STATISTICAL MECHANICS ON A 2D-RANDOM SURFACE

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Abstract. Various geometrical models first defined in the Euclidean plane or on a regular lattice have been briefly reviewed, including self-avoiding walks, random walk intersections, percolation and Ising clusters. These systems embody infinite sets of field operators defined in a natural way from the (fractal) geometry of these fluctuating critical systems. Their scaling behavior can be linked to that of associated conformal field theories.

These systems can also all be redefined on a *random lattice* or *surface*, instead of on a *regular* 2*D* lattice. They are then coupled to "quantum gravity", and live on the "world-sheet". The fact that all their new exponents on a random surface can then be related to those in the usual 2*D*-plane, although now well known in string theory, is worth publicizing in this *Physics in* 2*D* conference.

We illustrate it by some exact solutions in the case of polymers and branched polymers (animals) on a random fluid surface.

Introduction and summary

As an example of a statistical system defined on a random 2*D* lattice or surface, consider the Ising model [1]. On a fixed lattice graph $\mathcal{G}$, its partition function is defined as

$$\mathcal{Z}_{\text{Ising},\mathcal{G}}(K) = \sum_{\left\{\begin{array}{l}\sigma_i = \pm 1\\ i,j \in \mathcal{G}\end{array}\right\}} e^{K\sum_{\langle i,j\rangle}\sigma_i\sigma_j} \tag{1}$$

where K is the coupling constant between spins σ_i, σ_j at nearest neighbour sites $\langle i,j\rangle$ belonging to $\mathcal{G}$. Now, one lets the lattice fluctuate and sums over all possible bidimensional *planar* configurations of the lattice $\mathcal{G}$ made of, e.g., trivalent vertices (Fig.1).
The number of its vertices $|\mathcal{G}|$ (its "area") is also free to fluctuate. A double partition function is then defined as

$$Z(\beta,K) = \sum_{\text{planar } \mathcal{G}} e^{-\beta|\mathcal{G}|}\mathcal{Z}_{\text{Ising},\ \mathcal{G}}(K) \tag{2}$$

where the Ising partition function $\mathcal{Z}$ for each specific lattice realization $\mathcal{G}$ is weighted by a Gibbs area factor $e^{-\beta|\mathcal{G}|}$, β being the chemical potential for the number of lattice sites, the "cosmological constant" in string theory.

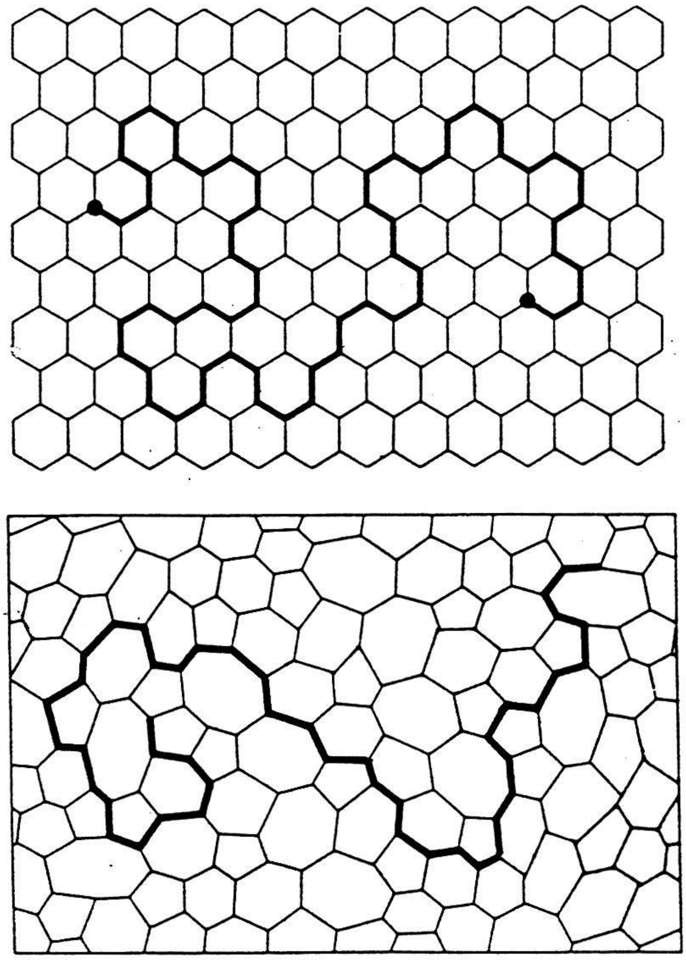

Figure 1: A regular hexagonal lattice graph, and a random graph made of trivalent vertices arbitrarily connected. Both bear in addition a self-avoiding walk, modelling a polymer.

The summation over planar lattices alone can lead to a critical behaviour when the average number of sites $|\mathcal{G}|$ is infinite, for a certain critical value βc [2]

$$Z(\beta) = \sum_{\textbf{planar } \mathcal{G}} \mathrm{e}^{-\beta|\mathcal{G}|} \qquad \text{is difined for } \beta > \beta_c \tag{3}$$

with

$$Z(\beta) \sim (\beta - \beta_c)^{2-\gamma_{\rm str}}, \qquad \beta \longrightarrow \beta_c^+ \tag{4}$$

where $\gamma_{\rm str}$ is the "string susceptibility exponent". Its simply means by inverting the sum (3), that the number of random lattices with a fixed number of sites (area) A and a fixed (planar) topology grows exponentially like

$$\mathcal{Z}(A) \sim e^{\beta cA} A^{\gamma_{\rm str}-3} \tag{5}$$

with a power law correction term governed by γ_{str}. Now, when mixed with the Ising model as in (2) the random surface develops also a double singular behavior at some critical values (β_c, K_c) for the pair (β, K) where the random surface is infinite and the Ising model is at its critical (Curie point) coupling constant K_c. Then, by mixing with the random surface critical behavior, the model acquires new values for its critical exponents like ν and γ. Conversely, the presence of the Ising model induces a new value for the susceptibility exponent γ_{str} of the random suface [1].

An astonishing result [3] is now that the values of the Ising exponents on the random $2D$ lattice are actually *related* to the standard Onsager values in two dimensions (namely on a fixed $2D$ lattice)! More precisely, if one relates the exponents ν and γ to the conformal scaling dimensions Δ_σ of the spin operator σ, and Δ_ε of the energy operator $\varepsilon = \sigma\sigma$, by

$$\nu^{-1} = 2 - 2\Delta_\varepsilon \quad \gamma = (2-\eta)\nu, \qquad \eta = 4\Delta_\sigma \tag{6}$$

one gets on a random surface the amazing relation due to Knizhnik, Polyakov and Zamolodchikov [3]

$$\Delta = \Delta^{\mathrm{grav}} \left(\frac{\Delta^{\mathrm{grav}} - \gamma_{str}}{1 - \gamma_{str}} \right) \tag{7}$$

where the corresponding Δ^{grav} are the values of the conformal dimensions of the same operators on the random surface (in presence of "gravity" due to the fluctuations of the metrics). This relation, which originates from the requirement of the global conformal invariance of the system (statistical model + random surface) [3,4], can be checked from the exact solution of the partition function (2) in the case of the Ising model. At this point, it should be mentionned the new remarkable fact that the double summation (lattice + statistical degrees of freedom) as in (2) actually can be solved exactly by *random matrix* techniques [1,5]. It can also be understood intuitively that relaxing a constraint by summing over all planar lattices actually makes the problem mathematically tractable (like the grand canonical sum in statistical mechanics).

This generalizes to any statistical system which has a (second order or more) critical point in two dimensions. One can always define it on an arbitrary planar lattice, sum over the lattices as in (2) (a case of annealed disorder) and get a new critical behaviour, still related by 2-dimensional conformal invariance to that on regular $2D$ lattices as in (7). This has now been checked in many cases, which have been solved exactly [1,6,7,8] on a random lattice by random matrix techniques [5], particularly in the case of polymers [7]. The relation (7) is found always to hold true as indicated in (Fig.2). The reader is referred to the relevant works for further information [6,7,8]. Let us also finally mention that we have specified the topology of the random surface (here planar for simplicity). It can be chosen freely but fixed (only γ_{str} depends on it). Resumming over all topologies is a main problem of string theories of elementary particles [9]. New progress has also appeared

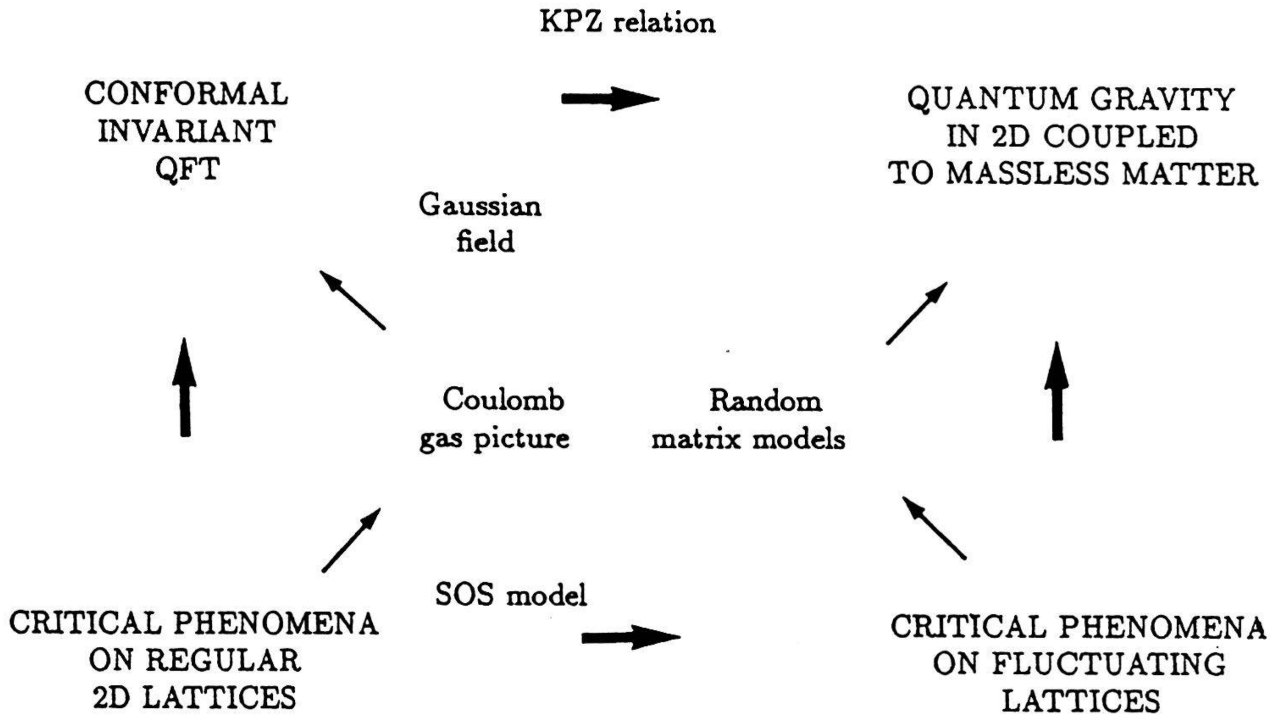

Figure 2: The general relation between critical phenomena in the Euclidean plane, and on a random surface (i.e. in presence of quantum gravity).

recently [10] in this case (the so-called $2D$ quantum gravity), essentially by using the same random matrix techniques in some finite size scaling way. But the mathematical problems encountered in the divergence of the topological series are still lagging [11].

Lastly, let us mention that one could also think of using the relation (7) between standard $2D$ critical exponents and those dressed by gravity, to determine the unknown solution to some problems on regular lattices in two dimensions. An example is that of branched polymers or "animals", where a supersymmetric trick [12] gave the $2D$ value of the configuration exponent θ, but not the size exponent ν. Even the conformal classification of this statistical model has been elusive. A recent work [13] solves exactly the model on a random surface (Fig.3), and finds a nice universality. Transferring the results to the standard plane as in (7) suggests a conformal central change $c = 1$, and a new value of exponent $\nu = 2/3$. But intriguing questions arise [13], concerning the comparison to numerical simulations, and the meaning of the conformal theory of animals, if its exists.

In summary, let us stress that as in (2), all $2D$ statistical mechanics can be redone on a random lattice (i.e. in presence of a fluctuating metrics).

It is then expected that the resulting model should be actually solvable in particular by matrix integrals. Near a critical point, the conformal relation (7) to Euclidean statistical mechanics should hold. It would be also very interesting to explore this avenue for other systems in $2D$, like first-order transition points, disordered systems (spin glasses...) or

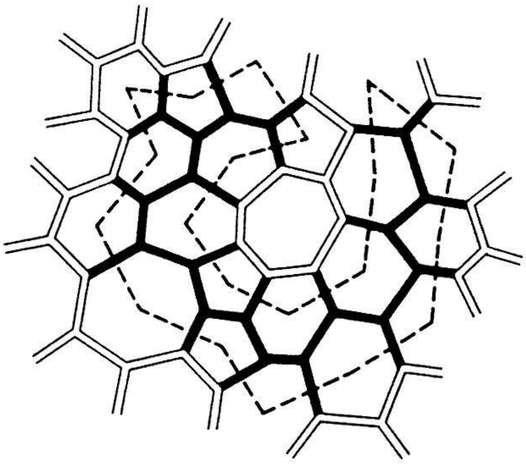

Figure 3: A random tree or branched polymer ("animal") on a random graph. The dotted line delimits the shape of the "animal".

even superconductors.

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