N=2-extended supersymmetries and Clifford algebras

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N=2-EXTENDED SUPERSYMMETRIES AND CLIFFORD ALGEBRAS

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Abstract

By searching for the largest numbers of one-parameter Lie algebras for one-dimensional supersymmetric harmonic oscillators, we study the impact of fermionic variables associated with fundamental Clifford algebras such as $C\ell_2$ and $C\ell_4$. Amongst the sets of associated generators we point out the largest closed superstructures identified as invariance or spectrum generating superalgebras. The additional supersymmetries which do not close under the generalized Lie product lead to new constants of motion. Direct connections with other recent contributions are also singled out.

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I. INTRODUCTION

The group-theoretical analysis of arbitrary differential equations has been originally proposed and developed a long time ago by Lie.[1,2] More recently, specific textbooks like those of Miller[3], Ovsiannikov[4] and Olver[5] deal with the developments of such a subject and contain a lot of interesting references.

Here we will mainly be concerned with the so-called "non classical Lie approach" as referred and described by Fushchich and Nikitin[6,7] when the accent is put on all the one-parameter Lie algebras and their collection leading to closed or open structures of symmetries admitted by systems of differential equations. Such considerations have to deal with Lie extended symmetries : they have already been applied to classical and quantum (wave) equations including the nonrelativistic as well as relativistic contexts. Specific equations such as the ones describing the nonrelativistic quantum free system and (isotropic) harmonic oscillator[8] as well as the relativistic Dirac, Weyl or Maxwell systems have particularly been studied[6,7,9,10] following some of the above mentioned works. In particular we have just extended[11] similar considerations to supersymmetric quantum physics[12] by taking the explicit example of the 1-dimensional harmonic oscillator and its supersymmetric wave equation[12,13] admitting very well known kinematical and dynamical supersymmetries.[14,15,16,17]

Let us come back on the concept of invariance of a wave equation under space-time transformations and search for the more general operator X ensuring that the concerned equation

$$\Delta\varphi = 0 \tag{1.1}$$

is invariant under the (infinitesimal) transformation $1 + i\,\varepsilon X$. This corresponds to the study of the associated kinematical symmetries. The resulting condition is φ-independent and writes

$$[\,\Delta\,, X\,] = \lambda\,\Delta \tag{1.2}$$

where λ is an arbitrary function. The problem of the general form for X can then be expressed in two ways :

- (i) either we ask for its general form by requiring that the one-parameter Lie substructures do altogether form a closed Lie structure as it is the case in the so-called classical Lie context (containing in particular amongst the above mentioned references those of Niederer[8], Rudra[9,10] and Durand[14]) ;

- (ii) or we do not ask for a closed structure as it is the case in the non classical Lie approach (as presented by Fushchich and Nikitin[6,7]).

The last context contains the preceding one and it leads to supplementary results connected with constants of motion for example.[6,7] In order to characterize such an approach, let us consider the equation (1.1) as describing a physical system through the wave function $\varphi \equiv \varphi(t,\underline{x})$ where we refer to $\underline{x} \equiv (x_1,x_2,\ldots,x_n)$ as the position of the system in a n-dimensional space. We then define the operators Δ and X respectively by

$$\Delta \equiv a(t\underline{x}) + a_{\mu}(t\underline{x})\partial_{\mu} + a_{\mu\nu}(t\underline{x})\partial_{\mu}\partial_{\nu} \tag{1.3}$$

and

$$X \equiv c(t\underline{x}) + b_{\mu}(t\underline{x})\partial_{\mu} \tag{1.4}$$

where summations on repeated indices are understood and where we refer for brevity to the whole set of partial derivatives by

$$\left\{\partial_{\mu}\right\} \equiv \left\{\frac{\partial}{\partial x_{\mu}}\right\} \equiv \left\{\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right\} \equiv \left\{\partial_t, \partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n}\right\} . \tag{1.5}$$

The condition (1.2) leads to the following system constraining the known functions a , a_{μ} , $a_{\mu\nu}$ in terms of the unknowns b_{μ} and c and of the arbitrary $\lambda(t,\underline{x})$:

$$[a,c] - b_{\mu}\left(\partial_{\mu}a\right) + a_{\mu}\left(\partial_{\mu}c\right) + a_{\mu\nu}\left(\partial_{\mu}\partial_{\nu}c\right) = \lambda a \ , \tag{1.6a}$$

$$\left[a,b_\mu\right]+\left[a_\mu,c\right]+a_\nu\left(\partial_\nu b_\mu\right)-b_\nu\left(\partial_\nu a_\mu\right)+a_{\nu\mu}\left(\partial_\nu c\right)+$$
$$+a_{\mu\nu}\left(\partial_\nu c\right)+a_{\nu\rho}\left(\partial_\nu\partial_\rho b_\mu\right)=\lambda a_\mu \ , \ \forall\, \mu \ , \tag{1.6b}$$

$$\frac{1}{2}\left[a_\mu \, , b_\rho\right]+\frac{1}{2}\left[a_\rho \, , b_\mu\right]+\left[a_{\mu\rho} \, , c\right]+a_{\nu\mu}\left(\partial_\nu b_\rho\right)+$$
$$+a_{\rho\nu}\left(\partial_\nu b_\mu\right)-b_\nu\left(\partial_\nu a_{\mu\rho}\right)=\lambda\, a_{\mu\rho} \ , \ \forall\, \mu,\rho \ , \tag{1.6c}$$

$$\left[a_{\mu\nu} \, , b_\rho\right]+\left[a_{\nu\rho} \, , b_\mu\right]+\left[a_{\rho\mu} \, , b_\nu\right]=0 \ , \ \forall\, \mu,\nu,\rho \ . \tag{1.6d}$$

Such a system is directly obtained by equating the corresponding orders in derivatives in both sides of Eq. (1.2). It shows that if the wave equation (1.1) is not scalar all the unknowns have to be developed in the corresponding matrix basis leading in such a case to complicated equations in general containing a very important number of unknown scalar functions.

We intend to exploit such an approach in connection with supersymmetric quantum mechanics[12] following the second way (ii) mentioned above after equation (1.2). This approach is more general than the one developed by Durand[14,17] and permits us to study the impact of different dimensions in the matrix realizations. Moreover it can be compared with another recent approach[11] also applied to supersymmetric quantum mechanics.

We take the 1-dimensional supersymmetric harmonic oscillator as the example which permits us to illustrate our developments. The corresponding results have evidently to deal with the so-called extended supersymmetries which will be here subtended by matrix equations and theories in the case of the simplest Clifford algebra[18] $\mathcal{C}\ell_2$ of order 4. In fact, in Sec. II, we consider the supersymmetric wave equation of the 1-dimensional harmonic oscillator (a matrix equation expressed in terms of 2X2-Pauli matrices) and find twenty-four (super)symmetries. They are interpreted in a specific way (clearly apparent in the following) as four times the six initial bosonic symmetries[8] of the usual 1-dimensional harmonic oscillator. The corresponding closed superalgebra contains only thirteen (super)symmetries as already known[15,16,17] and the further eleven ones can be discussed in connection with constants of motion. Other supersymmetric wave equations[17] which are also subtended by this simplest Clifford algebra can be studied in a complete parallel way. In Sec. III, we address ourselves to the same problem but by choosing a 4X4-realization so that the Clifford algebra now is $\mathcal{C}\ell_4$ of order 16. We correspondingly get 96 (= 16X6) supersymmetries and can draw parallel conclusions to the preceding case. If both Sec. II and III deal with the nonclassical Lie approach by treating explicit matrix equations, we come back in Sec. IV on the classical Lie approach applied to supersymmetric quantum mechanics by grading the generator $X \equiv (1.4)$ and the arbitrary function λ included in eq. (1.2). This method[11] enlightens the results of Sec. II and III in what concerns the respective closed superstructures. Sec. V is then devoted to comments and conclusions.

The units are chosen so that $m=1$, $\hbar=1$ but we maintain the angular frequency ω when harmonic oscillators are concerned. As nonrelativistic examples are only considered here we do not distinguish between co- and contravariant indices as it should be necessary if relativistic applications were studied with pseudo-euclidean metric tensors.

II. N=2-EXTENDED SUPERSYMMETRIES AND THE CLIFFORD ALGEBRA Cl_2

Let us consider the N=2-supersymmetric quantum mechanical context described by the Hamiltonian

$$H^{ss} = \{ Q , Q^\dagger \} \tag{2.1}$$

where the two Q-type supercharges are such that[12]

$$\{ Q , Q \} = \{ Q^\dagger , Q^\dagger \} = 0 \quad , \quad [H^{ss} , Q] = [H^{ss} , Q^\dagger] = 0 \ . \tag{2.2}$$

In terms of the superpotential $W(x)$, these conserved supercharges take the following forms

$$Q = \left(p + i \frac{dW}{dx} \right) \sigma_- \quad , \quad Q^\dagger = \left(p - i \frac{dW}{dx} \right) \sigma_+ \tag{2.3}$$

where, for 1-dimensional systems, we insist on the bosonic (p and x) and fermionic (σ_+ and σ_-) operators associated with the corresponding degrees of freedom according to

$$[p , x] = -i \quad , \quad \{ \sigma_+ , \sigma_- \} = I_2 \ . \tag{2.4}$$

We evidently get by remembering the Lie algebra su(2)-relation

$$[\sigma_+ , \sigma_-] = \sigma_3 \tag{2.5}$$

that

$$H^{ss} = \frac{1}{2} \left(p^2 + | W' |^2 \right) + \frac{1}{2} W'' \sigma_3 \tag{2.6}$$

so that the equation (1.1) takes here the form

$$\Delta\varphi \equiv \left(i\partial_t - H^{SS}\right)\varphi\,(t,x) = 0 \tag{2.7}$$

and is subtended by matrix considerations associated with the simplest Clifford algebra[18] $\mathcal{C}\ell_2$ of order 4 (= 2^2) :

$$\mathcal{C}\ell_2 \equiv (\, I_2 \,,\, \sigma_1 \,,\, \sigma_2 \,,\, \sigma_3 \,) \equiv (\, \sigma_0 \,,\, \sigma_+ \,,\, \sigma_- \,,\, \sigma_3 \,)\ . \tag{2.8}$$

As an explicit example let us consider the 1-dimensional harmonic oscillator.[12] Its supersymmetric version corresponds to the superpotential

$$W_{H.O.}(x) = \frac{1}{2}\ \omega x^2 \tag{2.9}$$

so that

$$H^{SS}_{H.O.} = \frac{1}{2}\left(p^2 + \omega^2 x^2\right) + \frac{1}{2}\,\omega\sigma_3 = H_B + H_F \tag{2.10}$$

where we recognize the bosonic and fermionic Hamiltonians as expected.[12,13,19,20] Eq. (2.7) explicitly becomes

$$\Delta\varphi\,(t,x) \equiv \left(i\partial_t + \frac{1}{2}\,\partial_x^2 - \frac{1}{2}\,\omega^2 x^2 - \frac{1}{2}\,\omega\sigma_3\right)\varphi(t,x) = 0 \tag{2.11}$$

and the operator (1.3) is here characterized by the only nonzero matrix quantities

$$a = -\frac{1}{2}\,\omega^2 x^2\,\sigma_0 - \frac{1}{2}\,\omega\sigma_3\ ,\ a_t = i\,\sigma_0\ ,\ a_{xx} = \frac{1}{2}\,\sigma_0\ . \tag{2.12}$$

Then the system (1.6) reduces to the only five following equations :

$$\frac{1}{2}\,\omega[\sigma_3\,,\,c] - b_x\omega^2 x - i\partial_t c - \frac{1}{2}\,\partial_x^2 c = \lambda\left(\frac{1}{2}\,\omega^2 x^2\,\sigma_0 + \frac{1}{2}\,\omega\sigma_3\right)\ ,$$

$$\frac{1}{2}\,\omega\left[b_x\,,\,\sigma_3\right] + i\partial_t b_x + \partial_x c + \frac{1}{2}\,\partial_x^2 b_x = 0\ ,$$

$$\frac{1}{2}\omega\left[b_t, \sigma_3\right] + i\partial_t b_t + \frac{1}{2}\partial_x^2 b_t = 2\lambda \,, \tag{2.13}$$

$$\partial_x b_x = \frac{\lambda}{2} \quad , \quad \partial_x b_t = 0 \ .$$

We then expand the unknowns b_x , b_t and c in the basis (2.8) and we solve the system (2.13). Some tedious calculations lead to twenty-four supersymmetries which fall after rearrangements into four classes written as follows when, for simplicity, we limit ourselves to the t=0-context :

$$\left\{H_B, C_\pm, I, P_\pm\right\}\sigma_0 \quad , \quad \left\{H_B, C_\pm, I, P_\pm\right\}\sigma_3 \quad , \tag{2.14a}$$

$$\left\{H_B, C_\pm, I, P_\pm\right\}\sigma_+ \quad , \quad \left\{H_B, C_\pm, I, P_\pm\right\}\sigma_- \quad . \tag{2.14b}$$

These results show the main role of the four independent elements of the $C\ell_2$-basis multiplying the six Niederer (bosonic) symmetries.[8] In eqs. (2.14), let us recall[8,15] that the generators $C_\pm$ and $P_\pm$ read for $t \neq 0$

$$C_\pm = \pm\frac{i}{2}\left[\exp\left(\mp 2i\omega t\right)\right](p \pm i\omega x)^2 \ , \ P_\pm = \pm \ i\left[\exp\left(\mp i\omega t\right)\right](p \pm i\omega x) \ . \tag{2.14c}$$

Amongst these twenty-four operators, only thirteen of them close under commutation and anticommutation and form the semi-direct sum osp(2/2) $\square$ sh(2/2) already obtained by Beckers and Hussin.[15] Indeed we notice (at t=0) the identifications

$$H_B \, , C_\pm \, , H_F \equiv \frac{\omega}{2}\sigma_3 \ , \ Q_\pm \equiv \sigma_\mp P_\pm \ , \ S_\pm \equiv \sigma_\pm P_\pm \tag{2.15}$$

leading to the orthosymplectic Lie algebra osp(2/2) (including the odd supercharges[12,21] $Q_\pm$ and $S_\pm$) while the five operators

$$I \, , P_\pm \ , \ T_\pm \equiv \sigma_\pm \tag{2.16}$$

generate the Heisenberg superalgebra sh(2/2) (including the odd operators $\sigma_\pm$).

The eleven supplementary operators can then be completely specified as follows : the five even ones write as

$$H_B\,\sigma_3 \;,\; P_\pm\,\sigma_3 \;,\; C_\pm\,\sigma_3 \tag{2.17}$$

and the six odd ones take the explicit forms combined in the following three pairs :

$$[\exp(\pm i\omega t)]\,\sigma_\mp H_B \;,\; [\exp(\pm i\omega t)]\,C_\pm\sigma_\mp \;,\; [\exp(\pm i\omega t)]\,C_\mp\sigma_\mp \;. \tag{2.18}$$

All the operators X_A contained in eqs. (2.15)-(2.18) lead to constants of motion C_A given by

$$C_A = \int \varphi^\dagger(t,x)\,X_A\,\varphi(t,x)\,dx \;,\; A = 1,\ldots,24 \;, \tag{2.19}$$

where the two-component wavefunction $\varphi(t,x)$ can be developed in an energy basis[15,16] in correspondence with both the σ_3-eigenvalues ($\varepsilon = \pm 1$).

Let us point out that similar considerations can evidently be developed for <u>other</u> supersymmetric wave equations subtended by the Clifford algebra $C\ell_2$. For example, as described in connection with spectrum generating superalgebras, we refer to some cases collected in D'Hoker-Vinet-Kostelecky[17] corresponding to other superpotentials. Let us mention the form

$$W(x) = \mu \ln x \tag{2.20}$$

and the superposition of the expressions (2.9) and (2.20), i.e. the so-called Calogero potential

$$W(x) = \mu \ln x + \frac{1}{2}\,\omega\,x^2 \;. \tag{2.21}$$

As shown hereafter (see Sec. IV), the corresponding results do contain those mentioned by D'Hoker et al.[17] but also additional ones.

III. N=2-EXTENDED SUPERSYMMETRIES AND THE CLIFFORD ALGEBRA $\mathcal{C}\ell_4$

Let us now come back on the study of the equation (2.11) but when we introduce 4 by 4 matrices generating the Clifford algebra $\mathcal{C}\ell_4$ with sixteen fundamental elements given, for example, through the following construction

$$\begin{gathered}\sigma_i \otimes I_2 \equiv \sigma_i \otimes \sigma_0 \equiv \sigma_{i0}\ ,\ I_2 \otimes \sigma_i \equiv \sigma_0 \otimes \sigma_i \equiv \sigma_{0i}\ , \\ \sigma_0 \otimes \sigma_0 \equiv \sigma_{00} \equiv I_4\ ,\ \sigma_i \otimes \sigma_j \equiv \sigma_{ij}\ ,\ i,j = 1,2,3\ .\end{gathered} \tag{3.1}$$

This doubling corresponds to a system of four equations written in a compact form as

$$\Delta \Phi(t,x) \equiv \left(i\partial_t + \frac{1}{2}\partial_x^2 - \frac{1}{2}\omega^2 x^2 - \frac{1}{2}\omega\sigma_{30}\right)\Phi(t,x) = 0 \tag{3.2}$$

allowing the corresponding unknowns b_x, b_t and c in the generator $X \equiv (1.4)$ to be expanded in the $\mathcal{C}\ell_4$-basis :

$$\mathcal{C}\ell_4 \equiv \{\sigma_{00}\,, \sigma_{0i}\,, \sigma_{i0}\,, \sigma_{ij}\}\ . \tag{3.3}$$

New tedious calculations associated with the resolution of the adapted system corresponding to Eqs. (2.13) lead us to ninety-six supersymmetries. As in Sec. II we have understood that the twenty-four supersymmetries can be seen as 6 times 4 with an explicit meaning of these numbers (6 for bosonic results[8] and 4 for the order of $\mathcal{C}\ell_2$), we get here that 96 = 6X16, keeping the same meaning for the six symmetries while 16 is the order of $\mathcal{C}\ell_4$. Indeed we can write these ninety-six supersymmetries as all the products between the members of the following two sets

$$\{H_B\,, C_\pm\,, I\,, P_\pm\} \text{ and } \{\sigma_{00}\,, \sigma_{0i}\,, \sigma_{i0}\,, \sigma_{ij}\}\ . \tag{3.4}$$

Let us notice that if we associate as usual the even (odd) character to the matrices $\sigma_0 \equiv I_2, \sigma_3$ (σ_1, σ_2 or $\sigma_\pm$) in $C\ell_2$, correspondingly we get in $C\ell_4$ the following eight even matrices $\{\mathcal{E}\} \equiv \{\sigma_{00}, \sigma_{03}, \sigma_{30}, \sigma_{33}, \sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22}\}$ and the following eight odd ones $\{\mathcal{O}\} \equiv \{\sigma_{01}, \sigma_{02}, \sigma_{10}, \sigma_{20}, \sigma_{23}, \sigma_{32}, \sigma_{31}, \sigma_{13}\}$. We thus have trivially constructed 48 (= 6X8) even generators as well as 48 (= 6X8) odd ones.

With the help of the Pauli algebraic properties

$$\sigma_i \, \sigma_j = i \, \varepsilon_{ijk} \, \sigma_k + \delta_{ij} \ , \tag{3.5}$$

it is easy to determine the structure relations according to a graded Lie product ensuring as usual that

$$[\mathcal{E},\mathcal{E}] \to \mathcal{E} \ , \ \{\mathcal{O},\mathcal{O}\} \to \mathcal{E} \ , \ [\mathcal{E},\mathcal{O}] \to \mathcal{O} \ . \tag{3.6}$$

We evidently recover the superalgebra osp (2/2) □ sh(2/2) generated by the thirteen operators (2.15) and (2.16) when the substitution $\sigma_\pm \to \sigma_\pm \otimes \sigma_0$ is effectively realized. Moreover it is possible to find eleven additional generators which form with the thirteen previous ones a closed superstructure. These eleven generators are the five even matrices

$$\sigma_{03} \ , [\exp(i\omega t)] \, \sigma_- \otimes \sigma_\pm \ , [\exp(-i\omega t)] \, \sigma_\pm \otimes \sigma_\pm \ , \tag{3.7}$$

and the six odd ones

$$\sigma_3 \otimes \sigma_\pm \ , \ P_+ \, \sigma_3 \otimes \sigma_\pm \ , \ P_- \, \sigma_3 \otimes \sigma_\pm \ . \tag{3.8}$$

We thus get a 24-dimensional superalgebra which can be identified as the semi-direct sum osp(4/2) □ sh(4/2). Without loss of generality, let us once again take t=0 and mention that this semi-direct sum corresponds to the following set of 17 generators for osp(4/2) :

$$osp(4/2) \equiv \{H_B \, , C_\pm \, , H_F \, , Q_\pm \, , S_\pm \, , \sigma_\pm \otimes \sigma_\pm \, , \sigma_3 \otimes \sigma_\pm \, P_\pm\} \tag{3.9}$$

and to the following set of 7 generators for sh(4/2) :

$$sh(4/2) \equiv \{P_\pm \, , T_\pm \, , \sigma_{00} \, , \sigma_3 \otimes \sigma_\pm\} \ . \tag{3.10}$$

In fact we recognize the Lie algebra $so(4) \oplus sp(2,\mathbb{R})$ as the even part of osp(4/2) by identifying $sp(2,\mathbb{R}) \sim so(2,1)$ with the three generators $(H_B,C_\pm)$ while the compact so(4)-subalgebra is directly obtained through the superposition of two commuting su(2)-subalgebras. The latter are generated by the respective combinations

$$\{\sigma_- \otimes \sigma_+ \, , \sigma_+ \otimes \sigma_- \, , \frac{1}{2} \, (\sigma_{03} - \sigma_{30})\} \tag{3.11a}$$

and

$$\{\sigma_+ \otimes \sigma_+ \, , \sigma_- \otimes \sigma_- \, , \frac{1}{2} \, (\sigma_{03} + \sigma_{30})\} \tag{3.11b}$$

by remembering that in the present context :

$$H_F = \frac{1}{2} \, \omega \, \sigma_{30} \ . \tag{3.12}$$

With respect to sh(4/2) given by the set (3.10) we evidently identify the even Lie algebra h(2) as generated by

$$h(2) \equiv \{P_\pm \, , \sigma_{00} \equiv I_4\} \ . \tag{3.13}$$

The (96-24)=seventy-two other supersymmetries lead to a corresponding set of constants of motion in the sense described by Fushchich and Nikitin.[6,7] They evidently contain the corresponding eleven constants of motion obtained from the operators (2.17) and (2.18).

IV. ON THE CLASSICAL LIE SUPERSYMMETRIES

The superstructures osp(2/2) □ sh(2/2) and osp(4/2) □ sh(4/2) obtained as closed superalgebras in Sect. II and III respectively can also be recovered through other developments[11] applied to 1-dimensional supersymmetric harmonic oscillators. Indeed, we have obtained parallel results by reconsidering the determination of the largest superalgebras of supersymmetries for the equation (2.7) but by grading the construction of the generator X ≡ (1.4) and the arbitrary function $\lambda(t,x)$ in the condition (1.2).

Let us recall[11] that we can see the fermionic Hamiltonian as expressed in terms of the odd variables $(\Psi,\overline{\Psi})$ in such a way that

$$H_F = \frac{\omega}{2}\,[\Psi,\overline{\Psi}] \tag{4.1}$$

with

$$\{\Psi,\overline{\Psi}\} = I \ , \ \{\Psi,\Psi\} = \{\overline{\Psi},\overline{\Psi}\} = 0 \ . \tag{4.2}$$

Moreover by requiring that

$$X = X_{\bar{0}} + X_{\bar{1}} \ , \ \lambda = \lambda_{\bar{0}} + \lambda_{\bar{1}}\left(\Psi + \overline{\Psi}\right) \ , \tag{4.3}$$

we have proposed to generalize (1.4) in the 1-dimensional context through the expressions

$$\begin{aligned} X_{\bar{0}} = {} & c(t,x) + b_t(t,x)\partial_t + b_x(t,x)\partial_x + b(t,x)\Psi\overline{\Psi} \\ & + c_1(t,x)\Psi\partial_\Psi + c_2(t,x)\overline{\Psi}\partial_{\overline{\Psi}} + d_1(t,x)\overline{\Psi}\partial_\Psi + d_2(t,x)\Psi\partial_{\overline{\Psi}} + e(t,x)\partial_\Psi\partial_{\overline{\Psi}} \end{aligned} \tag{4.4}$$

and

$$X_{\overline{1}} = \alpha_1(t,x)\Psi + \alpha_2(t,x)\overline{\Psi} + \beta_1(t,x)\Psi\partial_x + \beta_2(t,x)\overline{\Psi}\partial_x$$
$$+ \gamma_1(t,x)\partial_\Psi + \gamma_2(t,x)\partial_{\overline{\Psi}} + \delta_1(t,x)\partial_\Psi\partial_x + \delta_2(t,x)\partial_{\overline{\Psi}}\partial_x \ . \tag{4.5}$$

In fact, by rewriting the equations (2.11) <u>and</u> (3.2) on the single form

$$\Delta\left(t,x,p,\Psi,\overline{\Psi}\right)\chi(t,x) \equiv \left(i\partial_t + \frac{1}{2}\partial_x^2 - \frac{1}{2}\omega^2x^2 - \frac{1}{2}\omega\left[\Psi,\overline{\Psi}\right]\right)\chi(t,x) \ , \tag{4.6}$$

we can solve the system issued from the condition (1.2) with the expressions (4.3)-(4.5). By requiring explicitly that

$$\left\{\Psi,\partial_\Psi\right\} = \left\{\overline{\Psi},\partial_{\overline{\Psi}}\right\} = I \ , \ \left\{\Psi,\partial_{\overline{\Psi}}\right\} = \left\{\overline{\Psi},\partial_\Psi\right\} = 0 \ , \tag{4.7}$$

ensuring a correct effect of the corresponding operators on the wavefunction χ , we get <u>twenty-four</u> generators of one parameter-structures, twelve even and twelve odd operators which can be arranged and denoted as follows. The first twelve <u>even</u> generators absorb the <u>six</u> Niederer ones[8] (which are, let say, <u>purely bosonic</u>)

$$H_B \ , \ C_\pm \ , \ I \quad \text{and} \quad P_\pm \ , \tag{4.8}$$

already defined in eqs. (2.10) and (2.14c) <u>and</u> contain the <u>six</u> following ones[11] (which are <u>purely fermionic</u>) written as

$$H_F \equiv (4.1) = \frac{\omega}{2}\left[\Psi,\overline{\Psi}\right] \ , \ X_1 \equiv \left[\exp\,(i\omega t)\right]\overline{\Psi}\partial_\Psi \ , \ X_2 \equiv \left[\exp\,(-\,i\omega t)\right]\Psi\partial_{\overline{\Psi}} \ ,$$
$$X_3 \equiv \Psi\partial_\Psi - \overline{\Psi}\partial_{\overline{\Psi}} + \partial_\Psi\partial_{\overline{\Psi}} \ , \ X_4 \equiv \left[\exp\,(-i\omega t)\right]\left(\Psi\partial_\Psi - \Psi\overline{\Psi}\right) \ , \ X_5 \equiv \left[\exp(i\omega t)\right]\left(\overline{\Psi}\partial_{\overline{\Psi}} - \overline{\Psi}\Psi\right) . \tag{4.9}$$

The second series of the twelve <u>odd</u> generators absorbs the <u>six</u> odd operators referred in eqs. (2.15) and (2.16) but here given on the forms

$$Q_+ \equiv (p - i\omega x)\Psi \ , \qquad Q_- \equiv (p + i\omega x)\overline{\Psi}$$

$$S_+ \equiv [\exp(-2i\omega t)]\,(p + i\omega x)\Psi \ , \quad S_- \equiv [\exp(2i\omega t)]\,(p - i\omega t)\,\overline{\Psi} \ , \tag{4.10}$$

$$T_+ \equiv [\exp(-i\omega t)]\,\Psi \ , \qquad T_- \equiv \exp(i\omega t)\,\overline{\Psi} \ .$$

It also contains six additional (odd) generators given by the explicit expressions

$$X_6 \equiv \partial_\Psi - \overline{\Psi} \ ,$$
$$X_7 \equiv \partial_{\overline{\Psi}} - \Psi \ ,$$

$$X_8 \equiv \frac{1}{\sqrt{2}}\,[\exp(i\omega t)]\,(p - i\omega x)\,X_6 \ ,$$
$$X_9 \equiv \frac{1}{\sqrt{2}}\,[\exp(-i\omega t)]\,(p + i\omega x)\,X_7 \ , \tag{4.11}$$

$$X_{10} \equiv \frac{1}{\sqrt{2}}\,[\exp(-i\omega t)]\,(p + i\omega x)\,X_6 \ , \ X_{11} \equiv \frac{1}{\sqrt{2}}\,[\exp(i\omega t)]\,(p - i\omega x)\,X_7 \ .$$

Amongst these twenty-four operators (4.8)-(4.11), thirteen of them were expected[15,17] in connection with the superalgebra osp(2/2) □ sh(2/2) also recovered in Sec. II while the other eleven ones X_1, $,X_{11}$ (five even $X_1,\ldots,X_5$ and six odd $X_6,\ldots,X_{11}$) are new if they can be realized in a nontrivial way.

Let us now discuss the possible choices for the fermionic variables Ψ and $\overline{\Psi}$ entering in these eleven generators $X_1,\ldots,X_{11}$. It is easy to show that, inside the $\mathcal{C}\ell_2$-algebra (4.2), the possible realizations of Ψ and $\overline{\Psi}$ lead immediately to trivial or redundant X_B for B=1,...,11 so that we are left with the only closed superalgebra osp(2/2) □ sh(2/2) as already noticed. One of the (two) possible choices is the realization

$$\Psi \equiv \sigma_+ \ , \ \overline{\Psi} \equiv \sigma_- \ , \ \partial_\Psi \equiv \sigma_- \ , \ \partial_{\overline{\Psi}} \equiv \sigma_+ \tag{4.12}$$

according to the constraints (4.2) and (4.7) leading moreover to the properties

$$\left\{\partial_{\Psi}, \partial_{\overline{\Psi}}\right\} = I \quad , \quad \left\{\partial_{\Psi}, \partial_{\Psi}\right\} = \left\{\partial_{\overline{\Psi}}, \partial_{\overline{\Psi}}\right\} = 0 \ . \tag{4.13}$$

To such a choice correspond the expected expressions (4.10) in connection with the generators (2.15) and (2.16) given in the $C\ell_2$-context.

If we want to find a realization which does not trivialize the generators $X_1, \ldots, X_{11}$, we have to go to a $C\ell_4$-algebra completely consistent with the constraints (4.2), (4.7) <u>and</u> (4.13) included in our developments. In fact, we have associated four <u>commuting</u> $C\ell_2$-algebras to the set of fermionic operators $\left\{\Psi, \overline{\Psi}, \partial_{\Psi}, \partial_{\overline{\Psi}}\right\}$ which lead to the construction[22]

$$\overset{1}{C\ell_2} \oplus \overset{2}{C\ell_2} = \overset{1,2}{C\ell_3} \quad , \quad \overset{3}{C\ell_2} \oplus \overset{4}{C\ell_2} = \overset{3,4}{C\ell_3} \tag{4.14}$$

and

$$\overset{1,2}{C\ell_3} \oplus \overset{3,4}{C\ell_3} = C\ell_4 \ . \tag{4.15}$$

This clearly appears in our recent developments[11] through the necessary introduction of the operators ∂_{Ψ} and $\partial_{\overline{\Psi}}$ besides the initial Ψ and $\overline{\Psi}$-ones in the general expression of the graded generator X. Moreover with such a point of view, it is straightforward to understand that the doubling proposed in Sec. III has no meaning in the $C\ell_2$-context but presents an interest in the $C\ell_4$-context. An elegant way to superpose both contexts is to rewrite the Clifford relations (4.13) as

$$\left\{\partial_{\Psi}, \partial_{\overline{\Psi}}\right\} = \frac{d}{2} I \quad , \quad \left\{\partial_{\Psi}, \partial_{\Psi}\right\} = \left\{\partial_{\overline{\Psi}}, \partial_{\overline{\Psi}}\right\} = 0 \ , \tag{4.16}$$

where d is the dimension of the matrices. If d=2, we are led to choices such as the one given in eq. (4.12) and the eleven generators become trivial since we are playing with the only irreducible representation of the Clifford algebra $C\ell_2$. If d=4, we thus go to the only irreducible representation of the Clifford algebra $\mathcal{C}\ell_4$ constructed in eq. (4.15) and our eleven additional operators $X_1,...,X_{11}$ become nontrivial. An explicit realization of the last context is given for example by

$$\Psi \equiv \sigma_+ \otimes \sigma_0 \quad , \quad \overline{\Psi} \equiv \sigma_- \otimes \sigma_0 \tag{4.17a}$$

and

$$\partial_\Psi \equiv \sigma_- \otimes \sigma_0 + \sigma_3 \otimes \sigma_- \quad , \quad \partial_{\overline{\Psi}} \equiv \sigma_+ \otimes \sigma_0 + \sigma_3 \otimes \sigma_+ \ . \tag{4.17b}$$

With such a realization, eqs. (4.2), (4.7) and (4.16) are verified and the generators $X_1,...,X_{11}$ given in (4.9) and (4.11) are easily constructed. We immediately recover the eleven explicit forms (3.7) and (3.8) and realize in that way the connection between all these developments. The closed superstructure generated by the twenty-four operators (4.8)-(4.11) is consequently the superalgebra osp(4/2) $\square$ sh(4/2) when the matrices display an effective Clifford algebra $\mathcal{C}\ell_4$. The structure relations are evidently those[15] of osp(2/2) $\square$ sh(2/2) supplemented by the following nonzero ones where we have maintained the parameter d introduced in (4.16). In terms of evident (complex conjugate) considerations and for compactification in the structure relations, let us introduce the following notations in connection with eqs. (4.9) and (4.11) :

$$\begin{gathered} X_1 \equiv M_+ \ , \ X_2 \equiv M_- \ , \ X_4 \equiv N_- \ , \ X_5 \equiv N_+ \ , \ X_6 \equiv U_+ \ , \ X_7 \equiv U_- \ , \\ X_8 \equiv V_+ \ , \ X_9 \equiv V_- \ , \ X_{10} \equiv W_+ \ , \ X_{11} \equiv W_- \ . \end{gathered} \tag{4.18}$$

The supplementary structure relations are then

$$[H_F \, , M_\pm] = \mp\, \omega\, M_\pm \ , \ [H_F \, , N_\pm] = \mp\, \omega\, N_\pm \ ,$$

$$[X_3 \, , M_\pm] = \pm\left(\frac{d}{2} - 2\right) M_\pm \ , \ [X_3 \, , N_\pm] = \mp\, \frac{d}{2}\, N_\pm \ ,$$

$$[M_+ , M_-] = - X_3 + \frac{d}{2}\left(\frac{1}{\omega} H_F + \frac{1}{2} I\right) ,$$

$$[N_+ , N_-] = - X_3 + \left(\frac{d}{2} - 2\right)\left(\frac{1}{\omega} H_F + \frac{1}{2} I\right) , \tag{4.19a}$$

$$\{U_+ , U_-\} = \left(\frac{d}{2} - 1\right) I \ , \ \{V_\pm , W_\pm\} = \pm i \left(\frac{d}{2} - 1\right) C_\mp \ ,$$

$$\{U_\pm , W_\pm\} = \pm \frac{i}{\sqrt{2}} \left(\frac{d}{2} - 1\right) P_\mp = \{U_\pm , W_\mp\} \ ,$$

$$\{Q_\mp , V_\pm\} = \mp \omega M_\pm = - \{S_\mp , W_\pm\} \ ,$$

$$\tag{4.19b}$$

$$\{S_\pm , V_\pm\} = \mp \omega N_\mp = - \{Q_\pm , W_\pm\} \ ,$$

$$\{V_+ , V_-\} = \left(\frac{d}{2} - 1\right) H_B + \omega X_3 - \frac{\omega d}{4} I - H_F \ ,$$

$$\{W_+ , W_-\} = \left(\frac{d}{2} - 1\right) H_B - \omega X_3 - \frac{\omega d}{4} I + H_F$$

and

$$[M_\pm , T_\pm] = - U_\pm = [N_\mp , T_\mp] \ , \ [M_\pm , Q_\pm] = - V_\pm = [N_\mp , S_\mp] \ ,$$

$$[M_\pm , S_\pm] = - W_\pm = [N_\mp , Q_\mp] \ , \ [M_\pm , U_\mp] = \left(\frac{d}{2} - 1\right) T_\mp = [N_\pm , U_\pm] \ ,$$

$$[M_\pm , W_\mp] = \left(\frac{d}{2} - 1\right) S_\mp = [N_\pm , V_\pm] \ , \ [M_\pm , V_\mp] = \left(\frac{d}{2} - 1\right) Q_\mp = [N_\pm , W_\pm] \ ,$$

$$[X_3 , T_\pm] = \pm T_\pm \ , \ [X_3 , Q_\pm] = \pm T_\pm \ , \ [X_3 , S_\pm] = \pm S_\pm \ ,$$

$$[X_3 , U_\pm] = \pm \left(\frac{d}{2} - 1\right) U_\pm \ , \ [X_3 , V_\pm] = \pm \left(\frac{d}{2} - 1\right) V_\pm \ , \ [X_3 , W_\pm] = \pm \left(\frac{d}{2} - 1\right) W_\pm \ ,$$

$$[H_B , V_\pm] = \mp \omega V_\pm \ , \ [H_B , W_\pm] = \pm \omega W_\pm \ , \ [P_\pm , V_\pm] = - i \sqrt{2}\, \omega U_\pm \ ,$$

$$[P_\pm , W_\mp] = - i \sqrt{2}\, \omega\, U_\mp \ , \ [C_\pm , V_\pm] = - 2i\, \omega\, W_\pm \ , \ [C_\pm , W_\mp] = 2\, i\, \omega\, V_\mp \ , \quad (4.19c)$$

where we have distinguished the three blocks (4.19a-c) according to eqs. (3.6) respectively. As it has already been noticed that, when d=2, all the operators $M_\pm$, $N_\pm$, $U_\pm$, $V_\pm$, $W_\pm$ become trivial and X_3 is redundant ($X_3 \equiv \frac{1}{\omega} H_F + \frac{1}{2} I_2$), we immediately see that all the relations (4.19) disappear and that we are left with the structure osp(2/2) □ sh(2/2) as expected. When d=4, all these relations survive and we have the largest superalgebra osp(4/2) □ sh(4/2) associated with the only irreducible representation of the Clifford algebra $\mathcal{C}\ell_4$. Due to the N=2-supersymmetric context and the two fermionic variables Ψ and $\overline{\Psi}$, we get here the maximal superposition of four $\mathcal{C}\ell_2$-algebras leading to the algebra $\mathcal{C}\ell_4$ as mentioned in eqs. (4.14) and (4.15).

V. COMMENTS AND CONCLUSIONS

From Sec. II, III and IV, we learn that, in connection with the Clifford algebra $\mathcal{C}\ell_2$, it is possible to get 24 - 13 = 11 extra (super)symmetries with respect to the closed superstructure osp(2/2) $\square$ sh(2/2) <u>and</u> that, in connection with the Clifford algebra $\mathcal{C}\ell_4$, it is possible to get 96 - 24 = 72 extra (super)symmetries with respect to the closed superstructure osp(4/2) $\square$ sh(4/2) when 1-dimensional harmonic oscillators are concerned. These extra (super)symmetries lead to constants of motion according to eq. (2.19) for example while the <u>closed</u> superstructures were confirmed through nonclassical (see Sec. II and III) as well as classical Lie (see Sec. IV) approaches. This completes the results obtained by Durand[14] and by Beckers-Hussin[15] in the particular application we are concerned with.

The extension to n-dimensional supersymmetric harmonic oscillators is rather straightforward but tedious in both approaches. Let us only mention that the largest invariance superalgebra appearing in connection with the Clifford algebra $\mathcal{C}\ell_{4n}$ (dimension $d = 2^{2n}$) is the superstructure

$$[\mathrm{osp}(4/2) \oplus \mathrm{so}(n)] \,\square\, \mathrm{sh}(4n/2n) \tag{5.1}$$

reducing to

$$[\mathrm{osp}(2/2) \oplus \mathrm{so}(n)] \,\square\, \mathrm{sh}(2n/2n) \ , \tag{5.2}$$

according to recent results on largest kinematical superalgebras[16] when the Clifford algebra coming into the game is $\mathcal{C}\ell_{2n}$ ($d = 2^n$). In both contexts, we are dealing with 4n fermionic quantities $\{\Psi_j , \overline{\Psi}_j , \partial_{\Psi_j} , \partial_{\overline{\Psi}_j} \ , \ j = 1,\ldots,n\}$ and all the results of Sec. IV can be extended for arbitrary n. The operators corresponding to the superalgebra (5.1) are realized as follows :

$$H_B = \frac{1}{2}\left(p_j^2 + \omega^2 x_j^2\right) \ , \ C_+ = \frac{i}{2} \ [\exp(-2\, i\omega t)]\, (p_j + i\omega x_j)^2 \ ,$$

$$C_- = -\frac{i}{2} \ [\exp(-2i\omega t)]\, (p_j + i\omega x_j)^2 \ , \ H_F = \frac{1}{2}\ \omega\, [\Psi_j , \Psi_j] \ ,$$

$$Q_+ = \frac{1}{\sqrt{2}}\left(p_j - i\omega x_j\right)\Psi_j \ , \ Q_- = \frac{1}{\sqrt{2}}\left(p_j + i\omega x_j\right)\overline{\Psi}_j \ ,$$

$$S_+ = \frac{1}{\sqrt{2}}\left[\exp\left(-\,2i\omega t\right)\right]\left(p_j + i\omega x_j\right)\Psi_j \ , \ S_- = \frac{1}{\sqrt{2}}\left[\exp\left(2i\omega t\right)\right]\left(p_j - i\omega x_j\right)\Psi_j \ , \qquad (5.3a)$$

and

$$X_1 = \left[\exp\left(i\omega t\right)\right]\left(\overline{\Psi}_j\partial_{\Psi_j}\right) \ , \ X_2 = \left[\exp\left(-i\omega t\right)\right]\left(\Psi_j\partial_{\overline{\Psi}_j}\right) \ ,$$

$$X_3 = \Psi_j\partial_{\Psi_j} - \overline{\Psi}_j\partial_{\overline{\Psi}_j} + \partial_{\Psi_j}\partial_{\overline{\Psi}_j} \ , \ X_4 = \left[\exp\left(-i\omega t\right)\right]\left(\Psi_j\partial_{\Psi_j} - \Psi_j\overline{\Psi}_j\right) \ ,$$

$$X_5 = \left[\exp\left(i\omega t\right)\right]\left(\overline{\Psi}_j\partial_{\overline{\Psi}_j} - \overline{\Psi}_j\Psi_j\right) \ , \qquad (5.3b)$$

$$X_8 = \frac{1}{\sqrt{2}}\left[\exp\left(i\omega t\right)\right]\left(p_j - i\omega x_j\right)X_{6,j} \ , \ X_9 = \frac{1}{\sqrt{2}}\left[\exp\left(-\,i\omega t\right)\right]\left(p_j + i\omega x_j\right)X_{7,j} \ ,$$

$$X_{10} = \frac{1}{\sqrt{2}}\left[\exp\left(-\,i\omega t\right)\right]\left(p_j + i\omega x_j\right)X_{6,j} \ , \ X_{11} = \frac{1}{\sqrt{2}}\left[\exp\left(i\omega t\right)\right]\left(p_j - i\omega x_j\right)X_{7,j} \ ,$$

generate the osp(4/2)-superalgebra, while

$$J_{ij} \equiv x_i p_j - x_j p_i \ , \ i \neq j \ , \qquad (5.3c)$$

generate the orthogonal subalgebra so(n) and

$$\begin{gathered} P_{+,j} = i\left[\exp(-i\omega t)\right]\left(p_j + i\omega x_j\right) \ , \ P_- = -\,i\left[\exp(-i\omega t)\right]\left(p_j - i\omega x_j\right) \ , \\ T_{+,j} = \left[\exp\left(-i\omega t\right)\right]\Psi_j \ , \ T_{-,j} = \left[\exp(i\omega t)\right]\overline{\Psi}_j \ , \ I \ , \\ X_{6,j} = \partial_{\Psi_j} - \overline{\Psi}_j \ , \ X_{7,j} = \partial_{\overline{\Psi}_j} - \Psi_j \end{gathered} \qquad (5.3d)$$

generate the sh(4n/2n)-superalgebra. All the operators X_1 , X_2 , X_4 , X_5 , $X_{6,j}$, $X_{7,j}$, X_8 , X_9 , X_{10} , X_{11} become trivial and X_3 redundant when the $\mathcal{Cl}_{2n}$-context is required leading to the structure (5.2). Let us notice that for n=1,2,3, the superalgebra (5.1) has respectively the dimension 24, 31 or 39.

The present developments can also be applied to other supersymmetric systems besides the harmonic oscillator. If, after D'Hoker et al.[17], we consider the superpotentials (2.20) and (2.21) in the 1-dimensional context, we can show that the corresponding supersymmetric wave equations lead to new supersymmetries. In fact, in both cases, we can apply our method presented in Sec. IV and get the (closed) superalgebra osp(2/2) $\oplus$ su(1/1). Here the nonsimple superalgebra su(1/1) is generated by the operators X_3 , X_6 and X_7 characterized by the structure relations

$$\{X_6\,,X_7\} = \left(\frac{d}{2}-1\right)I \ , \ [X_3\,,X_6] = \left(\frac{d}{2}-1\right)X_6 \ , \ [X_3\,,X_7] = -\left(\frac{d}{2}-1\right)X_7 \ . \qquad (5.4)$$

Once again if they are realized in terms of 2 by 2 matrices they are trivial or redundant and we are left with the previous resuls[17] corresponding to osp(2/2) alone while realized in terms of 4×4 matrices the whole superalgebra works. These results can also be extended for arbitrary n.

One of us (J.B.) wants to dedicate this article to the memory of Professor Léon Van Hove.

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