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On Mourre's approach to spectral theory*)

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Abstract. We present an extension of Mourre's method for proving absence of singularly continuous spectrum of a self-adjoint operator H. We specify a large class of locally H-smooth operators and apply these results to N-body Schrödinger operators.

1. Introduction

Let H be a self-adjoint operator in a Hilbert space $\mathscr{H}$ and $E(\cdot)$ its spectral measure. Mourre's method for determining the principal spectral properties of H consists essentially of two points [1], [2]:

(i) find a self-adjoint operator A that is locally conjugate to H modulo a compact operator, more precisely such that, for suitable intervals J:

$$E(J)[iH, A]E(J) \geqslant aE(J) + K,$$

where $a > 0$ and K are a real constant and a compact operator respectively (depending on J);

(ii) replace the imaginary part of z in the resolvent $(H - z)^{-1}$ by $(\text{Im}\, z + N_\varepsilon)$, where N_ε is a certain self-adjoint operator (having the same sign as $\text{Im}\, z$) constructed from the commutator $[H, A]$ and depending on a parameter $\varepsilon \in (0, 1)$, with $N_\varepsilon \to 0$ as $\varepsilon \to 0$; then show that, for a suitable operator L, there is a differential inequality (as a function of ε) for the norm of $L(H - \lambda + i(\mu + N_\varepsilon))^{-1}L^*$ in such a way that the coefficients of this differential inequality are independent of λ and μ for λ in a suitable subinterval J_0 of J and $\mu \geqslant 0$. If these coefficients are not too singular as functions of ε near $\varepsilon = 0$, one obtains after integration that

$$\sup \{\|L(H - \lambda + i(\mu + N_\varepsilon))^{-1}L^*\| \mid \varepsilon \in (0, 1), \lambda \in J_0, \mu \geqslant 0\} < \infty, \tag{1}$$

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which implies local H-smoothness of L and hence the absolute continuity of the spectrum of H in J_0.

Part (ii) requires various technical assumptions on the domains of A, H and their commutators, in particular an assumption on the double commutator $[[H, A], A]$ which, in the applications to Schrödinger operators for example, entails a certain restriction on the potentials (see e.g. Remark 3 on page 523 of [2]). In trying to understand the origin of this restriction, we found that it can be by-passed by using more carefully the ideas of part (ii) of Mourre's arguments. In fact the derivation of (1) would go through if the coefficients appearing in Mourre's differential inequality were replaced by more singular functions of ε, and this extra freedom can be exploited by choosing auxiliary operators N_ε different from those used by Mourre and subsequent authors.

There are various possibilities for generalizing the operators N_ε used previously. We shall here present in detail the case where N_ε is constructed from $[H(\varepsilon), A]$ rather than from $[H, A]$, where $H(\varepsilon)$ is a suitable approximation of H. In applications to Schrödinger operators this allows one to dispense with the restriction on the potentials mentioned above and, for example, to recover essentially Lavine's smoothness result [3] by the Mourre method; in this case $H(\varepsilon)$ is obtained by introducing an ε-dependent cut-off on the potentials.

The organization of the paper is as follows. In Section 2 we introduce the general mathematical framework. In Section 3 we present a simple proof of the virial theorem and its implications for the point spectrum of H. In Section 4 we give the generalized form of part (ii) of Mourre's argument. In Section 5 we construct a large class of operators L that are locally H-smooth, and in Section 6 we give some applications to N-body Schrödinger operators. In order to limit the length of the paper, some of the proofs will be indicated only at a formal level; detailed and rigorous arguments may be found in [4]. For an extensive bibliography we refer to [5].

2. General framework

Our basic objects are a self-adjoint operator H in a complex separable Hilbert space $\mathscr{H}$ and a strongly continuous one-parameter group $\{W(\alpha)\}_{\alpha\in\mathbb{R}}$ of unitary operators leaving the domain of H invariant.

In the present section we introduce some additional quantities that can be defined in terms of $\mathscr{H}$, H and $\{W(\alpha)\}$, and we specify our notations. We denote by $(\cdot, \cdot)$ and $\|\cdot\|$ the scalar product and the norm in $\mathscr{H}$ respectively. We let $\mathscr{G}^1$ be the domain of H, provided with the norm

$$\|f\|_1 \equiv \|(I + |H|)f\|,$$

where I is the identity operator. We observe that the above norm is equivalent to the graph norm $(\|f\|^2 + \|Hf\|^2)^{1/2}$. For $s \in [0, 1]$, we denote by $\mathscr{G}^s$ the interpolation space between $\mathscr{G}^0 \equiv \mathscr{H}$ and $\mathscr{G}^1$; $\mathscr{G}^s$ coincides with the domain of the operator

$(I+|H|)^s$ in $\mathcal{H}$ and is provided with the norm

$$\|f\|_s = \|(I+|H|)^s f\|$$

(see e.g. [6], p. 44). We denote by $\mathcal{G}^{-s} \equiv (\mathcal{G}^s)^*$ the adjoint Hilbert space of $\mathcal{G}^s$ (the vector space of all anti-linear continuous mappings from $\mathcal{G}^s$ to $\mathbb{C}$, provided with the usual dual norm). We shall always identify $\mathcal{H}^* \equiv (\mathcal{G}^0)^*$ with $\mathcal{H} \equiv \mathcal{G}^0$ through the Riesz lemma, and we then have

$$\mathcal{G}^s \subset \mathcal{G}^t \subset \mathcal{H} \subset \mathcal{G}^{-t} \subset \mathcal{G}^{-s} \quad \text{for any} \quad 1 \geqslant s \geqslant t \geqslant 0,$$

with continuous and dense inclusions. We use the notation $\{E(\cdot)\}$ for the spectral measure of H.

Since each $W(\alpha)$ is assumed to leave $\mathcal{G}^1$ invariant, it follows by interpolation that each $\mathcal{G}^s$ $(0 \leqslant s \leqslant 1)$ is invariant under $\{W(\alpha)\}_{\alpha\in\mathbb{R}}$, and that the restriction of $\{W(\alpha)\}_{\alpha\in\mathbb{R}}$ to $\mathcal{G}^s$ defines a strongly continuous (with respect to $\|\cdot\|_s$) one-parameter group in $\mathcal{G}^s$ (the strong continuity is implied by the weak measurability ([7], Section 10.2) and the latter is easy to prove from the strong continuity in $\mathcal{H}$). By taking the adjoints of these one-parameter groups, one sees that $\{W(\alpha)\}_{\alpha\in\mathbb{R}}$ extends from $\mathcal{H}$ to a strongly continuous one-parameter group in $\mathcal{G}^{-s}$ for each $s \in [0, 1]$.

The generator A_t of the group $\{W(\alpha)\}_{\alpha\in\mathbb{R}}$ in $\mathcal{G}^t$ $(-1 \leqslant t \leqslant 1)$ is a closed densely defined operator (see [7], Section 10.3), and A_0 is self-adjoint. Formally we have $W(\alpha) = \exp(i\alpha A_t)$. We set $\mathscr{E} = D(A_1)$ and provide it with the graph norm

$$\|f\|_{\mathscr{E}} = (\|f\|_1^2 + \|A_1 f\|_1^2)^{1/2}.$$

Clearly

$$\mathscr{E} \subset \mathcal{G}^1 \subset \mathcal{H} \subset \mathcal{G}^{-1} \subset \mathscr{E}^*,$$

where the image of each embedding is dense in the respective space. The restriction of $\{W(\alpha)\}_{\alpha\in\mathbb{R}}$ to $\mathscr{E}$ defines a strongly continuous one-parameter group in $\mathscr{E}$. We denote by A the operator A_1 considered as defined on $\mathscr{E}$ with values in $\mathcal{G}^1$; hence A is a bounded operator from $\mathscr{E}$ to $\mathcal{G}^1$.

If $\mathcal{F}_1$, $\mathcal{F}_2$ are Hilbert spaces, we denote by $\mathcal{B}(\mathcal{F}_1, \mathcal{F}_2)$ the Banach space of all linear continuous operators from $\mathcal{F}_1$ to $\mathcal{F}_2$. We set $\mathcal{B}(\mathcal{F}) = \mathcal{B}(\mathcal{F}, \mathcal{F})$ and we denote by $\|\cdot\|$ and $\|\cdot\|_{s,s'}$ the norm in $\mathcal{B}(\mathcal{H})$ and in $\mathcal{B}(\mathcal{G}^s, \mathcal{G}^{s'})$ respectively. An operator $T \in \mathcal{B}(\mathcal{F}, \mathcal{F}^*)$ is said to be *symmetric* if $\langle f, Tg\rangle = \overline{\langle g, Tf\rangle}$ for all $f, g \in \mathcal{F}$, where $\langle \cdot, \cdot \rangle : \mathcal{F} \times \mathcal{F}^* \to \mathbb{C}$ is defined by $\langle f, \varphi\rangle = \varphi(f)$, $f \in \mathcal{F}$, $\varphi \in \mathcal{F}^*$.

All operators that we shall consider will be well defined as elements of $\mathcal{B}(\mathscr{E}, \mathscr{E}^*)$. If T is an operator in $\mathcal{B}(\mathcal{G}^s, \mathcal{G}^t)$ for some $s, t \in [-1, 1]$, then its restriction to $\mathscr{E}$ clearly belongs to $\mathcal{B}(\mathscr{E}, \mathscr{E}^*)$. On the other hand, if an operator S in $\mathcal{B}(\mathscr{E}, \mathscr{E}^*)$ has an extension to a bounded operator from $\mathcal{G}^s$ to $\mathcal{G}^t$ for example, then we say that $S \in \mathcal{B}(\mathcal{G}^s, \mathcal{G}^t)$ and use the same symbol S for this extension.

If $T : \mathcal{G}^s \to \mathcal{G}^{-t}$ is symmetric and continuous, with $s \geqslant t \geqslant 0$, then its adjoint T^* is in $\mathcal{B}(\mathcal{G}^t, \mathcal{G}^{-s})$. Since T^* coincides with T on $\mathcal{G}^s \subset \mathcal{G}^t$, it follows that T has an extension to an operator in $\mathcal{B}(\mathcal{G}^t, \mathcal{G}^{-s})$; hence (with the convention made above) one has $T \in \mathscr{B}(\mathscr{G}^t, \mathscr{G}^{-s})$ and

$$\|T\|_{t,-s} \equiv \|T^*\|_{t,-s} = \|T\|_{s,-t}. \tag{2}$$

Similarly the symmetric bounded operator A from $\mathscr{E}$ to $\mathscr{G}^1$ introduced before extends to a bounded operator (also denoted by A) from $(\mathscr{G}^1)^* \equiv \mathscr{G}^{-1}$ to $\mathscr{E}^*$; in other terms $A \in \mathscr{B}(\mathscr{E}, \mathscr{G}^1) \cap \mathscr{B}(\mathscr{G}^{-1}, \mathscr{E}^*)$. In particular, if $S \in \mathscr{B}(\mathscr{G}^1, \mathscr{G}^{-1})$, then the commutator $[A, S]$ is well defined in $\mathscr{B}(\mathscr{E}, \mathscr{E}^*)$. Also there is no need to make any distinction between the operators A_t and A.

Finally we observe that, if θ is a bounded function of compact support, then $\theta(H) \in \mathscr{B}(\mathscr{G}^s, \mathscr{G}^t)$ for any $s, t \in [-1, 1]$. In particular, if J is a bounded set, then $E(J) \in \mathscr{B}(\mathscr{G}^{-1}, \mathscr{G}^1)$.

3. The virial theorem. The point spectrum

We begin with an auxiliary result and then prove the virial theorem.

Proposition 1. *Let $T \in \mathscr{B}(\mathscr{G}^1, \mathscr{G}^{-1})$ and $-1 \leq t \leq s \leq +1$. For $\alpha \in \mathbb{R}$, define $T_\alpha = W(\alpha)TW(-\alpha)$. Assume that $[A, T] \in \mathscr{B}(\mathscr{G}^s, \mathscr{G}^t)$. Then*

(a) *One has for any $\alpha, \beta \in \mathbb{R}$:*

$$T_\alpha - T_\beta = -i \int_\alpha^\beta W(\tau)[A, T]W(-\tau)\, d\tau, \tag{3}$$

where the right-hand side is defined as a strong integral in $\mathscr{B}(\mathscr{G}^s, \mathscr{G}^t)$.

(b) *The following relation holds, as a strong limit in $\mathscr{B}(\mathscr{G}^s, \mathscr{G}^t)$:*

$$[A, T] = \lim_{\alpha \to 0} -i\alpha^{-1}[W(\alpha), T]. \tag{4}$$

(c) *In particular, if $s = +1$ and $t = -1$, T_α is strongly continuously differentiable with respect to α in $\mathscr{B}(\mathscr{G}^1, \mathscr{G}^{-1})$, and $dT_\alpha/d\alpha\,|_{\alpha=0} = i[A, T]$.*

Proof. We indicate only a formal proof. (a) is formally clear, it suffices to integrate $dT_\tau/d\tau$ over $[\alpha, \beta]$ and to observe that the derivative makes sense in $\mathscr{B}(\mathscr{E}, \mathscr{E}^*)$. (b) is then easy to derive by using the result of (a) with $\beta = 0$ and the fact that $W(\alpha) \to I$ as $\alpha \to 0$ strongly in $\mathscr{G}^s$:

$$[A, T] = \lim_{\alpha \to 0} [A, T]W(\alpha) = \lim_{\alpha \to 0} \frac{1}{\alpha} \int_0^\alpha d\tau W(\tau)[A, T]W(-\tau)W(\alpha)$$

$$= \lim_{\alpha \to 0} \frac{1}{i\alpha} \{W(\alpha)TW(-\alpha) - T\}W(\alpha)$$

$$= \lim_{\alpha \to 0} \frac{1}{i\alpha} [W(\alpha), T],$$

where all limits are in $\mathscr{B}(\mathscr{G}^s, \mathscr{G}^t)$. ■

Proposition 2 (Virial Theorem). *Let H be a self-adjoint operator in $\mathscr{H}$ with domain $\mathscr{G}^1$, and assume that $B \equiv i[H, A]$ belongs to $\mathscr{B}(\mathscr{G}^1, \mathscr{G}^{-1})$. Let $f \in \mathscr{G}^1$ be an eigenvector of H* (i.e. *$Hf = \lambda f$ for some $\lambda \in \mathbb{R}$*). *Then $\langle f, Bf\rangle = 0$.*

Proof. For $\alpha \in \mathbb{R}$, set $f_\alpha = W(\alpha)f$ and $H_\alpha = W(\alpha)HW(-\alpha)$. Then $H_\alpha f_\alpha = \lambda f_\alpha$, so that (for $\alpha \neq 0$):

$$\langle f, \alpha^{-1}(H_\alpha - H)f_\alpha\rangle = \langle f, \alpha^{-1}(\lambda - H)f_\alpha\rangle = \alpha^{-1}\overline{\langle f_\alpha, (\lambda - H)f\rangle} = 0.$$

Since H_α is a symmetric operator in $\mathscr{B}(\mathscr{G}^1, \mathscr{G}^{-1})$, this implies that

$$\langle f_\alpha, (\alpha)^{-1}(H_\alpha - H)f\rangle = 0 \qquad \forall \alpha \neq 0.$$

Now, as $\alpha \to 0$, $f_\alpha \to f$ strongly in $\mathscr{G}^1$ and $\alpha^{-1}[H_\alpha - H]f \to -Bf$ strongly in $\mathscr{G}^{-1}$ by Proposition 1(c). ■

Proposition 3. *Let H be as in Proposition 2 and assume that $B \equiv i[H, A]$ belongs to $\mathscr{B}(\mathscr{G}^1, \mathscr{G}^{-1})$. Let J be a bounded Borel set in $\mathbb{R}$ and assume that there are a number $a \in (0, \infty)$ and a compact operator K in $\mathscr{H}$ such that (as an operator inequality in $\mathscr{H}$):*

$$E(J)BE(J) \geqslant aE(J) + K. \tag{5}$$

Then H has at most a finite number of eigenvalues in J, and each of these eigenvalues is of finite multiplicity.

Proof. (i) If f is an eigenvector of H with associated eigenvalue in J and such that $\|f\| = 1$, then (since $E(J)f = f$) the hypothesis (5) implies together with Proposition 2 that $(f, Kf) \leqslant -a < 0$.

(ii) Now assume that there is an infinite orthonormal sequence $\{f_n\}$ of eigenvectors of H with associated eigenvalues $\{\lambda_n\}$ in J. Then $(f_n, Kf_n) \to 0$ as $n \to \infty$, since $\{f_n\}$ converges weakly to zero in $\mathscr{H}$ and K is compact. This contradicts the fact that $(f_n, Kf_n) \leqslant -a < 0$ for each n. ■

4. Absence of singularly continuous spectrum

The purpose of this section is to prove the following theorem:

Proposition 4. *Let H be a self-adjoint operator in $\mathscr{H}$ with domain $\mathscr{G}^1$. Set $B = i[H, A]$ and let J be a bounded interval. Assume that*

(α) $B \in \mathscr{B}(\mathscr{G}^1, \mathscr{G}^{-1})$ and that the Mourre estimate (5) is satisfied for some $a \in (0, \infty)$ and some compact operator K in $\mathscr{H}$,

(β) there is a family $\{H(\varepsilon)\}_{0 \leqslant \varepsilon \leqslant 1}$ of symmetric operators belonging to $\mathscr{B}(\mathscr{G}^1, \mathscr{H})$ such that

(β_1) the mapping $[0, 1] \ni \varepsilon \mapsto H(\varepsilon) \in \mathscr{B}(\mathscr{G}^1, \mathscr{H})$ is strongly continuous, and $H(0) \equiv H$,

(β_2) for each $\varepsilon \in (0, 1]$, the operator $B_\varepsilon \equiv i[H(\varepsilon), A]$ belongs to $\mathscr{B}(\mathscr{G}^1, \mathscr{G}^{-1/2})$, is strongly continuously differentiable as a $\mathscr{B}(\mathscr{G}^1, \mathscr{G}^{-1/2})$-valued function of ε on

(0, 1), and there are constants $C<\infty$ *and* $\delta>0$ *such that*

$$\left\| \frac{d}{d\varepsilon} B_\varepsilon \right\|_{1,-1/2} \leq C\varepsilon^{-1+\delta}, \tag{6}$$

(β_3) *for each* $\varepsilon \in (0, 1]$ *one has* $[B_\varepsilon, A] \in \mathscr{B}(\mathscr{G}^1, \mathscr{G}^{-1})$ *and, for some constants* $C<\infty$ *and* $\delta>0$:

$$\|[B_\varepsilon, A]\|_{1,-1} \leq C\varepsilon^{-1+\delta}. \tag{7}$$

Then H *has no singularly continuous spectrum, and the number of eigenvalues of* H *in* J *(multiplicities counted) is finite.*

Proof. The finiteness of the number of eigenvalues in J has been established in Proposition 3. To prove the absence of singularly continuous spectrum, it suffices to show that each $\lambda_0 \in J$ which is not an eigenvalue of H has a neighbourhood which does not contain any singularly continuous spectrum. To do this, we shall show in the remainder of this section that the hypotheses of the proposition imply that each λ_0 as above has a neighbourhood J_0 such that

$$\sup_{\lambda \in J_0, \mu \in (0,1)} \|(I+A_0^2)^{-1}(H-\lambda+i\mu)^{-1}(I+A_0^2)^{-1}\| < \infty. \tag{8}$$

In fact the validity of (8) is a special case of the much more general Proposition 5 (take $L_\varepsilon = (I+A_0^2)^{-1}$ and observe that (33) is satisfied with $\nu=0$, since the restriction of $A: \mathscr{G}^{-1} \to \mathscr{E}^*$ to $D(A_0)$, the domain of A_0 in $\mathscr{H}$, coincides with A_0). Now (8) implies that $(I+A_0^2)^{-1}E(J_0)$ is an H-smooth operator, so that the range of $E(J_0)(I+A_0^2)^{-1}$ is contained in the absolutely continuous subspace of H (see [8], Theorems XIII.30 and XIII.23). Since the range of $E(J_0)(I+A_0^2)^{-1}$ is dense in $E(J_0)\mathscr{H}$, it follows that the spectrum of H in J_0 is purely absolutely continuous. ■

In the lemma below we give some consequences of the hypotheses of Proposition 4. These hypotheses are assumed to be satisfied throughout this section.

Lemma 1. (a) *We have* $B \in \mathscr{B}(\mathscr{G}^1, \mathscr{G}^{-1/2})$ *and*

$$\|B - B_\varepsilon\|_{1,-1/2} \leq C\delta^{-1}\varepsilon^\delta. \tag{9}$$

(b) *If* $\lambda_0 \in J$ *is not eigenvalue of* H, *there exists an open interval* J_0, *a number* $\varepsilon_1 \in (0, 1)$ *and a function* $\varphi \in C_0^\infty(\mathbb{R})$ *such that* $\lambda_0 \in J_0$, $\varphi(\lambda)=1$ *for all* $\lambda \in J_0$, $0 \leq \varphi(\lambda) \leq 1$ *for all* $\lambda \in \mathbb{R}$ *and*

$$\varphi(H)B_\varepsilon\varphi(H) \geq (a/2)\varphi(H)^2 \qquad \forall \varepsilon \in (0, \varepsilon_1). \tag{10}$$

(c) *If* φ *is the function determined in* (b), *set*

$$\Phi = \varphi(H), \qquad \Phi^\perp = 1 - \Phi \tag{11}$$

and, for $\varepsilon \in (0, \varepsilon_1)$:

$$N_\varepsilon = (2\varepsilon/a)\Phi B_\varepsilon \Phi. \tag{12}$$

Then (for some constant $c < \infty$):

$$N_\varepsilon \geqslant \varepsilon \Phi^2, \tag{13}$$

$$\|N_\varepsilon\| \leqslant c\varepsilon, \tag{14}$$

and there is $\varepsilon_0 \in (0, \varepsilon_1)$ such that for all $\varepsilon \in (0, \varepsilon_0)$, all $\lambda \in J_0$, all $\mu \geqslant 0$ and all $f \in \mathscr{G}^1$:

$$\|[H - \lambda \pm i(N_\varepsilon + \mu)]f\| \geqslant ((\varepsilon^2/16) + \mu^2)^{1/2} \|f\|. \tag{15}$$

Proof. (a) As a consequence of (6), $\{B_\varepsilon\}$ is Cauchy in $\mathscr{B}(\mathscr{G}^1, \mathscr{G}^{-1/2})$ as $\varepsilon \to 0$. But the limit of this Cauchy sequence is just B, because $B_\varepsilon \to B$ in $\mathscr{B}(\mathscr{E}, \mathscr{E}^*)$ as $\varepsilon \to 0$ by (β_1).

(b) Let $\{J_n\}_{n=1}^\infty$ be a sequence of open subintervals of J such that $J_{n+1} \subset J_n$ for all n and $\bigcap_{n=1}^\infty J_n = \{\lambda_0\}$. Since λ_0 is not an eigenvalue of H, we have $s\text{-lim}\, E(J_n) = 0$ as $n \to \infty$, hence (since K is compact) $\lim \|E(J_n)KE(J_n)\| = 0$. We fix n such that $\|E(J_n)KE(J_n)\| < a/4$ and observe that $\|E(J_n)(B_\varepsilon - B)E(J_n)\| \to 0$ as $\varepsilon \to 0$ by (a). We choose ε_1 such that this last norm is less than $a/4$ for all $\varepsilon \in (0, \varepsilon_1)$ and then have for these values of ε(pre- and post-multiply (5) by $E(J_n)$ and observe that $E(J_n)E(J) = E(J_n)$):

$$\begin{aligned} E(J_n)B_\varepsilon E(J_n) &= E(J_n)(B_\varepsilon - B)E(J_n) + E(J_n)BE(J_n) \\ &\geqslant -a/4 + aE(J_n) + E(J_n)KE(J_n) \geqslant -a/2 + aE(J_n). \end{aligned} \tag{16}$$

Now it suffices to take for J_0 any subinterval of J_n containing λ_0 such that the closure of J_0 is contained in J_n, and then to choose φ with the required properties such that $\operatorname{supp} \varphi \subset J_n$. (10) then follows upon pre- and post-multiplying (16) by $\varphi(H)$, since $\varphi(H)E(J_n) = \varphi(H)$ if φ is as indicated above.

(c) The inequality (13) follows immediately from (10) and (12), and (14) holds with

$$c = (2/a)\, \|\varphi(H)B\varphi(H)\| + \tfrac{1}{2}\, \|\varphi(H)\|^2,$$

because $\|E(J_n)(B - B_\varepsilon)E(J_n)\| < a/4$ if $\varepsilon < \varepsilon_1$. To prove (15), we first notice that for $f \in D(H)$ and $\mu \geqslant 0$:

$$\begin{aligned} \|[H - \lambda \pm i(N_\varepsilon + \mu)]f\|^2 &= \|(H - \lambda \pm iN_\varepsilon)f\|^2 + \mu^2 \|f\|^2 + 2\mu(f, N_\varepsilon f) \\ &\geqslant \|(H - \lambda \pm iN_\varepsilon)f\|^2 + \mu^2 \|f\|^2, \end{aligned} \tag{17}$$

since $N_\varepsilon \geqslant 0$ by (13). On the other hand we have, by using again (13):

$$\begin{aligned} \varepsilon \|f\|^2 &\leqslant 2\varepsilon \|\Phi f\|^2 + 2\varepsilon \|\Phi^\perp f\|^2 \\ &\leqslant 2\varepsilon \|\Phi f\|^2 + 2 \operatorname{Re}(f, [\mp i(H - \lambda) + N_\varepsilon - \varepsilon\Phi^2]f) + 2\varepsilon \|\Phi^\perp f\|^2 \\ &\leqslant 2 \|f\| \, \|(H - \lambda \pm iN_\varepsilon)f\| + 2\varepsilon \|f\| \, \|\Phi^\perp f\|. \end{aligned} \tag{18}$$

Furthermore, since

$$\kappa \equiv \sup_{\lambda \in J_0} \|(H - \lambda)^{-1}\Phi^\perp\| < \infty,$$

we have by (14)

$$\|\Phi^\perp f\| = \|(H-\lambda)^{-1}\Phi^\perp(H-\lambda \pm iN_\varepsilon)f \mp i(H-\lambda)^{-1}\Phi^\perp N_\varepsilon f\|$$
$$\leq \kappa \, \|(H-\lambda \pm iN_\varepsilon)f\| + \varepsilon\kappa c \, \|f\|. \tag{19}$$

By choosing $\varepsilon_0 \in (0, \varepsilon_1)$ so small that $2\varepsilon_0\kappa < 1$ and $\varepsilon_0\kappa c < 1/8$, we find from (19) and (18) that

$$\varepsilon \, \|f\| \leq 3 \, \|(H-\lambda \pm iN_\varepsilon)f\| + \varepsilon/4 \, \|f\|.$$

This, together with (17), leads to (15). ■

In what follows, we always let J_0, ε_0, Φ and N_ε be the objects defined in Lemma 1. In the next lemma we study the commutators of Φ and N_ε with A.

Lemma 2. (a) *The operator* $[(H+i)^{-1}, A]$ *belongs to* $\mathscr{B}(\mathscr{H}, \mathscr{G}^{1/2}) \cap \mathscr{B}(\mathscr{G}^{-1/2}, \mathscr{H})$.

(b) *The operator* $[\Phi, A]$ *belongs to* $\mathscr{B}(\mathscr{G}^{-1/2}, \mathscr{G}^{1/2})$.

(c) *The operator* $[N_\varepsilon, A]$ *belongs to* $\mathscr{B}(\mathscr{G}^{-1/2}, \mathscr{G}^{1/2})$ *and is given by*

$$[N_\varepsilon, A] = 2\varepsilon/a\{[\Phi, A]B_\varepsilon\Phi + \Phi B_\varepsilon[\Phi, A] + \Phi[B_\varepsilon, A]\Phi\}. \tag{20}$$

Proof. (i) The relation

$$[e^{iHt}, A] = \int_0^t e^{iHs}Be^{iH(t-s)} \, ds$$

implies together with Lemma 1(a) that $[\exp(iHt), A] \in \mathscr{B}(\mathscr{G}^1, \mathscr{G}^{-1/2})$ and

$$\|[e^{iHt}, A]\|_{1,-1/2} \leq \|B\|_{1,-1/2} \, |t|.$$

Now let $\psi: \mathbb{R} \to \mathbb{C}$ be such that $(1+|t|)\tilde{\psi}(t) \in L^1(\mathbb{R}; dt)$, where $\tilde{\psi}$ denotes the Fourier transform of ψ. Then

$$[\psi(H), A] = (2\pi)^{-1/2} \int_{-\infty}^{\infty} [e^{iHt}, A]\tilde{\psi}(t) \, dt,$$

so that $[\psi(H), A] \in \mathscr{B}(\mathscr{G}^1, \mathscr{G}^{-1/2})$.

(ii) The following relation is formally evident:

$$[(H+i)^{-1}, A] = i(H+i)^{-i}B(H+i)^{-1}. \tag{21}$$

It can be rigorously justified by the methods of (i) by noticing that the Fourier transform of $\psi(\lambda) = (\lambda + i)^{-1}$ is $\tilde{\psi}(t) = -i(2\pi)^{1/2}\exp(-t)\chi_{[0,\infty]}(t)$, where χ_Ω is the characteristic function of Ω. The operator on the r.h.s. of (21) is bounded from $\mathscr{H}$ to $\mathscr{G}^{1/2}$ and from $\mathscr{G}^{-1/2}$ to $\mathscr{H}$, since $(H+i)^{-1}$ is bounded from $\mathscr{G}^s$ to $\mathscr{G}^{s+1}$ and $B \in \mathscr{B}(\mathscr{G}^1, \mathscr{G}^{-1/2}) \cap \mathscr{B}(\mathscr{G}^{1/2}, \mathscr{G}^{-1})$ by Lemma 1(a) and (2). This proves (a).

(iii) Next, let $\theta \in C_0^\infty(\mathbb{R})$ and define ψ by $\psi(\lambda) = (\lambda + i)^2\theta(\lambda)$. Then

$$[\theta(H), A] = [(H+i)^{-1}\psi(H)(H+i)^{-1}, A]$$
$$= [(H+i)^{-1}, A]\psi(H)(H+i)^{-1} + (H+i)^{-1}[\psi(H), A](H+i)^{-1}$$
$$+ (H+i)^{-1}\psi(H)[(H+i)^{-1}, A]. \tag{22}$$

By (a) and the result of (i), each term on the r.h.s. is bounded from $\mathscr{H}$ to $\mathscr{G}^{1/2}$. Hence $[\theta(H), A] \in \mathscr{B}(\mathscr{H}, \mathscr{G}^{1/2})$ for each $\theta \in C_0^\infty(\mathbb{R})$. By using this fact (for $\psi(H)$) together with (a) in (22), one sees that $[\theta(H), A] \in \mathscr{B}(\mathscr{G}^{-1/2}, \mathscr{G}^{1/2})$. In particular we have (b).

(iv) The identity (20) is formally evident, since $N_\varepsilon = 2\varepsilon a^{-1}\Phi B_\varepsilon \Phi$. Each term on the r.h.s. of (20) is in $\mathscr{B}(\mathscr{G}^{-1/2}, \mathscr{G}^{1/2})$, since $\Phi \in \mathscr{B}(\mathscr{G}^{-1}, \mathscr{G}^1)$, $B_\varepsilon \in \mathscr{B}(\mathscr{G}^1, \mathscr{G}^{-1/2}) \cap \mathscr{B}(\mathscr{G}^{1/2}, \mathscr{G}^{-1})$, $[B_\varepsilon, A] \in \mathscr{B}(\mathscr{G}^1, \mathscr{G}^{-1})$ by hypothesis and $[\Phi, A] \in \mathscr{B}(\mathscr{G}^{-1/2}, \mathscr{G}^{1/2})$. (A rigorous proof of (20) can be obtained by writing $[N_\varepsilon, A] = \lim_{\alpha\to 0}(-i\alpha)^{-1}[W(\alpha), N_\varepsilon]$ in $\mathscr{B}(\mathscr{E}, \mathscr{E}^*)$, expanding $[W(\alpha), \Phi B_\varepsilon \Phi]$ into a sum of three terms and using Proposition 1(b), cf. [4]). ■

The inequality (15) shows that, if $\varepsilon \in (0, \varepsilon_0)$ and $\mu \geq 0$, then $H - \lambda \pm i(N_\varepsilon + \mu)$ are isomorphisms of $\mathscr{G}^1$ onto $\mathscr{H}$. Hence we can define, for these values of ε and μ:

$$G_\varepsilon \equiv G_\varepsilon(\lambda, \mu) = (H - \lambda + i(N_\varepsilon + \mu))^{-1}. \tag{23}$$

It is clear that G_ε and G_ε^* belong to $\mathscr{B}(\mathscr{H}, \mathscr{G}^1)$; their boundedness between $\mathscr{H}$ and $\mathscr{G}^1$ follows from the closed graph theorem or from the identity

$$HG_\varepsilon = I + (\lambda - iN_\varepsilon - i\mu)G_\varepsilon. \tag{24}$$

The Mourre method consists in proving the boundedness for small ε of a family $\{F_\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ of operators of the form

$$F_\varepsilon \equiv F_\varepsilon(\lambda, \mu) = L_\varepsilon G_\varepsilon L_\varepsilon^*, \tag{25}$$

where $\{L_\varepsilon\}_{0<\varepsilon<\varepsilon_0}$ is a suitable auxiliary family of operators in $\mathscr{B}(\mathscr{H})$, $\lambda \in J_0$ and $\mu \geq 0$. The next lemma contains some estimates on the operators introduced above.

Lemma 3. *There is a finite constant c such that for all* $\varepsilon \in (0, \varepsilon_0)$:

$$\|G_\varepsilon\|_{0,1} + \|G_\varepsilon^*\|_{0,1} \leq c/\varepsilon, \tag{26}$$

$$\|\Phi^\perp G_\varepsilon\|_{0,1} + \|\Phi^\perp G_\varepsilon^*\|_{0,1} \leq c, \tag{27}$$

$$\|\Phi G_\varepsilon L_\varepsilon^*\|_{0,1} + \|\Phi G_\varepsilon^* L_\varepsilon^*\|_{0,1} \leq c\varepsilon^{-1/2}\,\|F_\varepsilon\|^{1/2}, \tag{28}$$

$$\|G_\varepsilon L_\varepsilon^*\|_{0,1} + \|G_\varepsilon^* L_\varepsilon^*\|_{0,1} \leq c\varepsilon^{-1/2}\,\|F_\varepsilon\|^{1/2} + c\,\|L_\varepsilon\|. \tag{29}$$

Proof. We show how to estimate the first term in these inequalities. The proof for the second term is similar in each case.

(15) implies that $\|G_\varepsilon\| \leq 4\varepsilon^{-1}$, whereas (24) and (14) show that $\|HG_\varepsilon\| \leq 1 + (|\lambda| + |\mu| + c\varepsilon)8(\varepsilon + \mu)^{-1}$. These inequalities imply (26).

The second resolvent equation implies that

$$\Phi^\perp G_\varepsilon = \Phi^\perp G_0[I - iN_\varepsilon G_\varepsilon].$$

Since $\|\Phi^\perp G_0(\lambda, \mu)\| < \infty$ and $\|H\Phi^\perp G_0(\lambda, \mu)\| < \infty$ if $\lambda \in J_0$ and since $\|N_\varepsilon G_\varepsilon\| \leq 4c$ as seen above, (27) follows.

Next we use (13) and the first resolvent equation to write (recall that $\mu \geqslant 0$):

$$L_\varepsilon G_\varepsilon^* \Phi^2 G_\varepsilon L_\varepsilon^* \leqslant \frac{1}{\varepsilon} L_\varepsilon G_\varepsilon^* (N_\varepsilon + \mu) G_\varepsilon L_\varepsilon^*$$

$$= (1/2i\varepsilon) L_\varepsilon (G_\varepsilon^* - G_\varepsilon) L_\varepsilon^* = (1/2i\varepsilon)(F_\varepsilon^* - F_\varepsilon).$$

Hence $\|\Phi G_\varepsilon L_\varepsilon^*\|^2 = \|L_\varepsilon G_\varepsilon^* \Phi^2 G_\varepsilon L_\varepsilon^*\| \leqslant \varepsilon^{-1} \|F_\varepsilon\|$. Since φ has compact support, we have $\|H\Phi G_\varepsilon L_\varepsilon^*\| \leqslant c_0 \|\Phi G_\varepsilon L_\varepsilon^*\|$, which implies (28).

Finally (29) follows from (28) and (27) by observing that $G_\varepsilon L_\varepsilon^* = \Phi G_\varepsilon L_\varepsilon^* + \Phi^\perp G_\varepsilon L_\varepsilon^*$. ■

We now derive Mourre's differential inequality in the present context.

Lemma 4. *Assume that the family $\{L_\varepsilon\}_{0<\varepsilon<\varepsilon_0} \in \mathscr{B}(\mathscr{H})$ is weakly C^1 as a function of ε when considered as a $\mathscr{B}(\mathscr{G}^1, \mathscr{H})$-valued function and such that $L_\varepsilon A \in \mathscr{B}(\mathscr{G}^1, \mathscr{H})$. Then there is a finite constant c such that for all $\varepsilon \in (0, \varepsilon_0)$, all $\lambda \in J_0$ and all $\mu \geqslant 0$:*

$$\left\|\frac{d}{d\varepsilon} F_\varepsilon\right\| \leqslant c\varepsilon^{-\beta} \|F_\varepsilon\| + c(\varepsilon^{-1/2} \|F_\varepsilon\|^{1/2} + \|L_\varepsilon\|)\left(\|L_\varepsilon\| + \|L_\varepsilon A\|_{1,0} + \left\|\frac{d}{d\varepsilon} L_\varepsilon\right\|_{1,0}\right),$$

with $\beta = 1 - \delta < 1$.

Proof. We denote derivatives with respect to ε by a prime. All derivatives below are in the weak sense, and the use of the rule for differentiating a product is justified since a weakly C^1 function is norm continuous. Now

$$\begin{aligned} F'_\varepsilon &= L'_\varepsilon G_\varepsilon L_\varepsilon^* + L_\varepsilon G_\varepsilon L_\varepsilon'^* - iL_\varepsilon G_\varepsilon N'_\varepsilon G_\varepsilon L_\varepsilon^* \\ &= L'_\varepsilon G_\varepsilon L_\varepsilon^* + L_\varepsilon G_\varepsilon L_\varepsilon'^* - (2i\varepsilon/a) L_\varepsilon G_\varepsilon \Phi B'_\varepsilon \Phi G_\varepsilon L_\varepsilon^* \\ &\quad - (2i/a) L_\varepsilon G_\varepsilon \Phi B_\varepsilon \Phi G_\varepsilon L_\varepsilon^*. \end{aligned} \tag{30}$$

The norm of the first term on the r.h.s. (and similarly that of the second one) is bounded by $\|L'_\varepsilon\|_{1,0} \|G_\varepsilon L_\varepsilon^*\|_{0,1} \leqslant \|L'_\varepsilon\|_{1,0}[c\varepsilon^{-1/2} \|F_\varepsilon\|^{1/2} + c \|L_\varepsilon\|]$ by (29), whereas that of the third term is bounded by

$$2\varepsilon/a \|L_\varepsilon G_\varepsilon \Phi\|_{-1,0} \|B'_\varepsilon\|_{1,-1} \|\Phi G_\varepsilon L_\varepsilon^*\|_{0,1} \leqslant \frac{2}{a} \varepsilon \frac{c^2}{\varepsilon} \|F_\varepsilon\| C\varepsilon^{-1+\delta} = c_1 \varepsilon^{-\beta} \|F_\varepsilon\|,$$

by (28) and (6).

To estimate the last term in (30) we use the identity

$$\begin{aligned} L_\varepsilon G_\varepsilon \Phi B_\varepsilon \Phi G_\varepsilon L_\varepsilon^* &= L_\varepsilon G_\varepsilon B_\varepsilon G_\varepsilon L_\varepsilon^* - L_\varepsilon G_\varepsilon \Phi^\perp B_\varepsilon \Phi G_\varepsilon L_\varepsilon^* \\ &\quad - L_\varepsilon G_\varepsilon \Phi B_\varepsilon \Phi^\perp G_\varepsilon L_\varepsilon^* - L_\varepsilon G_\varepsilon \Phi^\perp B_\varepsilon \Phi^\perp G_\varepsilon L_\varepsilon^*. \end{aligned} \tag{31}$$

The norm of the second term on the r.h.s. (and similarly that of the third term) is bounded by

$$\|L_\varepsilon\| \|G_\varepsilon \Phi^\perp\|_{-1,0} \|B_\varepsilon\|_{1,-1} \|\Phi G_\varepsilon L_\varepsilon^*\|_{0,1} \leqslant c_2 \|L_\varepsilon\| \varepsilon^{-1/2} \|F_\varepsilon\|^{1/2},$$

where we have used (27), (28) and the fact that $\|B_\varepsilon\|_{1,-1} \le c_3$ by Lemma 1(a). Similarly the norm of the last term in (31) is seen to be bounded by $c_4 \|L_\varepsilon\|^2$.

It remains to estimate the first term on the r.h.s. of (31). By using the identity

$$B_\varepsilon = (B_\varepsilon - B) + i[H - \lambda + i(N_\varepsilon + \mu), A] + [N_\varepsilon, A]$$

and Lemma 2(c), this term may be rewritten as

$$\begin{aligned} L_\varepsilon G_\varepsilon B_\varepsilon G_\varepsilon L_\varepsilon^* = {} & L_\varepsilon G_\varepsilon (B_\varepsilon - B) G_\varepsilon L_\varepsilon^* + iL_\varepsilon A G_\varepsilon L_\varepsilon^* - iL_\varepsilon G_\varepsilon A L_\varepsilon^* \\ & + (2\varepsilon/a) L_\varepsilon G_\varepsilon [\Phi, A] B_\varepsilon \Phi G_\varepsilon L_\varepsilon^* + (2\varepsilon/a) L_\varepsilon G_\varepsilon \Phi B_\varepsilon [\Phi, A] G_\varepsilon L_\varepsilon^* \\ & + (2\varepsilon/a) L_\varepsilon G_\varepsilon \Phi [B_\varepsilon, A] \Phi G_\varepsilon L_\varepsilon^*. \end{aligned} \tag{32}$$

The norm of the second plus third term is majorized by

$$2\|L_\varepsilon A\|_{1,0} [\|G_\varepsilon L_\varepsilon^*\|_{0,1} + \|G_\varepsilon^* L_\varepsilon^*\|_{0,1}] \le c_5 \|L_\varepsilon A\|_{1,0} (\varepsilon^{-1/2} \|F_\varepsilon\|^{1/2} + \|L_\varepsilon\|),$$

while that of the first term may be estimated as follows by using (9) and (29):

$$\begin{aligned} \|G_\varepsilon L_\varepsilon^*\|_{0,1} \|G_\varepsilon^* L_\varepsilon^*\|_{0,1} \|B_\varepsilon - B\|_{1,-1} & \le c_6 (\varepsilon^{-1} \|F_\varepsilon\| + \|L_\varepsilon\|^2) \varepsilon^\delta \\ & \le c_6 (\varepsilon^{-\beta} \|F_\varepsilon\| + \|L_\varepsilon\|^2). \end{aligned}$$

The norm of the fourth (and similarly of the fifth) term is bounded by

$$\begin{aligned} & 2\varepsilon/a \|L_\varepsilon\| \|G_\varepsilon\| \|[\Phi, A]\|_{-1/2,0} \|B_\varepsilon\|_{1,-1/2} \|\Phi G_\varepsilon L_\varepsilon^*\|_{0,1} \\ & \quad \le c_7 \|L_\varepsilon\| \varepsilon^{-1/2} \|F_\varepsilon\|^{1/2} \end{aligned}$$

(use (26), Lemmas 2(b) and 1(a) as well as (28)). Finally the norm of the last term in (32) can be estimated by the following expression by using (28) and (7):

$$\begin{aligned} & 2\varepsilon/a \|\Phi G_\varepsilon L_\varepsilon^*\|_{0,1} \|\Phi G_\varepsilon^* L_\varepsilon^*\|_{0,1} \|[B_\varepsilon, A]\|_{1,-1} \\ & \quad \le c_8 \varepsilon (\varepsilon^{-1} \|F_\varepsilon\|) \varepsilon^{-1+\delta} \equiv c_8 \varepsilon^{-\beta} \|F_\varepsilon\|. \quad \blacksquare \end{aligned}$$

Proposition 5. *Assume that the hypotheses of Proposition* 4 *and of Lemma* 4 *are satisfied. Assume furthermore that there are constants* $c \in (0, \infty)$ *and* $\nu < \frac{1}{2}$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$:

$$\|L_\varepsilon\| + \|L_\varepsilon A\|_{1,0} + \|L_\varepsilon'\|_{1,0} \le c\varepsilon^{-\nu}. \tag{33}$$

Then one has

$$\sup \{\|L_\varepsilon G_\varepsilon(\lambda, \mu) L_\varepsilon^*\| \mid 0 < \varepsilon < \varepsilon_0, \lambda \in J_0, \mu \ge 0\} < \infty. \tag{34}$$

Proof. We set $\gamma = \max(\beta, \nu + \frac{1}{2})$. From Lemma 4 and the inequality $x^{1/2} \le 1 + x$ $(x \ge 0)$ we obtain that

$$\|F_\varepsilon'\| \le c_1 \varepsilon^{-\beta} \|F_\varepsilon\| + c_1 \varepsilon^{-1/2-\nu} \|F_\varepsilon\|^{1/2} + c_1 \varepsilon^{-2\nu} \le 2c_1 \varepsilon^{-\gamma} (\|F_\varepsilon\| + 1). \tag{35}$$

This implies the boundedness of $\|F_\varepsilon\|$ by repeated integration. (By introducing the inequality $\|F_\varepsilon\| \le c\varepsilon^{-b}$ into (35), one finds that $\|F_\varepsilon\| \le c'(1 + \varepsilon^{-b+1-\gamma})$; now the first of these inequalities holds for $b = 2\nu + 1$, because $\|F_\varepsilon\| \le \|L_\varepsilon\|^2 \|G_\varepsilon\| \le c\varepsilon^{-2\nu-1}$; since $1 - \gamma > 0$, one obtains (34) after a finite number of steps). $\blacksquare$

5. Local smoothness

We continue to use the notations of Section 4 and denote by $\sigma_p(H)$ the set of all eigenvalues of H. By using the results of Section 4, one can now prove the local H-smoothness of various operators and the existence of the boundary values of $(H-z)^{-1}$ in $\mathscr{B}(\mathscr{H}_L, \mathscr{H}_L^*)$ when z approaches the real axis, where $\mathscr{H}_L$ is a suitable auxiliary Hilbert space. We refer to [4] for a discussion of boundary values and present here some results on local smoothness.

Definition. Let $\nu \in [0, \frac{1}{2})$. An operator $L \in \mathscr{B}(\mathscr{G}^{1/2}, \mathscr{H})$ is said to belong to the class $\mathscr{L}_\nu$ if there is a family $\{L_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$ of operators in $\mathscr{B}(\mathscr{H})$ such that

(α) L_ε converges to L weakly in $\mathscr{B}(\mathscr{G}^{1/2}, \mathscr{H})$ as $\varepsilon \to 0$,

(β) The mapping $\varepsilon \mapsto L_\varepsilon$ is weakly C^1 on $(0, \varepsilon_0)$ when considered with values in $\mathscr{B}(\mathscr{G}^1, \mathscr{H})$,

(γ) $L_\varepsilon A \in \mathscr{B}(\mathscr{G}^1, \mathscr{H})$ for each $\varepsilon \in (0, \varepsilon_0]$,

(δ) there is a constant c such that the inequality (33) is satisfied for all $\varepsilon \in (0, \varepsilon_0]$.

Proposition 6. *Assume that all hypotheses of Proposition 4 are satisfied. Let $\nu \in [0, \frac{1}{2})$ and let L be an operator belonging to the class $\mathscr{L}_\nu$. Then for each closed subinterval J_1 of J such that $J_1 \cap \sigma_p(H) = \varnothing$, one has*

$$\sup_{\lambda\in J_1, \mu>0} \|L(H-\lambda+i\mu)^{-1}L^*\| < \infty. \tag{36}$$

In particular L is H-smooth on J_1.

Proof. Since J_1 is compact, it suffices to show that each $\lambda_0 \in J_1$ has a neighbourhood J_0 in which (36) holds (with $\lambda \in J_0$). Thus we may assume that we are in the situation of Proposition 5, in particular that (34) is true.

(i) We first observe that (33) implies that

$$\|L - L_\varepsilon\|_{1,0} \leq c(1-\nu)^{-1}\varepsilon^{1-\nu}. \tag{37}$$

Next assume that $\lambda \in J_0$, $\mu > 0$. Then, by the second resolvent equation

$$\begin{aligned}\|(H-\lambda+i\mu)^{-1} - G_\varepsilon(\lambda, \mu)\|_{0,1} &\leq \|(H-\lambda+i\mu)^{-1}\|_{0,1}\, \|N_\varepsilon\|\, \|G_\varepsilon(\lambda, \mu)\| \\ &\leq c(\mu)\varepsilon,\end{aligned}$$

where the last inequality follows from (14), the fact that $\|G_\varepsilon(\lambda, \mu)\| \leq |\mu|^{-1}$ (cf. (15)) and $\|(H-\lambda+i\mu)^{-1}\|_{0,1} \leq c_1(1+|\mu|^{-1})$ for all $\lambda \in J_0$ (use (24)). Similarly one finds that $\|(H-\lambda+i\mu)^{-1*} - G_\varepsilon(\lambda, \mu)^*\|_{0,1} \leq c(\mu)\varepsilon$. Hence $G_\varepsilon(\lambda, \mu)$ converges to $(H-\lambda+i\mu)^{-1}$ in the norm of $\mathscr{B}(\mathscr{H}, \mathscr{G}^1)$ and in that of $\mathscr{B}(\mathscr{G}^{-1}, \mathscr{H})$. By interpolation one then obtains that, for $\lambda \in J_0$ and $\mu > 0$:

$$\lim_{\varepsilon\to 0} \|G_\varepsilon(\lambda, \mu) - (H-\lambda+i\mu)^{-1}\|_{-1/2, 1/2} = 0. \tag{38}$$

(ii) Now write

$$L_\varepsilon G_\varepsilon L_\varepsilon^* = LG_\varepsilon L^* + (L_\varepsilon - L)G_\varepsilon L^* + L_\varepsilon G_\varepsilon(L_\varepsilon^* - L^*). \tag{39}$$

Since $L \in \mathscr{B}(\mathscr{G}^{1/2}, \mathscr{H})$, hence $L^* \in \mathscr{B}(\mathscr{H}, \mathscr{G}^{-1/2})$, the first term on the r.h.s. converges in the norm of $\mathscr{B}(\mathscr{H})$ to $L(H-\lambda+i\mu)^{-1}L^*$, by virtue of (38). The second term converges weakly to zero in $\mathscr{B}(\mathscr{H})$ because $L_\varepsilon - L \to 0$ weakly in $\mathscr{B}(\mathscr{G}^{1/2}, \mathscr{H})$ by assumption (α). The last term in (39) converges to zero in the norm of $\mathscr{B}(\mathscr{H})$; this follows from (37), (29), the boundedness of $\|F_\varepsilon\|$, (33) and the assumption that $v < \frac{1}{2}$:

$$\|L_\varepsilon G_\varepsilon(L_\varepsilon^* - L^*)\| \leqslant \|L_\varepsilon - L\|_{1,0}\,\|G_\varepsilon^* L_\varepsilon^*\|_{0,1} \leqslant c_1 \varepsilon^{1-v}(\varepsilon^{-1/2}\,\|F_\varepsilon\|^{1/2} + \|L_\varepsilon\|)$$
$$\leqslant c_2 \varepsilon^{1/2-v}.$$

Thus $L_\varepsilon G_\varepsilon L_\varepsilon^* \to L(H-\lambda+i\mu)^{-1}L^*$ weakly in $\mathscr{B}(\mathscr{H})$. This implies that $\|L(H-\lambda+i\mu)^{-1}L^*\| \leqslant \liminf \|L_\varepsilon G_\varepsilon L_\varepsilon^*\|$ (see [9], p. 151), so that (36) follows from (34). ■

It remains to indicate more explicitly a class of operators L belonging to $\mathscr{L}_v$. This is the purpose of Proposition 7, which will be preceded by some preliminary considerations.

Lemma 5. *Let X be a positive self-adjoint operator in $\mathscr{H}$ such that $\{\exp(-tX)\}_{t\geqslant 0}$ leaves $\mathscr{G}^1$ invariant. Assume that there are finite constants M and $m > 0$ such that $\|\exp(-tX)\|_{1,1} \leqslant M \exp(-mt)$ for all $t \geqslant 0$. Then $X^{-\omega} \in \mathscr{B}(\mathscr{G}^s)$ for each $\omega > 0$ and each $s \in [-1, 1]$, and one has*

$$\lim_{\varepsilon\to 0}(I+\varepsilon X)^{-\omega} = I \qquad \text{strongly in } \mathscr{B}(\mathscr{G}^s), \tag{40}$$

$$\|(I+\varepsilon X)^{-\omega}\|_{s,s} \leqslant M \qquad \forall \varepsilon > 0, \tag{41}$$

$$\left\|\left(\frac{\varepsilon X}{I+\varepsilon X}\right)^{\omega}\right\|_{s,s} \leqslant c(\omega) \qquad \forall \varepsilon > 0. \tag{42}$$

Proof. (i) Since $\exp(-tX)$ is a bounded self-adjoint operator in $\mathscr{H}$, one also has $\|\exp(-tX)\|_{-1,-1} \leqslant M\exp(-mt)$, hence by interpolation

$$\|\exp(-tX)\|_{s,s} \leqslant Me^{-mt} \qquad \forall s \in [-1, +1]. \tag{43}$$

As explained in Section 2 for the group $\{W(\alpha)\}_{\alpha\in\mathbb{R}}$, one finds that $\{\exp(-tX)\}_{t\geqslant 0}$ is a strongly continuous semi-group in $\mathscr{B}(\mathscr{G}^s)$. Also observe that (43), for $s = 0$, implies that $X \geqslant m > 0$ as an operator in $\mathscr{H}$.

We shall use the following integral representations for powers of a strictly positive self-adjoint operator S in $\mathscr{H}$:

$$S^{-\omega} = \frac{1}{\Gamma(\omega)}\int_0^\infty x^{\omega-1}e^{-xS}\,dx \qquad (\omega > 0), \tag{44}$$

$$S^{\rho} = \frac{\sin\pi\rho}{\pi}\int_0^\infty S(S+x)^{-1}x^{\rho-1}\,dx \qquad (0<\rho<1). \tag{45}$$

(44) holds on $\mathscr{H}$ and (45) on the domain of S^ρ. These relations are easily proved by working in a spectral representation of S and observing that, by simple

changes of variables, (44) and (45) are true when S is replaced by a positive number.

(ii) We first use (44) with $S = a + bX$, where $a \geq 0$ and $b > 0$. The integral on the r.h.s. will then exist in $\mathscr{B}(\mathscr{G}^s)$ for any $s \in [-1, +1]$, by virtue of (43). Hence $(a + bX)^{-\omega} \in \mathscr{B}(\mathscr{G}^s)$ for each $s \in [-1, +1]$ and each $\omega > 0$, and

$$\|(a + bX)^{-\omega}\|_{s,s} \leq \frac{M}{\Gamma(\omega)} \int_0^\infty x^{\omega-1} e^{-(a+bm)x}\, dx = M(a + bm)^{-\omega}. \tag{46}$$

In particular we have (41). Similarly one has

$$\|(I + \varepsilon X)^{-\omega} - I\|_{s,s} \leq \frac{1}{\Gamma(\omega)} \left\| \int_0^\infty x^{\omega-1} e^{-x} [e^{-\varepsilon x X} - I]\, dx \right\|_{s,s}.$$

This converges to zero as $\varepsilon \to 0$ by the Lebesgue dominated convergence theorem, which gives (40).

(iii) Next, by using (45) with $S = \varepsilon X(I + \varepsilon X)^{-1}$, $\varepsilon > 0$, one finds that

$$\left[\frac{\varepsilon X}{I + \varepsilon X}\right]^\rho = \frac{\sin \pi\rho}{\pi} \int_0^\infty \frac{dx}{x^{1-\rho}(1+x)} \left[I - \frac{\beta(x)}{\beta(x) + \varepsilon X}\right],$$

where $\beta(x) = x(1+x)^{-1}$. This implies (42) for $0 < \omega < 1$ (with $c(\omega) = 1 + M$) after observing that, by virtue of (46):

$$\|\beta(x)(\beta(x) + \varepsilon X)^{-1}\|_{s,s} \leq M\beta(x)(\beta(x) + \varepsilon m)^{-1} \leq M.$$

To see that (42) holds for general $\omega > 0$, it suffices to write $\omega = n + \delta$ with n integer and $0 \leq \delta < 1$ and to observe that, by virtue of (41):

$$\|(\varepsilon X)^n (I + \varepsilon X)^{-n}\|_{s,s} = \|[I - (I + \varepsilon X)^{-1}]^n\|_{s,s} < \infty. \quad \blacksquare$$

Proposition 7. *Let X be a positive self-adjoint operator in $\mathscr{H}$ satisfying the conditions of Lemma 5 and such that $X^{-1}A \in \mathscr{B}(\mathscr{G}^1, \mathscr{G}^{1/2})$. Let $\rho \in [0, \frac{1}{2})$ and let R be an operator in $\mathscr{B}(\mathscr{G}^\rho, \mathscr{H})$. Then, if $\alpha > \frac{1}{2}$, the operator $L \equiv RX^{-\alpha}$ belongs to $\mathscr{L}_\nu$, where $\nu = \max(\rho, 1 - \alpha)$. In particular, if H satisfies the hypotheses of Proposition 4, then $RX^{-\alpha}$ is H-smooth on each compact subset of $J \backslash \sigma_p(H)$.*

Proof. By virtue of Proposition 6, it suffices to show that $L \in \mathscr{L}_\nu$. Since $X^{-\alpha} \in \mathscr{B}(\mathscr{G}^\rho)$ by Lemma 5, we clearly have $L \in \mathscr{B}(\mathscr{G}^\rho, \mathscr{H}) \subset \mathscr{B}(\mathscr{G}^{1/2}, \mathscr{H})$.

We set $G = (I + |H|)$ and observe that, for $\varepsilon > 0$:

$$\|G^a(I + \varepsilon G)^{-b}\|_{s,s'} \equiv \|G^{a+s'-s}(I + \varepsilon G)^{-b}\| \leq \varepsilon^{s-s'-a} \quad \text{if} \quad 0 \leq a + s' - s \leq b \tag{47}$$

and

$$\|G^a(I + \varepsilon G)^{-b}\|_{s,s'} \leq 1 \quad \text{if} \quad a + s' - s \leq 0 \quad \text{and} \quad b \geq 0. \tag{48}$$

We also set $\tau = \max(1-\alpha, 0)$ and define, for $\varepsilon > 0$:

$$R_\varepsilon = R(I+\varepsilon G)^{-\nu}, \tag{49}$$

$$T_\varepsilon = X^{-\alpha}(I+\varepsilon X)^{-\tau}, \tag{50}$$

$$L_\varepsilon = R_\varepsilon T_\varepsilon. \tag{51}$$

Clearly all these operators belong to $\mathscr{B}(\mathscr{H})$, and we must show that the conditions (α)–(δ) in the definition of $\mathscr{L}_\nu$ are satisfied.

(i) From Lemma 5 one obtains that $T_\varepsilon \to X^{-\alpha}$ strongly in $\mathscr{B}(\mathscr{G}^{1/2})$ as $\varepsilon \to 0$. Since $\rho < \frac{1}{2}$ and $(I+\varepsilon G)^{-\nu} \to I$ strongly in $\mathscr{B}(\mathscr{G}^{1/2})$, one sees that $L_\varepsilon \to L$ strongly in $\mathscr{B}(\mathscr{G}^{1/2}, \mathscr{H})$, which proves (α).

The validity of (γ) is easily established by writing $L_\varepsilon A = R_\varepsilon(T_\varepsilon X)(X^{-1}A)$ and observing that the first two factors on the r.h.s. are in $\mathscr{B}(\mathscr{H})$ and that $X^{-1}A \in \mathscr{B}(\mathscr{G}^1, \mathscr{G}^{1/2}) \subset \mathscr{B}(\mathscr{G}^1, \mathscr{H})$.

To prove (β), it suffices to observe that the function $\varepsilon \mapsto L_\varepsilon$ is strongly C^1 in $\mathscr{B}(\mathscr{H})$ on $(0, \varepsilon_0)$; in fact one has

$$\frac{d}{d\varepsilon} L_\varepsilon = R'_\varepsilon T_\varepsilon + R_\varepsilon T'_\varepsilon = -\nu RG(I+\varepsilon G)^{-\nu-1} T_\varepsilon$$
$$- \tau\varepsilon^{-\tau} R(I+\varepsilon G)^{-\nu}(\varepsilon X)^\tau (I+\varepsilon X)^{-\tau-1}. \tag{52}$$

(ii) It remains to prove that L_ε satisfies (33). We estimate separately each term on the l.h.s. of (33). First we have

$$\|L_\varepsilon\| \leqslant \|R\|_{\nu,0} \, \|(I+\varepsilon G)^{-\nu}\|_{0,\nu} \, \|(I+\varepsilon_0 X)^\alpha X^{-\alpha}\| \leqslant c\varepsilon^{-\nu}$$

by (47). Next we write

$$\|L_\varepsilon A\|_{1,0} \leqslant \|R\|_{1/2,0} \, \|(I+\varepsilon G)^{-\nu}\|_{1/2,1/2} \, \|\varepsilon^{-\tau}(\varepsilon X)^\tau (I+\varepsilon X)^{-\tau}\|_{1/2,1/2}$$
$$\cdot \|X^{-[\tau-(1-\alpha)]}\|_{1/2,1/2} \, \|X^{-1}A\|_{1,1/2}.$$

By using (47) and (42) and by observing that $\tau - (1-\alpha) \geqslant 0$, one sees that $\|L_\varepsilon A\|_{1,0} \leqslant c\varepsilon^{-\tau} \leqslant c\varepsilon^{-\nu}$, since $\tau \leqslant \nu$ and $\varepsilon < 1$.

Finally (52) implies that

$$\|L'_\varepsilon\|_{1,0} \leqslant \|R\|_{\nu,0} \, \|G(I+\varepsilon G)^{-\nu-1}\|_{1,\nu} \, \|T_\varepsilon\|_{1,1}$$
$$+ \tau\varepsilon^{-\tau} \|R\|_{\nu,0} \, \|(I+\varepsilon G)^{-\nu}\|_{1,\nu} \, \|(\varepsilon X)^\tau (I+\varepsilon X)^{-\tau}\|_{1,1} \, \|(I+\varepsilon X)^{-1}\|_{1,1}.$$

The first term is bounded by $c_1\varepsilon^{-\nu}$ (use (47) and Lemma 5) and the second one by $c_2\varepsilon^{-\tau} \leqslant c_2\varepsilon^{-\nu}$ (use (48) and Lemma 5). ■

6. N-body Schrödinger operators

In this last section we indicate how the preceding abstract results can be applied to N-body Schrödinger operators. We restrict ourselves to the usual case of N-body Schrödinger operators in the center-of-mass frame with *local* pair potentials. A proof of the Mourre estimate (5) for such Hamiltonians can be found in [10] (see also [2] for a somewhat smaller class of potentials). It is possible to extend all these results to N-body Schrödinger operators with non-local potentials and with many-body interactions; for these extensions and further details on the special case treated below we refer to [4].

If $\mathscr{H} = L^2(\mathbb{R}^n)$, we let $Q = (Q_1, \ldots, Q_n)$ and $P = (P_1, \ldots, P_n)$ be the self-adjoint n-component position and momentum operator respectively (Q_j is multiplication by x_j, P_j is differentiation: $P_j = -i\partial/\partial x_j$), and we set

$$\langle Q\rangle = \left(I + \sum_{j=1}^{n} Q_j^2\right)^{1/2}, \qquad \langle P\rangle = \left(I + \sum_{j=1}^{n} P_j^2\right)^{1/2}.$$

For $r \in [0, 2]$, we let $\mathscr{H}^r \equiv \mathscr{H}^r(\mathbb{R}^n)$ be the domain of $\langle P\rangle^r$, with norm $\|f\|_{\mathscr{H}^r} = \|\langle P\rangle^r f\|_{L^2}$, and we denote by $\mathscr{H}^{-r}$ the adjoint Hilbert space of $\mathscr{H}^r$. The norm of an operator T in $\mathscr{B}(L^2(\mathbb{R}^n))$ will be denoted simply by $\|T\|$.

We denote by D the differential operator

$$D = -i/2 \sum_{j=1}^{n} (x_j\, \partial/\partial x_j + \partial/\partial x_j x_j). \tag{53}$$

To each $\chi \in C_0^\infty(\mathbb{R}^n)$ we associate a function $\hat{\chi} \in C_0^\infty(\mathbb{R}^n)$ by

$$\hat{\chi}(x) = \sum_{k=1}^{n} x_k \chi_{,k}(x), \quad \text{where} \quad \chi_{,k}(x) = \partial/\partial x_k \chi(x).$$

We observe that, for $\varepsilon > 0$:

$$\frac{d}{d\varepsilon}\chi(\varepsilon Q) = \varepsilon^{-1}\hat{\chi}(\varepsilon Q), \tag{54}$$

$$[\chi(\varepsilon Q), D] = i\hat{\chi}(\varepsilon Q). \tag{55}$$

Lemma 6. *Let $\varphi \in C_0^\infty(\mathbb{R}^n)$, $\psi \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$ and $\kappa \in [0, 1]$. Then there is a finite constant c such that for all $\varepsilon \in (0, 1)$ and $j = 1, \ldots, n$:*

(a) $\|\varphi(\varepsilon Q) Q_j P_j \langle Q\rangle^{-\kappa}\|_{\mathscr{B}(\mathscr{H}^{-1}, \mathscr{H}^{-2})} \leq c\varepsilon^{-1+\kappa}$,

(b) *$\varphi(\varepsilon Q)$ is a strongly continuous $\mathscr{B}(\mathscr{H}^2)$-valued function of ε and converges strongly in $\mathscr{B}(\mathscr{H}^2)$ to $\varphi(0)I$ as $\varepsilon \to 0$.*

(c) $\|\psi(\varepsilon Q)\langle Q\rangle^{-\kappa}\|_{\mathscr{B}(\mathscr{H}^{-1})} \leq c\varepsilon^{\kappa}$.

Proof. The proof can be based on the following observation: if T is a symmetric operator and $s > 0$, then

$$\|T\|_{\mathscr{B}(\mathscr{H}^{-s})} = \|T\|_{\mathscr{B}(\mathscr{H}^{s})} = \|\langle P\rangle^s T \langle P\rangle^{-s}\|. \tag{56}$$

For $s = 2$, we get

$$\|T\|_{\mathscr{B}(\mathscr{H}^{-2})} \leq \|T\| + \|[P^2, T]\langle P\rangle^{-2}\|, \tag{57}$$

whereas for $s=1$:

$$\|T\|_{\mathscr{B}(\mathscr{H}^{-1})} \leqslant \|T\langle P\rangle^{-1}\| + \sum_{k=1}^{n} \|P_k T\langle P\rangle^{-1}\|$$

$$\leqslant (n+1)\,\|T\| + \sum_{k=1}^{n} \|[P_k, T]\|. \tag{58}$$

The norms on the r.h.s. of (57) and (58) can be explicitly evaluated when T is a function of Q. We shall use the relation

$$\partial/\partial x_k \langle x\rangle^{-\kappa} \equiv \partial/\partial x_k (1+x^2)^{-\kappa/2} = -\kappa x_k (1+x^2)^{-1}\langle x\rangle^{-\kappa}. \tag{59}$$

We have for example from (58) and (59):

$$\|\varphi(\varepsilon Q) Q_j \langle Q\rangle^{-\kappa}\|_{\mathscr{B}(\mathscr{H}^{-1})} \leqslant (n+1)\,\|\varphi(\varepsilon x) x_j \langle x\rangle^{-\kappa}\|_{L^\infty(\mathbb{R}^n)}$$

$$+ \sum_{k=1}^{n} \|\varphi_{,k}(\varepsilon x)\varepsilon x_j \langle x\rangle^{-\kappa}\|_{L^\infty(\mathbb{R}^n)} + \|\varphi(\varepsilon x)\langle x\rangle^{-\kappa}\|_{L^\infty(\mathbb{R}^n)}$$

$$+ \sum_{k=1}^{n} \kappa\,\|\varphi(\varepsilon x) x_j x_k (1+x^2)^{-1}\langle x\rangle^{-\kappa}\|_{L^\infty(\mathbb{R}^n)}. \tag{60}$$

The last three terms are bounded by a constant c independent of ε, whereas for the first term on the r.h.s. we have

$$\|\varphi(\varepsilon x) x_j \langle x\rangle^{-\kappa}\|_{L^\infty(\mathbb{R}^n)} = \varepsilon^{-1} \sup_{y\in\mathbb{R}^n} |y_j|\,|\varphi(y)|\,[\varepsilon^{-2}(\varepsilon^2+y^2)]^{-\kappa/2}$$

$$\leqslant \varepsilon^{-1+\kappa} \sup_{y\in\mathbb{R}^n} |y_j|\,|y|^{-\kappa}\,|\varphi(y)| \equiv c\varepsilon^{-1+\kappa},$$

where c is finite since $\kappa \leqslant 1$. Hence, for $s=1$:

$$\|\varphi(\varepsilon Q) Q_j \langle Q\rangle^{-\kappa}\|_{\mathscr{B}(\mathscr{H}^{-s})} \leqslant c\varepsilon^{-1+\kappa}. \tag{61}$$

In a similar way one finds from (57) that (61) also holds for $s=2$.

Now, by commuting P_j and $\langle Q\rangle^{-\kappa}$ and by using (59), one sees that

$$\|\varphi(\varepsilon Q) Q_j P_j \langle Q\rangle^{-\kappa}\|_{\mathscr{B}(\mathscr{H}^{-1},\mathscr{H}^{-2})}$$
$$\leqslant \|\varphi(\varepsilon Q) Q_j \langle Q\rangle^{-\kappa}\|_{\mathscr{B}(\mathscr{H}^{-2})} + \kappa\,\|\varphi(\varepsilon Q) Q_j \langle Q\rangle^{-\kappa}\cdot Q_j\langle Q\rangle^{-2}\|_{\mathscr{B}(\mathscr{H}^{-1})}. \tag{62}$$

Since $Q_j\langle Q\rangle^{-2}$ is bounded as an operator in $\mathscr{H}^{-1}$, it follows from (61) that each term on the r.h.s. of (62) is $0(\varepsilon^{-1+\kappa})$, which proves (a).

The proof of (c) is analogous; it suffices to derive the analogue of the inequality (60) and to observe that the last supremum in the inequality below is finite since $\psi(y)=0$ near $y=0$:

$$\|\psi(\varepsilon x)\langle x\rangle^{-\kappa}\|_{L^\infty(\mathbb{R}^n)} = \sup_{y\in\mathbb{R}^n} |\psi(y)|\,[\varepsilon^{-2}(\varepsilon^2+y^2)]^{-\kappa/2}$$

$$\leqslant \varepsilon^{\kappa} \sup_{y\in\mathbb{R}^n} |\psi(y)|\,|y|^{-\kappa}.$$

Finally (b) is easily obtained by the same type of arguments. ■

Lemma 7. *Let $u: \mathbb{R}^n \to \mathbb{R}$ be a measurable function such that the operator $U \equiv u(Q)$ maps $\mathcal{H}^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$ and such that, for some $\kappa > 0$, the operators $\langle Q \rangle^\kappa U$ and $\langle Q \rangle^\kappa [U, D]$ belong to $\mathcal{B}(\mathcal{H}^2(\mathbb{R}^n), \mathcal{H}^{-1}(\mathbb{R}^n))$. Set $\delta = \min(1, \kappa)$ and let $\theta \in C_0^\infty(\mathbb{R}^n)$ be real and such that $\theta(x) = 1$ for $|x| \leqslant 1$. Define $U(\varepsilon)$ by*

$$U(\varepsilon) = \theta(\varepsilon Q)U = U\theta(\varepsilon Q). \tag{63}$$

Then:

(i) *$U(\varepsilon)$ is a strongly continuous $\mathcal{B}(\mathcal{H}^2(\mathbb{R}^n), L^2(\mathbb{R}^n))$-valued function of ε on $[0, 1]$, with $U(0) = U$,*

(ii) *the commutator $[U(\varepsilon), D]$ is strongly C^1 in $\mathcal{B}(\mathcal{H}^2(\mathbb{R}^n), \mathcal{H}^{-1}(\mathbb{R}^n))$ for $\varepsilon \in (0, 1)$, and*

$$\|d/d\varepsilon[U(\varepsilon), D]\|_{\mathcal{B}(\mathcal{H}^2, \mathcal{H}^{-1})} \leqslant c\varepsilon^{-1+\delta}, \tag{64}$$

(iii) *one has*

$$\|[[U(\varepsilon), D], D]\|_{\mathcal{B}(\mathcal{H}^2, \mathcal{H}^{-2})} \leqslant c\varepsilon^{-1+\delta}, \tag{65}$$

where c is some finite constant.

Proof. (i) follows immediately from Lemma 6(b). For (ii) and (iii) we remark that

$$[U(\varepsilon), D] = \theta(\varepsilon Q)[U, D] + i\hat{\theta}(\varepsilon Q)U, \tag{66}$$

$$d/d\varepsilon[U(\varepsilon), D] = \varepsilon^{-1}\hat{\theta}(\varepsilon Q)[U, D] + i\varepsilon^{-1}\hat{\hat{\theta}}(\varepsilon Q)U, \tag{67}$$

$$[[U(\varepsilon), D], D] = 2i\hat{\theta}(\varepsilon Q)[U, D] - \hat{\hat{\theta}}(\varepsilon Q)U - \theta(\varepsilon Q)D[U, D] + \theta(\varepsilon Q)[U, D]D. \tag{68}$$

(ii) is obtained from (67) by noticing that $\hat{\theta}, \hat{\hat{\theta}} \in C_0^\infty(\mathbb{R}^n\backslash\{0\})$ and that, if $\langle Q \rangle^\delta T \in \mathcal{B}(\mathcal{H}^2, \mathcal{H}^{-1})$ and $\psi \in C_0^\infty(\mathbb{R}^n\backslash\{0\})$, then by Lemma 6(c):

$$\|\psi(\varepsilon Q)T\|_{\mathcal{B}(\mathcal{H}^2, \mathcal{H}^{-1})} \leqslant \|\psi(\varepsilon Q)\langle Q \rangle^{-\delta}\|_{\mathcal{B}(\mathcal{H}^{-1})} \|\langle Q \rangle^\delta T\|_{\mathcal{B}(\mathcal{H}^2, \mathcal{H}^{-1})} \leqslant c\varepsilon^\delta.$$

The same reasoning shows that the first two terms on the r.h.s. of (68) are $0(\varepsilon^\delta)$ in $\mathcal{B}(\mathcal{H}^2, \mathcal{H}^{-1}) \subset \mathcal{B}(\mathcal{H}^2, \mathcal{H}^{-2})$, so that it only remains to prove that the last two terms in (68) are $0(\varepsilon^{-1+\delta})$ in $\mathcal{B}(\mathcal{H}^2, \mathcal{H}^{-2})$.

For the third term on the r.h.s. of (68) we have

$$\begin{aligned}&\|\theta(\varepsilon Q)D[U, D]\|_{\mathcal{B}(\mathcal{H}^2, \mathcal{H}^{-2})} \\ &\quad \leqslant \sum_{j=1}^n \|\theta(\varepsilon Q)Q_jP_j\langle Q \rangle^{-\delta}\|_{\mathcal{B}(\mathcal{H}^{-1}, \mathcal{H}^{-2})} \|\langle Q \rangle^\delta [U, D]\|_{\mathcal{B}(\mathcal{H}^2, \mathcal{H}^{-1})} \\ &\qquad + n/2 \|\theta(\varepsilon Q)[U, D]\|_{\mathcal{B}(\mathcal{H}^2, \mathcal{H}^{-2})}.\end{aligned}$$

The first term on the r.h.s. is $0(\varepsilon^{-1+\delta})$ by Lemma 6(a), the second one is $0(1)$ since $\theta(\varepsilon Q)$ is a continuous function of ε in $\mathcal{B}(\mathcal{H}^{-2})$ as a consequence of Lemma 6(b). Finally the last term in (68) may be estimated in the same way by choosing a function θ_0 in $C_0^\infty(\mathbb{R}^n)$ such that $\theta_0\theta = \theta$ and writing

$$\begin{aligned}&\|\theta(\varepsilon Q)[U, D]D\|_{\mathcal{B}(\mathcal{H}^2, \mathcal{H}^{-2})} = \|\theta(\varepsilon Q)[U, D]D\theta_0(\varepsilon Q)\|_{\mathcal{B}(\mathcal{H}^2, \mathcal{H}^{-2})} \\ &\quad \leqslant \|\theta_0(\varepsilon Q)D[U, D]\|_{\mathcal{B}(\mathcal{H}^2, \mathcal{H}^{-2})} \|\theta(\varepsilon Q)\|_{\mathcal{B}(\mathcal{H}^2)}. \quad \blacksquare\end{aligned}$$

We now apply these results to N-body Schrödinger operators in the center-of-mass frame. We refer to [2] or [11] for the relevant definitions. The Hilbert space is $\mathcal{H} = L^2(\mathbb{R}^{n(N-1)})$, the free Hamiltonian is denoted by H_0 and the pair potentials V_{kl} $(1 \leq k < l \leq N)$ are measurable real-valued functions defined on $\mathbb{R}^n$. We denote by $Q^{(l)}$ the n-component position operator of the l-th particle.

Proposition 8. *Let H be an N-body Schrödinger operator in the center-of-mass Hilbert space $\mathcal{H} = L^2(\mathbb{R}^{n(N-1)})$:*

$$H = H_0 + \sum_{1 \leq k < l \leq N} V_{kl},$$

where $V_{kl} = v_{kl}(Q^{(k)} - Q^{(l)})$. Assume that each $v_{kl} : \mathbb{R}^n \to \mathbb{R}$ has the following properties:

(i) *$v_{kl}(Q)$ is a compact operator in $\mathcal{B}(\mathcal{H}^2(\mathbb{R}^n), L^2(\mathbb{R}^n))$,*

(ii) *$[v_{kl}(Q), D]$ is a compact operator in $\mathcal{B}(\mathcal{H}^2(\mathbb{R}^n), \mathcal{H}^{-2}(\mathbb{R}^n))$,*

(iii) *v_{kl} can be decomposed into $v_{kl} = u_{kl} + w_{kl}$ in such a way that u_{kl}, w_{kl} are real, satisfy $u_{kl}(Q)$, $w_{kl}(Q) \in \mathcal{B}(\mathcal{H}^2(\mathbb{R}^n), L^2(\mathbb{R}^n))$ and*

(α) *$\langle Q \rangle^\kappa u_{kl}(Q)$and $\langle Q \rangle^\kappa [u_{kl}(Q), D]$ belong to $\mathcal{B}(\mathcal{H}^2(\mathbb{R}^n), \mathcal{H}^{-1}(\mathbb{R}^n))$ for some $\kappa > 0$,*

(β) *$[w_{kl}(Q), D] \in \mathcal{B}(\mathcal{H}^2(\mathbb{R}^n), \mathcal{H}^{-1}(\mathbb{R}^n))$ and*

$[[w_{kl}(Q), D], D] \in \mathcal{B}(\mathcal{H}^2(\mathbb{R}^n), \mathcal{H}^{-2}(\mathbb{R}^n))$.

Then:

(a) *H has no singularly continuous spectrum, the set $\tau(H)$ of thresholds of H is closed and countable, the eigenvalues of H not belonging to $\tau(H)$ are of finite multiplicity and their accumulation points are contained in $\tau(H)$.*

(b) *Let $\alpha > \frac{1}{2}$ and $s < 1$. Let R be an operator in $\mathcal{B}(\mathcal{H}^s(\mathbb{R}^n), L^2(\mathbb{R}^n))$ and denote by $\mathbf{Q}^2$ the operator of multiplication by $\sum_{j=1}^{n(N-1)} x_j^2$ in $L^2(\mathbb{R}^{n(N-1)})$. Then the operator $R(I + \mathbf{Q}^2)^{-\alpha/2}$ is H-smooth on each compact subset of $\mathbb{R} \setminus \{\tau(H) \cup \sigma_p(H)\}$.*

Proof. The properties of $\tau(H)$ and the Mourre estimate for compact subintervals J of $\mathbb{R} \setminus \tau(H)$ are proved in [10]. The other assertions of the proposition follow from Propositions 4 and 7 combined with Lemma 7 by setting $X = (1 + \mathbf{Q}^2)^{1/2}$, $U_{kl}(\varepsilon) = \theta(\varepsilon(Q^{(k)} - Q^{(l)}) u_{kl}(Q^{(k)} - Q^{(l)})$ (with $\theta : \mathbb{R}^n \to \mathbb{R}$ as in Lemma 7), $W_{kl} = w_{kl}(Q^{(k)} - Q^{(l)})$ and

$$H(\varepsilon) = H_0 + \sum_{1 \leq k < l \leq N} (W_{kl} + U_{kl}(\varepsilon)) \tag{69}$$

and by choosing for A the operator

$$A = -\frac{i}{2} \sum_{j=1}^{n(N-1)} (x_j \, \partial/\partial x_j + \partial/\partial x_j x_j). \tag{70}$$

It suffices to observe that the norm in $\mathcal{G}^\rho$ is equivalent to that in $\mathcal{H}^{2\rho}(\mathbb{R}^{n(N-1)})$, that $i[H_0, A] = 2H_0$ and that $[f(Q^{(k)} - Q^{(l)}), A] = [f(Q^{(k)} - Q^{(l)}), D] \otimes I$ in the tensor product decomposition $L^2(\mathbb{R}^{n(N-1)}) = L^2(\mathbb{R}^n) \otimes L^2(\mathbb{R}^{n(N-2)})$, where the variable in the first factor is $x^{(k)} - x^{(l)}$ and where D denotes the operator (53) with respect to this variable, whereas the variables in the second factor are relative coordinates between the center of mass of the pair (k, l) and the remaining $N-2$ particles. ■

The potentials w_{kl} in Proposition 8 cover the class considered in [2], and for u_{kl} one may for example take a sum of functions $u: \mathbb{R}^n \to \mathbb{R}$ satisfying one of the following conditions, for some $\kappa > 0$:

(α) $u \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$, $\langle Q\rangle^\kappa u(Q) \in \mathscr{B}(H^2(\mathbb{R}^n), L^2(\mathbb{R}^n))$ and

$$\langle Q\rangle^\kappa \sum_{k=1}^n Q_k u_{,k}(Q) \in \mathscr{B}(\mathscr{H}^2(\mathbb{R}^n), \mathscr{H}^{-1}(\mathbb{R}^n)),$$

where the derivatives of u are in the sense of distributions.

(β) $\langle Q\rangle^{1+\kappa} u(Q) \in \mathscr{B}(\mathscr{H}^2(\mathbb{R}^n), L^2(\mathbb{R}^n))$.

The condition (α) is satisfied in particular if u is C^1 and for $j = 1, \ldots, n$:

$$|u(x)| \leq c\langle x\rangle^{-\kappa} \quad \text{and} \quad |\partial/\partial x_j u(x)| \leq c\langle x\rangle^{-\kappa-1}.$$

For $n \leq 3$, (β) is equivalent to the requirement that

$$\sup_{x\in\mathbb{R}^n} \langle x\rangle^{1+\kappa}\left[\int_{|x-y|\leq 1} |u(y)|^2\, dy\right]^{1/2} < \infty.$$

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