Helvetica Physica Acta, Vol. **62** (1989) 389–397
0018–0238/89/040389–09$1.50 + 0.20/0

Vacuum polarization and the trace anomaly

By K. J. Abraham

Institut für Theoretische Physik, Universität Bern, Sidlerstrasse 5, 3012 Bern, Switzerland*)

(23. X. 1988)

Abstract. The trace of the energy momentum tensor (henceforth temt) in gauge theories coupled to spinors has been known for a long time. From the physical meaning of the energy-momentum tensor it is clear that the temt is renormalization group invariant, although this is not obviously so. In this paper we consider a simplified theory consisting of a massless Dirac spinor coupled to a Maxwell field, and show that the temt is renormalization group invariant to all orders in perturbation theory. The proof proceeds as follows: we express the renormalization group invariance of the temt as a set of relations between the coefficients of β functions and anomalous dimensions (valid in a certain subtraction scheme) and then prove by diagrammatic techniques that these relations indeed hold. The properties of the vacuum polarization tensor will turn out to be crucial to the proof.

Introduction

It has long been known that the temt has measurable phenomenological consequences, e.g. in the analysis of electron proton deep inelastic scattering. Hence it is an important consistency requirement that all the properties of the temt, that one expects to be true on the grounds that the energy momentum tensor is a symmetry generator, are actually fulfilled. This includes naturally, renormalization group invariance. The field theory we consider is QED in the limit that the electron mass vanishes. We go to this limit only to avoid complications due to operator mixings which would distract from the main line of the argument. The theory is dimensionaly regularized. In order to simplify the differential equation expressing renormalization group invariance as much as possible the S-Matrix and renormalized composite operators are defined in the minimal subtraction scheme. (In particular no momentum dependent subtraction constants are added to Greens functions of composite operators.) In this scheme the differential equation may be transformed into an algebraic one which relates the divergent pieces of different Greens functions. This algebraic equation may be independently verified by comparing appropriate sets of Feynman diagrams, thereby proving that the temt is renormalization group invariant.

*) Address after November 1988: NIKHEF-H, P.O. Box 41882, 1009 DB Amsterdam, The Netherlands.

Part I

By considering the temt of Non-Abelian Gauge Theories [1] and QED [2] in the appropriate limits, the temt of the theory we are considering[1]) can be seen to be $\beta(\alpha)F^2$ where, as is usual in dimensionally regularized theories the β function is defined by $\alpha\beta(\alpha) = \mu\dfrac{\partial\alpha}{\partial\mu}$, where μ is the mass scale introduced by dimensional regularization. Renormalization group invariance of the trace anomaly requires that

$$\mu\frac{\partial[\beta(\alpha)F^2]}{\partial\mu} = 0. \tag{1}$$

$\beta(\alpha)$ depends on μ implicitly through α and explicitly through finite pieces of counterterms. The explicit dependence may be made to vanish by adopting a suitable scheme, e.g. minimal subtraction. In such a scheme

$$\mu\frac{\partial\beta}{\partial\mu} = \alpha\beta(\alpha)\frac{\partial\beta(\alpha)}{\partial\alpha}$$

If γ is the anomalous dimension of F^2 then

$$\mu\frac{\partial F^2}{\partial\mu} = -\gamma F^2.$$

With these definitions Equation (1) can be written as

$$\frac{\partial\beta(\alpha)}{\partial\alpha}\alpha - \gamma = 0 \tag{2}$$

If $\beta(\alpha) = \sum_n \beta_n\alpha^n$ and $\gamma(\alpha) = \sum_n \gamma_n\alpha^n$ with the summations running over all integers greater than zero, then equation (2) can be written as

$$n\beta_n = \gamma_n\ \forall n; \qquad n = 1, 2, \ldots \tag{3}$$

This is the all loop identity we wanted to prove.

However, Feynman diagram calculations isolate not β_n and γ_n directly, but rather the divergent pieces of Greens functions. Hence equation (3) must be rewritten before it can be effectively used. To do so we go back to the definition of γ; γ is defined by $\gamma = \dfrac{\mu}{Z_f}\dfrac{\partial Z_f}{\partial\mu}$ where Z_F is defined by the $F_0^2 = Z_F F^2$; the subscript o denoting a bare quantity. If the Greens function used to define Z_F (which will be specified later) is also minimally renormalized; then by our previous reasoning, Z_F depends on μ only through α. Hence

$$\gamma = \left(\frac{\mu}{Z_F}\right)\left(\frac{\partial Z_F}{\partial\alpha}\right)\left(\frac{\partial\alpha}{\partial\mu}\right) = \frac{1}{Z_F}\left(\frac{\partial Z_F}{\partial\alpha}\right)\alpha\beta(\alpha) \tag{4}$$

---

[1]) As always, α is the coupling constant and F^2 is the square of the Field Strength.

If $\epsilon = \dfrac{4 - \text{no. of dimensions}}{2}$; then by virtue of our subtraction scheme Z_F is a Laurent series in ϵ with only negative powers of ϵ and 1 as the term independent of ϵ. Since Z_F diverges as ϵ vanishes we need in equation (4) not only the non-vanishing β function in four dimensions but also the pieces which vanish with ϵ. We now evaluate these pieces.

We go to the defining equation of the bare charge α_0 which is $\alpha_0 = \mu^{2\epsilon} Z_3^{-1} \alpha$; where the limit $\epsilon \to 0$ is assumed. (By definition the bare and renormalized gauge potentials are related by $A_0^\mu = Z_3^{1/2} A^\mu$). Using the independence of bare quantities on μ we have

$$0 = 2\epsilon Z_3^{-1}\alpha + \mu\alpha \frac{\partial Z_3^{-1}}{\partial \mu} + Z_3^{-1}\mu \frac{\partial \alpha}{\partial \mu} \tag{5}$$

By repeating our earlier reasoning we see that Z_3, and hence Z_3^{-1}, like Z_F must be a Laurent series in ϵ with only negative powers of ϵ and 1 as the term independent of ϵ. Hence $\mu \dfrac{\partial Z_3^{-1}}{\partial \mu}$ has singularities as ϵ vanishes. Extracting the non-singular terms in equation (5) and demanding that they seperately add up to 0 gives

$$\alpha\beta(\alpha) = -2\epsilon\alpha + \beta_1\alpha^2 + \beta_2\alpha^3 + \cdots \tag{6}$$

We have determined the ϵ dependent pieces of $\beta(\alpha)$. But equation (5) contains still more information, which we now extract.

Let the residue of the simple pole in Z_3 be $\sum_n b_n \alpha^n$ where the sum over n runs as usual over integers larger than zero. Then the residue of the simple pole of Z_3^{-1} must be $-\sum_n b_n \alpha^n$; implying

$$Z_3^{-1} = 1 - \sum_n \frac{b_n \alpha^n}{\epsilon} + \cdots$$

Hence

$$\mu \frac{\partial Z_3^{-1}}{\partial \mu} = -\sum_n \frac{n b_n \alpha^{n-1}}{\epsilon} \left(\mu \frac{\partial \alpha}{\partial \mu} \right) + \cdots$$

Substituting $\mu \left(\dfrac{\partial \alpha}{\partial \mu} \right)$ from equation (6) we have

$$\mu \frac{\partial Z_3^{-1}}{\partial \mu} = \sum_n 2n b_n \alpha^n + \cdots \tag{7}$$

Since the right hand side of equation (5) is a Laurent series in ϵ the coefficient of each power of ϵ of must vanish separately if the series as a whole is to vanish. Substituting the residue of the simple pole in Z_3^{-1} and equations (7) and (6) into equation (5) we see that the pieces independent of ϵ in equation (5) vanish if

$$\beta_n = -2n b_n \tag{8}$$

We have succeeded in expressing the β_n, which are the coefficients in the β function in terms of the b_n which are obtained by calculating vacuum polarization graphs. A text-book calculation [3] of b_1 and b_2 shows that equation (8) correctly reproduces the first two terms in the expansion of the β function.

We now rewrite γ in terms of the divergent pieces of Z_F. Let the residue of the simple pole in ϵ in the expansion of Z_F be $\sum_n g_n \alpha^n$. Then

$$Z_F^{-1} = 1 - \sum_n \frac{g_n \alpha^n}{\epsilon} + \cdots$$

Substituting this expansion as well as equation (6) into the defining equation for γ (equation (4)) we get, after ignoring the pieces which are singular as $\epsilon \to 0$

$$\gamma_n = -2ng_n \tag{9}$$

where γ_n was defined just preceeding equation (3). Substituting equations (9) and (8) into equation (2) we get a new form of our all loop identity

$$nb_n = g_n \ \forall n; \qquad n = 1, 2, \ldots \tag{10}$$

which expresses the renormalization group invariance of the temt.

In part II we will rederive equation (10) by expressing the divergent pieces of $\langle 0| \, TF^2 A_\mu A_\nu \, |0\rangle$ in terms of the singular pieces of vacuum polarization graphs. This independent check on the validity of equation (10) completes our proof that the temt is indeed renormalization group invariant to all orders in perturbation theory.

Part II

We begin by considering the tree-level Greens function $\langle 0| \, TF^2 A_\mu A_\nu \, |0\rangle$. This generates the Feynman rule for the vertex F^2, which we will denote by the symbol $\otimes$. It is easy to show [4] that in the limit that the composite operator transfers zero momentum, the tree-level Greens function is, with a suitable normalization merely the free Landau gauge photon propagator. This gives us a diagrammatic prescription for going to the zero-momentum transfer limit in a higher order Feynman diagram; remove the composite operator outright and join the resulting photon legs to form a single free Landau gauge propagator. We now apply this prescription to some arbitrary graph contributing to $\langle 0| \, TF^2 A_\mu A_\nu \, |0\rangle$. The graph has two external photon lines, a composite operator vertex, and at non-zero momentum transfer is dependent on two external momenta. Setting the inserted momentum to zero leaves a graph with two external photon propagators, dependent on one external momentum, and from the diagrammatic prescription just described, completely independent of any composite operator insertions. Such a graph is a vacuum polarization graph, by definition.

The renormalized composite operator Greens function is related to the bare one by a scale factor which is $\dfrac{Z_F}{Z_3}$, provided only one particle irreducible (1PI) graphs are taken into account. Z_F depends on the g_n and Z_3 on the b_n defined in part 1. In the zero momentum transfer limit, since we recover vacuum polarization graphs, the scale factor is Z_3 dependent. Hence in the zero momentum transfer limit, provided infra red singularities are identified and segregated, it is possible to express Z_F in terms of Z_3 and hence g_n in terms of b_n. If equation (10) is recovered, then we will have independently rederived an identity which was originally derived assuming the renormalization group invariance of the temt. This we will do and thereby prove that the temt is renormalization group invariant to all orders in perturbation theory.

Before beginning with the formal proof we consider the 3 loop Feynman graphs shown in the Appendix. They differ only in that one graph, graph II, has an F^2 vertex inserted between B and C. Heuristically, this graph could have been obtained from the vacuum polarization graph, graph I, by making the appropriate insertion. We could also have generated another composite operator graph from graph I by inserting the F^2 vertex between D and E instead of between B and C. This exhausts all possibilities from graph I. This process can be repeated for all distinct three loop "parent" vacuum polarization graphs. To $\mathcal{O}(\alpha^n)$ each vacuum polarization graph generates $(n-1)$ composite operator graphs since each vacuum polarization graph to this order has $(n-1)$ internal photon propagators. In fact, to $\mathcal{O}(\alpha^n)$ there are $(n-1)$ times as many graphs contributing to $\langle 0| TF^2A_\mu A_\nu |0\rangle$ as to vacuum polarization. This can be seen as follows.

Vacuum polarization to $\mathcal{O}(\alpha^n)$, neglecting overall constants and irrelevant arguments is given by

$$\langle 0| TA_\mu A_\nu \int d^4x^1 \cdots d^4x^{2n} \bar{\Psi}\gamma^{\mu_1}\Psi A_{\mu_1}(x^1) \cdots \bar{\Psi}\gamma^{\mu_{2n}}\Psi A_{\mu_{2n}}(x^{2n}) |0\rangle \tag{11}$$

$\langle 0| TF^2A_\mu A_{nu} |0\rangle$ to the same order in perturbation theory (once again neglecting irrelevant constants and arguments) is given by

$$\left\langle 0 \middle| TF^2A_\mu A_\nu \int d^4x^1 \cdots d^4x^{2n} \bar{\Psi}\gamma^{\mu_1}\Psi A_{\mu_1}(x^1) \cdots \bar{\Psi}\gamma^{\mu_{2n}}\Psi A_{\mu_{2n}}(x^{2n}) \middle| \right\rangle \tag{12}$$

Since the free F^2 vertex does not contain any spinorial fields the spinorial Wick contractions are identical in equations (11) and (12). This leaves the gauge potentials to be contracted among themselves. The external gauge potentials are accounted for first. This leaves in equation (11)

$$\frac{{}^{2n-2}C_2 \times {}^{2n-4}C_2 \times \cdots \times 1}{(n-1)!} \tag{13}$$

remaining Wick contractions among the gauge potentials to be performed. For $\langle 0| TF^2A_\mu A_\nu |0\rangle$ (equation (12)), there remain (after the external gauge potentials have been accounted for) ${}^{2n-2}C_2$ ways of Wick contracting the F^2 vertex. The remaining $(n-2)$ gauge potentials may be mutually contracted in

$$\frac{{}^{2n-4}C_2 \times {}^{2n-6}C_2 \times \cdots \times 1}{(n-2)!} \text{ ways.}$$

The total number of non-spinorial Wick contractions is then

$$ {}^{2n-2}C_2 \times \frac{{}^{2n-4}C_2 \times {}^{2n-6}C_2 \times \cdots \times 1}{(n-2)!} \tag{14} $$

Comparing equations (13) and (14) we see immediately that there are to $\mathcal{O}(\alpha^n)$, $(n-1)$ times as many composite operator graphs as vacuum polarization graphs. This suggests that it is possible to generate all possible composite operator graphs of a given order by making insertions of F^2 in the internal photon propagators of vacuum polarization graphs of the same order. We now show that this is indeed the case.

We begin by repeating the sequence of Wick contractions which we used to decompose equations (11) and (12). First the spinor fields were fully Wick contracted into loops and sets of loops which were identical for both cases. Among the gauge potentials the external propagators were accounted for first. In equation (11) this left $(2n-2)$ gauge potentials to be pairwise mutually contracted; each pair generating an internal photon propagator on contraction, each set of contracted pairs generating a single Feynman graph with $(n-1)$ internal photon propagators, different sets of contracted pairs generating different Feynman graphs, all possible sets of contracted pairs generating all possible Feynman graphs derivable from given spinorial Wick contractions. In equation (12) the same sets of pairs can be used to exhaust all possible remaining Wick contractions, provided each pair generates not only an internal photon propagator but also two internal propagators threaded through the F^2 vertex. This corresponds to contracting the vertex. Since each set contains $(n-1)$ pairs, all of whom must be threaded in succession through the vertex in order to exhaust all possibilities, each set generates not one, but $(n-1)$ separate Feynman graphs which differ from one another only in that each time a different pair is threaded through the F^2 vertex. But this amounts to inserting the F^2 vertex in each pair in succession, which in turn amounts to inserting the vertex into each internal photon propagator in the corresponding vacuum polarization graph in succession, as each contracted pair represents an internal photon propagator in the corresponding vacuum polarization graph. This is just what we wanted to show.

We now make a choice of gauge, the Landau gauge. We consider the $(n-1)$ Feynman graphs which can be generated by making F^2 insertions in the internal photon propagators of a given vacuum polarization graph, in the limit that no momentum is transferred by the insertion. Then, as a consequence of our prescription for going to the limit of zero momentum transfer, all the graphs generated can be set equal to another, and equal to the "parent" vacuum polarization graph, provided the internal momenta are suitably relabelled. (Since we are not dealing with an anomalous vertex, this can be without violating any of the Ward Identities we require.) Hence we see that in going to the zero momentum transfer limit we generate the "parent" vacuum polarization graph with an additional combinatorical weight of $(n-1)$. This process of making F^2 insertions in vacuum polarization graphs and going to the zero momentum transfer limit can obviously be repeated for all vacuum polarization graphs. Since we have shown that all graphs contributing to $\langle 0| TF^2 A_\mu A_\nu |0\rangle$ may be generated by making F^2 insertions in vacuum polarization graphs, it now follows that going to the zero momentum transfer limit merely reproduces vacuum polarization with an additional combinatorical weight $(n-1)$.

In practice though, $\langle 0| TF^2 A_\mu A_\nu |0\rangle$ at zero momentum transfer is not infra-red safe. We will show that the infra-red divergences can be understood in terms of the ultra-violet divergences of the S-Matrix. This will also provide a recipe for removing the infra-red divergences. To illustrate these remarks we now turn to the two Feynman graphs shown in the Appendix.

In the zero momentum transfer limit, and with a suitable choice of gauge (the Landau gauge), the two graphs are equal. However, in order to cancel the non-local singularities lower order ultra-violet counterterms must be added. Simple inspection reveals that the permissible counterterms are different for the different graphs. Graph I for example, requires a counterterm to compensate for the two loop vertex renormalization sub-graph ABCDE. It is not permissible to add the same counterterm to graph II, the one contributing to $\langle 0| TF^2 A_\mu A_\nu |0\rangle$. (Both the graphs of course, requires a common one loop vacuum polarization counterterm to compensate for the vacuum polarization bubble CD.) At non-zero momentum transfer the two graphs are distinct, and the mis-match between permissible sets of counterterms is then only to be expected. However, at zero-momentum transfer the mis-match persists, even though the two graphs are no longer distinct.

A similar mis-match may be found by considering many other pairs of graphs to three loops or any other higher order in perturbation theory. Given the fact that a mis-match is inevitable, we see that evaluating our composite operator Greens function at zero momentum transfer amounts (up to a factor of $(n-1)$ to $\mathcal{O}(\alpha^n)$) to evaluating vacuum polarization graphs without the full complement of lower order ultra-violet counterterms. This results in uncancelled non-local divergences.

We claim that these non-local divergences are infra-red. If they were not so, i.e. that they were ultra-violet divergences, then it would imply that F^2 is not multiplicatively renormalizable. In our massless theory this is clearly not the case. Hence our claim.

Before explaining how both the local and non-local infra-red divergences may be segregated, we draw some general conclusions about the nature of the ultra-violet divergences based on the requirement of multiplicative renormalizability. The Greens function calculated at zero momentum transfer is a second rank tensor dependent on one momentum which is the same as the momentum of the two external legs. The product of the two external propagators and the divergent pieces of the Greens function must have the same functional form as at tree-level; this is a condition imposed by multiplicative renormalizability. But the tree level Greens function is merely the free Landau gauge photon propagator. Hence the divergent pieces of the Greens function must be transverse, in addition to being local. This transversality requirement plays an important role in what follows.

Since our zero momentum transfer calculation is related to that of an S-Matrix element, vacuum polarization, we know that the full complement of ultra-violet lower order counterterms is required to completely cancel non-local divergences. The missing counterterms are then added in the guise of infra-red counterterms. This identification is permissible as we have already reasoned that the non-local divergences are infra-red divergences. The complete set of lower order counterterms together with the requirement of transversality is enough to fully specify the divergent pieces which are (since we are now computing vacuum polarization with no missing lower order counterterms) those of vacuum polarization, modulo a factor $(n-1)$ at $\mathcal{O}(\alpha^n)$.

As a precaution against local infra-red divergences the non-zero external momentum is chosen to be non-exceptional. The theorem of Poggio and Quinn [5] then guarentees infra-red finiteness. Our recipe of completing the complement of lower order ultra-violet counterterms and holding the external momentum non-exceptional then ensures cancellation not only of non-local divergences, but also of local infra-red divergences. The remaining divergences are therefore local and purely ultra-violet. They are related to those of vacuum polarization by the additional combinatorical weight of $(n-1)$. We are now in a position to evaluate the g_n in terms of the b_n and thereby check the validity of equation (10).

At tree-level $\dfrac{Z_F}{Z_3}$ is 1. We have just shown how to higher orders in perturbation theory, $\dfrac{Z_F}{Z_3}$ is related to the divergent pieces of vacuum polarization which however are themselves contained in Z_3. Hence we have from the definition of the b_n

$$\frac{Z_F}{Z_3} = 1 + \sum_n \frac{(n-1)b_n\alpha^n}{\epsilon} + \cdots$$

Using this and the definition of Z_3,

$$Z_F = \left(1 + \sum_n \frac{(n-1)b_n\alpha^n}{\epsilon} + \cdots\right)\left(1 + \sum_n \frac{b_n\alpha^n}{\epsilon} + \cdots\right) = 1 + \sum_n \frac{nb_n\alpha^n}{\epsilon} + \cdots \tag{15}$$

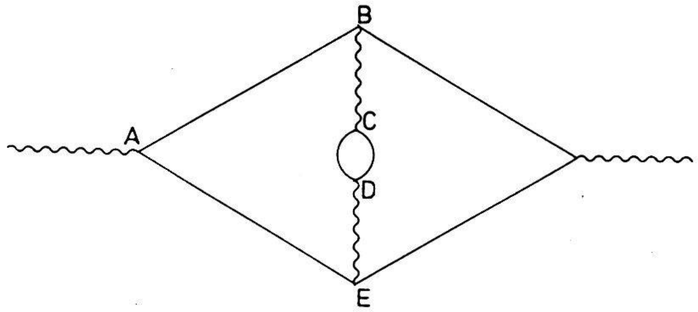
This implies, however from the definition of g_n, that

$$nb_n = g_n \ \forall n; \qquad n = 1, 2, \ldots \tag{16}$$

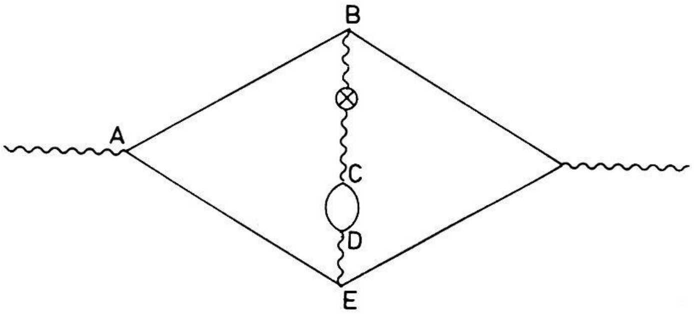
But this is identical with equation (10) which was derived assuming the renormalization group invariance of the temt.

Hence we have recovered an identity which is a consequence of the renormalization group invariance of the temt. This proves that the temt is renormalization group invariant to all orders in perturbation theory.

Appendix



GRAPH I



GRAPH II

Acknowledgements

I am indebted to Profs. H. Leutwyler and P. Minkowski for pointing out numerous ambiguities in a preliminary version of this paper. I would also like to thank Dr. J. Soto for many stimulating discussions. My thanks also go to Mr. Damian Lisik for drawing the Feynman diagrams in the Appendix. Finally, I am grateful to the Erziehungsdirektion of the Canton of Bern for financial support during the period in which this work was carried out.

REFERENCES

[1] COLLINS J. C., DUNCAN A., and JOGLEKAR S. L., Phys. Rev. *D 16* 438 (1977); NIELSEN N. K., Nucl. Phys. *B80*, 372 (1978).
[2] ADLER S. L., COLLINS J. C., and DUNCAN A., Phys. Rev. *D15* No. 6 1712 (1977); CREWTHER R. J., Phys. Rev. Lett. *28*, 1421 (1972); CHANOWITZ and ELLIS J., Phys. Lett. *40 B*, 397 (1972), Phys. Rev. *D 7*, 2490 (1973); MINKOWSKI P., University of Bern Pre-Print BUTP 14/76.
[3] ITZYKSON C. and ZUBER J. B., "Quantum Field Theory" McGraw Hill Book Company, (1980).
[4] TARRACH R., Nucl. Phys. *B 196*, 45 (1982).
[5] POGGIO E. C. and QUINN H. R., Phys. Rev. *D 14* 578 (1976).