where $b(\gamma_1 \dots \gamma_n) = \sum_{\alpha=1}^n b(\gamma_\alpha)$ and

$$b(\gamma) = \begin{cases} 1, & \text{if (0,1) inside } \gamma \\ 0, & \text{otherwise.} \end{cases}$$

Thus $b(\gamma_1 \dots \gamma_n)$ counts the number of contours passing through the bond $< (0,1)o >$.

4 Discussion of main observations.

As $L \to \infty$, the measures $\mu_V^\pm$ converge weakly to the infinite volume Gibbs measures $\mu^\pm$. For β large enough, $\mu^+ \neq \mu^-$ are the +, respectively − phase of the Ising model. More details can be found in [SI].

A standard application of ferromagnetic correlation inequalities [FKG] implies that the restriction of $\mu^\pm$ to the integer lattice $\mathbf{Z}$ coincides with the weak limit

$$\lim_{L \to \infty} \nu_\Lambda^\pm = \nu^\pm. \tag{4.1}$$

Clearly, $h_\Lambda^\pm(\xi)$ determines $\nu_\Lambda^\pm$ and is continuous in $\xi \in \Omega$. Moreover, from the martingale convergence theorem [BIL], the conditional probability distributions (2.4) converge $\mu^\pm$-a.s. as $L \uparrow \infty$.

Nevertheless, Schonmann has shown that there is no translation invariant summable interaction for which $\nu^\pm$ is a Gibbs measure [SCH]. Heuristically this can be understood by giving an example of what can happen [DvES]. For $1 \leq |x| \leq l$, let the $\{\xi_x\}$ be alternating ± 1 so that $\xi_1 = -\xi_{-1}$ and for $l < |x| \leq L$, take $\xi_x = +1$. In the projected measure ν^+, conditioning on this ξ makes the spin at the origin $+1$ (with large probability). Now change ξ in its tail by flipping (to -1) all spins at sites $l < |x| \leq L$. Energy-entropy arguments tell us that no matter how large l is, we can take $L = L(l)$ so that for this last conditioning, the spin at the origin prefers the $-$ state (analog of Lemma 1 in [SCH]). In other words, such arrangements show that for $L \uparrow \infty$, (2.4) is not continuous in $\xi \in \Omega$. This heuristic argument can be made into a proof, at least for sufficiently low temperatures [vEFS]. That indeed these are the bad configurations is also apparent from point 3 below: if large clusters of minuses are not allowed, then continuity for the ν^+ will be restored.

Formally, the limit of h_Λ^+ as $L \uparrow \infty$ is given by

$$h^+(\xi) = -2\beta\xi_o(\xi_1 + \xi_{-1}) - 4\beta\xi_o - \\ 2\sum_{n=1}\frac{1}{n!}\sum_{\gamma_1\dots\gamma_n}\psi_n^T(\gamma_1\dots\gamma_n)\prod_{\alpha=1}^{n} z_\xi(\gamma_\alpha)[1-\exp(4\beta\xi_o b(\gamma_1\dots\gamma_n))], \tag{4.2}$$

where we sum over all contours in the upper half plane. Clearly, the uniform convergence on Ω, $h_\Lambda^+ \to h^+$ would define ν^+ as a Gibbs measure with a continuous version of the conditional probabilities given by

$$\nu^+(\sigma_o = \xi_o \mid \sigma = \xi \text{ on } \mathbf{Z}\backslash o) = \frac{1}{1+\exp(h^+(\xi))}. \tag{4.3}$$

All information about the interaction potential is then contained in $h^+(\xi)$.

1. Relation with Proposition 2 in [SCH]:

Schonmann shows that if ν^+ is a bona fide Gibbs measure, so is ν^- and with respect to the same Hamiltonian. In our framework, the question of $\nu^\pm$ being Gibbsian can be formulated as follows. Consider the finite volume Hamiltonian $\mathcal{H}_\Lambda^\pm$ for the Gibbs measures $\nu_\Lambda^\pm$. As L grows, $\mathcal{H}_\Lambda^\pm(\xi)$ will of course change and it is only if this change in $\mathcal{H}_\Lambda^\pm(\xi)$ is not global but restricted to an effective boundary term that we can establish $\nu^\pm$ as a Gibbs measure in the thermodynamic limit. It would imply that the difference between $\mathcal{H}_\Lambda^+(\xi)$ and $\mathcal{H}_\Lambda^-(\xi)$ is, in the limit $L \uparrow \infty$, physically irrelevant, or, that $\mid \mathcal{H}_\Lambda^+(\xi) - \mathcal{H}_\Lambda^-(\xi) \mid$ is subextensive, that is $o(\mid \Lambda \mid)$. But,

$$-\frac{1}{2}[\mathcal{H}_\Lambda^+(\xi) - \mathcal{H}_\Lambda^-(\xi)] = \beta(\xi_L + \xi_{-L}) + \log\frac{Z_W^{+,\xi}(\beta)}{Z_W^{+,-\xi}(\beta)}$$

(where $-\xi$ refers to the configuration ξ with all spins flipped) and the last term is extensive of order L as can be seen by making an expansion like (3.6).
Another way of seeing this happen is to look at the expression (4.2) for $h^+(\xi)$. Formally $h^+(\xi) = h^-(-\xi)$ and if h_Λ^+ converges uniformly to h^+ on Ω then so does h_Λ^- to h^-. But in that event, certainly $h^+(\xi) \neq h^-(\xi)$ and this would contradict that $\nu^\pm$ are both Gibbs measures for the same interaction. Hence, from Proposition 2 in [SCH], ν^+ cannot be a Gibbs measure corresponding to a well defined h^+ on Ω.

2. What goes wrong in the low temperature expansion:

The reason why $h_\Lambda^+(\xi)$ does not converge uniformly in $\xi \in \Omega$ can be seen most easily by restricting the sum over contours $\gamma_1 \ldots \gamma_n$ in (3.8) to those contours which form rectangles of height one touching the set Λ with their lower side, i.e., we take W only one layer high, keeping its horizontal width equal to L. The contours γ_α can then be characterized by maximally connected subsets or chains A_α in $\mathbf{Z}$, containing the sites enclosed by γ_α. We have that

$$\mid \gamma_\alpha \mid = 2 \mid A_\alpha \mid +2$$
$$c(\xi, \gamma_\alpha) = \sum_{x \in A_\alpha} \frac{1-\xi_x}{2}.$$

We have that two chains are connected iff their union is again a chain. We are then interested in whether the sum over such contours satisfies a bound of the form

$$\mid \sum_{\substack{A_1 \ldots A_n \subset \mathbf{Z} \\ \ni 0}} \psi_n^T(A_1 \ldots A_n) \prod_{\alpha=1}^n \exp[-4\beta - 2\beta \sum_{x \in A_\alpha} (1+\xi_x)] \mid \leq n! C^n \tag{4.4}$$

for some $C < 1$.
If yes, then also the total sum in (3.8) and its uniform convergence as $L \uparrow \infty$ would be controlled, but clearly (4.4) fails. If indeed, there is a large cluster (say of length R) where all $\xi_x = -1$ then the sum in (4.4) is at least $n! O((e^{-4\beta} R)^n)$. But for typical ξ in Ω (with respect to ν^+) minus clusters of all sizes are present. For this reason h_Λ^+ will not converge uniformly.
What happens in (4.4) is that we try to make a low temperature expansion for the one-dimensional Ising model in a magnetic field $\{1+\xi_x, x \in \mathbf{Z}\}$. If, in the extreme, $\xi_x = -1$ for all $x \in \mathbf{Z}$, then we are dealing with the one-dimensional Ising model in zero magnetic field and clearly the low temperature description as a perturbation from its ground states must fail.

3. How to restore the Gibbs character:

To restore the Gibbsian nature of ν^+, we must restrict to a set of configurations for which (4.4) holds for all β large enough. Then on that restricted ensemble the relative energy h^+ will be defined and continuous. Optimally, that set would have large measure with respect to ν^+. Unfortunately, we do not know how to reconciliate all those requirements. Here we only show how to produce a Gibbs measure starting from the h_Λ^+ on a restricted ensemble of ν^+-measure zero. This Gibbs measure will however globally resemble ν^+. On any finite region, they get closer together as the temperature goes down.

Let N be a positive integer and define

$$\Omega_N^{\pm} = \{-1,+1\}^{\mathbf{Z}} \setminus \bigcup_{x\in\mathbf{Z}} \bigcap_{a=0}^{N} \{\xi_{x+a} = \mp 1\}. \tag{4.5}$$

The set $\Omega_N^{\pm}$ contains the configurations ξ on $\mathbf{Z}$ for which there is no sequence of more than N successive minus (plus) spins. It is a (relatively) compact set. We have that $\mu^{\pm}(\Omega_N^{\pm}) = 0$ whenever $\beta < \infty$. Of course, the $\nu^{\pm}$-probability that in a given finite interval $A \subset \mathbf{Z}$ there are no seqences of more than N successive minus (plus) spins, goes to one as $\beta \uparrow \infty$.

Proposition 1 *Let N be given. There exists $\beta_N < \infty$ such that for all $\beta \geq \beta_N$ $h^+(\xi) = \lim_{L\to\infty} h_\Lambda^+(\xi)$ is well defined for ξ in Ω_N.*
Moreover, $h_\Lambda^+ \to h^+$ uniformly on Ω_N.

Remarks:

(a) The Proposition remains true if we replace the set $\Omega_N^{\pm}$ by the larger set

$$\Omega_{N,T}^{\pm} = \{-1,+1\}^{\mathbf{Z}} \setminus \bigcup_{\substack{x\in\mathbf{Z}:\\ |x|\geq T}} \bigcap_{a=0}^{N} \{\xi_{x+a} = \mp 1\}, \tag{4.6}$$

for some $T \geq 0$. This additional complication does not change the basic idea of the proof presented below.

(b) An expression for h^+ was given in (4.2), where it is written as an absolutely convergent series. The dependence of $h^{\pm}(\xi)$ on ξ_x for $|x|$ large, is exponentially small.

(c) It is clear that we can define the relative energy not only for flipping one spin in the one-dimensional layer $\mathbf{Z}$, but also for a spinflip in any finite subset.

(d) As N grows, β_N also gets larger. Thus if we allow a larger set of configurations, the Gibbsian character of the projection is established only at lower temperatures. The situation becomes better at lower temperatures because the non-Gibbsian character of the projection is basically an entropy effect (cf. the discussion above (4.2)).

(e) Consider the Ising model with + boundary conditions and a uniform magnetic field $m > 0$. Then, via the low temperature expansion, the Gibbsian character of the projection measure can be established at least for sufficiently low temperatures. It is in fact sufficient to put a magnetic field $m > 0$ only on the layers just on top and below Λ. It is easy to check that in this case the weight of a contour will be

$$z_{\xi,m}(\gamma) = \exp[-2\beta \mid \gamma \mid +4\beta c(\gamma,\xi) - 2\beta m d(\gamma)],$$

where

$$\begin{aligned} d(\gamma) &= \mathrm{card}\{x \in \mathbf{Z} : (x,1) \quad \text{inside} \quad \gamma\} \\ &\geq c(\gamma,\xi). \end{aligned}$$

From this last inequality, it follows that

$$z_{\xi,m}(\gamma) \leq \exp(-\beta m \mid \gamma \mid).$$

This exponential decay of the weight factor (comparable with (5.2)) is enough to guarantee uniform convergence of the cluster expansion for the relative energy for β large enough (dependent on the value of m). The argument is the same as the one used in the proof of proposition 1.

We can now use the $\Omega_N^{\pm}$ as state space for a new Gibbs measure, or if wished, a Gibbs measure on Ω with some hard core interaction, excluding "bad" configurations. Let ν_N^+ be the Gibbs measure on Ω_N^+ which is obtained in the thermodynamic limit from the restriction of ν_Λ^+ to Ω_N^+. What it has in common with the original $\nu^{\pm}$ is the subject of the following

Proposition 2 *(a) Let A be a finite subset of $\mathbf{Z}$.*

$$\mid \langle \prod_{x\in A} \xi_x \rangle_{\nu_N^+} - \langle \prod_{x\in A} \xi_x \rangle_{\nu^+} \mid \leq \langle \prod_{x\in A} \xi_x \rangle_{\nu^+} O(e^{-4\beta N}).$$

(b) ν_N^+ has the cluster property

$$\mid \langle \xi_0 \xi_r \rangle_{\nu_N^+} - \langle \xi_0 \rangle_{\nu_N^+}^2 \mid \leq \langle \xi_0 \rangle_{\nu_N^+}^2 O(e^{-4\beta r}).$$

5 Proofs.

Proof of Proposition 1:

Note that the number of sites in the inside of a closed contour γ that have a ξ as nearest neighbor, is strictly less than $\frac{|\gamma|}{2}$. So for any contour γ and for any configuration ξ in Ω_N, we have that

$$c(\gamma,\xi) \leq \frac{N}{N+1} \frac{\mid \gamma \mid}{2}, \tag{5.1}$$

so that

$$z_\xi(\gamma) \leq \exp(-2\beta \frac{|\gamma|}{N+1}). \tag{5.2}$$

We now take β large enough so that

$$C = \sum_{\gamma \ni i^*} \exp(|\gamma|) \exp(-2\beta \frac{|\gamma|}{N+1}) < 1, \tag{5.3}$$

where we sum over all closed contours in the upper half plane containing some fixed site i^* of the dual lattice.
Then, for $\xi \in \Omega_N$, the expansion for h_Λ^+ is easily seen to converge absolutely and uniformly in Λ (see e.g. [PF]). That defines

$$\lim_\Lambda h_\Lambda^+(\xi) = h^+(\xi) = -2\beta\xi_o(\xi_1 + \xi_{-1}) - 4\beta\xi_o \tag{5.4}$$
$$-2\sum_{n=1} \frac{1}{n!} \sum_{\gamma_1 \cdots \gamma_n} \psi_n^T(\gamma_1 \ldots \gamma_n) \prod_{\alpha=1}^{n} z_\xi(\gamma_\alpha)[1 - \exp(4\beta\xi_o b(\gamma_1 \ldots \gamma_n))].$$

To prove the uniform convergence, choose $\epsilon > 0$. We need to show that

$$\sup_{\xi \in \Omega_N} | h_\Lambda^+(\xi) - h_{\Lambda'}^+(\xi) | < \epsilon$$

for Λ, Λ' large enough.
Let W, W' be the two-dimensional volumes corresponding to Λ and Λ' and to lengths $L > L' \gg 1$ respectively. Then for $\xi, \xi^o \in \Omega_N$

$$| h_\Lambda^+(\xi) - h_{\Lambda'}^+(\xi) | \leq 4 \max_{\eta \in \{\xi, \xi_o\}} \sum_{n=1} \frac{1}{n!} \sideset{}{^*}\sum_{\substack{\gamma_1 \cdots \gamma_n \in \Gamma_W \\ (0,1) \text{ inside some } \gamma}} | \psi_n^T(\gamma_1 \ldots \gamma_n) | \prod_{\alpha=1}^{n} z_\eta(\gamma_\alpha),$$

where the starred sum indicates that there is at least one contour which is not contained in W'. Because there is also a contour that contains $(0,1)$ in its inside and the $\gamma_1 \ldots \gamma_n$ are mutually connected (see (3.7)), we have

$$\prod_{\alpha=1}^{n} z_\eta(\gamma_\alpha) \leq \exp(-2\beta \frac{L'}{N+1}).$$

Thus, at low enough temperatures,

$$\sup_{\xi \in \Omega_N} | h_\Lambda^+(\xi) - h_{\Lambda'}^+(\xi) | \leq K \exp(-\beta \frac{L'}{N+1}),$$

for some constant $K < \infty$. This completes the proof of the Proposition.

□

Proof of Proposition 2:

By definition of ν^+ we have that $\langle \prod_{x\in A} \xi_x \rangle_{\nu^+} = \langle \prod_{x\in A} \xi_x \rangle_{\mu^+}$ and $\langle \prod_{x\in A} \xi_x \rangle_{\nu_N^+} = \langle \prod_{x\in A} \xi_x \rangle_{\mu_N^+}$, where μ_N^+ is the restriction of μ^+ to the configurations $\{\sigma \in \Theta : \sigma_{(.,0)} \in \Omega_N^+\}$. So, we can use the cluster expansion on $\mathbb{Z}^2$ to calculate the difference of expectation values. Our result is then proven in the same way as Lemma 4.3 in [PF], since the cluster expansion for $\langle \prod_{x\in A} \xi_x \rangle_{\mu_N^+}$ and $\langle \prod_{x\in A} \xi_x \rangle_{\mu^+}$ differ only in contours with lenght larger than $2N$.
Part b) follows from making a similar expansion for the correlation functions. In this expansion the points o and r keep in touch via a set of mutually connected contours which contains both points. The argument is then the same as in the proof of the uniform convergence in Proposition 1.

□

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A Appendix: The high temperature regime.

It is fairly easy to show that at high temperatures (say in the Dobrushin-Shlosman regime [DS]) the projection of the unique Gibbs measure for the two-dimensional Ising model is again a Gibbs measure on the one-dimensional layer. To obtain explicit information about its Hamiltonian we use the high temperature expansion. For this we consider the same volumes V,W and Λ, but this time with free boundary conditions. Using the superscript f to indicate this choice and for the same quantities as in (2.6)-(2.7), we have that for any configuration ξ on Λ

$$\exp(-h_\Lambda^f(\xi)) = \exp(2\beta\xi_o(\xi_1 + \xi_{-1}))[\frac{Z_W^{f,\xi}(\beta)}{Z_W^{f,\xi^o}(\beta)}]^2, \tag{A.1}$$

where

$$Z_W^{f,\xi}(\beta) = \sum_{\sigma_i=\pm1, i\in W} \exp(-\beta H_W^{f,\xi}(\sigma)),$$

$$H_W^{f,\xi}(\sigma) = - \sum_{\substack{<ij> \\ i\in W, j\in W\cup\Lambda}} \sigma_i\sigma_j \,|_{\sigma=\xi \text{ on } \Lambda} \,.$$

Let Ξ_W denote the collection of nearest neighbor connected paths of bonds $< ij >$, with $i \in W$ and $j \in W \cup \Lambda$. For $c \in \Xi_W$, ∂c contains those lattice sites which occur in an odd number of bonds of c. The total number of bonds in c is $| c |$. Then

$$- h_\Lambda^f(\xi) = 2\beta\xi_o(\xi_1 + \xi_{-1}) + 2 \sum_{\substack{A\subset\Lambda \\ A\ni o}} J_A^L \prod_{x\in A} \xi_x, \tag{A.2}$$

where

$$J_A^L = \sum_{n=1} \frac{1}{n!} \sum{}^* \psi_n^T(c_1 \dots c_n) \prod_{\alpha=1}^{n} (\tanh \beta)^{|c_\alpha|}. \tag{A.3}$$

The starred sum $\sum^*$ is over paths $\{c_\alpha\}_{\alpha=1}^n \subset \Xi_W$ (one of them containing the origin) with $\partial c_\alpha \subset \mathbf{Z}$ and for which the symmetric difference

$$\partial c_1 \Delta \dots \Delta \partial c_n = A.$$

For β small enough, $J_A = \lim_{L\to\infty} J_A^L$ is well defined for any $A \subset \mathbf{Z}$ and is obtained by taking in (A3) paths on the entire upper half plane. For example, $J_{\{0,1\}} = \beta^3 + \frac{1}{15}\beta^7 + O(\beta^9)$. Moreover $\sum_{A\subset\mathbf{Z}} | J_A |< \infty$. So, in the high temperature regime, the projected measure is a Gibbs measure for the translation invariant summable interaction determined by (A.2) in the limit $L \uparrow \infty$. It is also straightforward to deduce that this interaction is quasi-local in the sense that $| J_A |$ decays exponentially fast in the diameter of the set A.

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