Zeitschrift: Helvetica Physica Acta
Band: 60 (1987)
Heft: 8

Artikel: On the behaviour of an accelerated clock
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DOI: https://doi.org/10.5169/seals-115884

Helvetica Physica Acta, Vol. 60 (1987) 1024–1037

0018–0238/87/081024–14\$1.50 + 0.20/0

On the behaviour of an accelerated clock

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(28. III 1987)

Abstract. The 'clock hypothesis' of Special Relativity has been checked in the concrete example of the lifetime of an unstable particle. We show that two particles with the same energy have nearly the same decay-rate independently of their acceleration. In the example of the CERN g-2-experiment with a centripetal acceleration of 10^{18} g for the muons the here computed deviation from the behaviour of an ideal clock is really exceedingly small: namely less than 10^{-25}!

1. Introduction

The twin- (or clock-) paradox is one of the most perplexing and perhaps also one of the most discussed consequences of Special Relativity. Even after one has understood the asymmetry of the motion with the consequence that the twin who remained on the Earth (and so always in the same reference frame) has aged more than the other one who moved (and had at least for some time changed his reference frame), there remains another problem: Is it correct to integrate the expression for the differential of the proper time $d\tau = \sqrt{1 - v^2/c^2}\, dt$ over time in an *accelerated* frame in order to obtain the total proper time τ?

The so-called 'clock-hypothesis' says that in nature there are 'ideal clocks' with the property that their timekeeping is independent of their acceleration [1] or at least does not depend on it in a measureable way [2]. Hence such a clock would always show its proper time $\tau = \int_0^t \sqrt{1 - v(t')^2/c^2}\, dt'$. Einstein himself assumed that the behaviour of his measuring-rods and clocks did not depend upon the history of their previous motion ([3], p. 34).

Whether such ideal clock really exist must be decided either by experiment or by a suitable theory which includes the clock-mechanism. Mechanical clocks evidently are easily influenced by strong inertial forces. So we consider in this publication an unstable elementary particle that can be used as a clock because of its characteristic mean life time, and this 'clock' will be examined for its suitability as an 'ideal clock'.

Although this problem goes beyond the framework of Special Relativity, there is no necessity to use any General Relativity! We can assume namely that the particle is moving around a classical orbit in a constant magnetic field, such as the muons of the g-2-experiment of the 1970's at CERN [4]. *Quantum* *mechanically* this 'orbiting' can be described as a Landau-level with a fixed (albeit very high) quantum number in an *inertial frame* (the lab. frame), and with the theory of weak interactions the life time of the particle in the magnetic field can be approximately computed. This lifetime can then be compared with that one of a free particle with the same energy in order to check the clock hypothesis.

Unless otherwise noted, we put $\hbar = c = 1$ throughout the calculation.

2. Free decay

To simplify the calculation we choose the (hypothetical) decay

$$\mu_s^- \to e_s^- + \nu_s$$

of a scalar muon into a scalar electron and a scalar neutrino. Analogous to the phenomenological Fermi-coupling of the weak interaction as the low-energy limit ($E_\mu \ll m_w$) of the Glashow–Salam–Weinberg model, we start from the simple linear interaction

$$H_{\text{int}} = f \cdot \int d^3x e_s^* \cdot \mu_s \cdot \nu_s^*. \tag{1}$$

In lowest order perturbation theory the field-operators are expanded as usual to plane waves with the creation – and annihilation operators as coefficients, e.g.

$$\mu_s(x) = (2\pi)^{-3/2} \int d\Omega(p)(a(p)e^{-ipx} + b^*(p)e^{ipx}) \tag{2}$$

$$\ldots$$

with the invariant measure $d\Omega(p) = d^3p/2p^0$; $p^0 = E$, $px = p^0x^0 - \vec{p}\vec{x}$.

By neglecting the electron-mass with respect to the muon-mass, we readily obtain for the free decay-rate:

$$\Gamma_0 = \frac{f^2}{16\pi p^0} = \frac{f^2}{16\pi m_\mu} \cdot \frac{1}{\gamma}. \tag{3}$$

3. Decay in the magnetic field

We presume a constant magnetic field $\vec{B}$ in the z-direction. In order to illustrate the symmetry (classical orbits, if $v_z = 0$), we choose cylindrical coordinates and the gauge

$$A^0 = 0, \qquad \vec{A} = \frac{B\rho}{2} \vec{e}_\varphi. \tag{4}$$

The Klein–Gordon equation yields the same wave-function as the

Schrödinger equation [5]:

$$\psi_{nl,p_z}(\vec{x}) = C_{nl} e^{-\rho^2/2\rho_0^2} \rho^{l-n} L_n^{l-n}(\rho^2/\rho_0^2) e^{i(n-l)\varphi} \frac{e^{ip_z \cdot z}}{\sqrt{2\pi}} \tag{5}$$

with

$$C_{nl} = (-i)^n \sqrt{\frac{n!}{\pi l!}} \rho_0^{n-l-1} \quad \text{and} \quad \rho_0 = \sqrt{\frac{2}{eB}}, \tag{5a}$$

but with relativistic Landau–levels

$$E_{n,p_z} = \sqrt{m^2 + (2n+1)eB + p_z^2} \tag{6}$$

independently of l. So we can choose as the initial state a wave-packet with $l = 0$ and p_z concentrated around 0:

$$\psi_0(x) = \int dp_z f(p_z) \psi_{n,p_z}(x) \tag{7}$$

with $\int dp_z \, |f(p_z)|^2 = 1$ (in order to have $\|\psi_0\| = 1$) and

$$\psi_{n,p_z}(x) = i^n (\pi n!)^{-1/2} \rho_0^{-(n+1)} e^{-\rho^2/2\rho_0^2} \rho^n e^{-in\varphi} \frac{e^{ip_z \cdot z}}{\sqrt{2\pi}} e^{-iEn,p_z \cdot t}, \tag{8}$$

where we have chosen the *negative* z-direction as field-direction (in order to maintain the same convention as in [6]) and so substituted φ by $-\varphi$.

The field operators of the charged particles are expanded to the complete system (5) of solutions of the Klein–Gordon equation, while the field operator of the neutrino remains unchanged. In this way we have instead of (2):

$$\mu_s(x) = \int dp_z \sum_{n,l} \frac{1}{\sqrt{2E_n, p_z}} (a_{nl,p_z} \psi_{nl,p_z}(x) + b_{nl,p_z}^* \psi_{nl,p_z}^*(x)) \tag{9}$$

with the anti-commutation relations:

$$\{a_{nl,p_z}, a_{mk,p_z'}^*\} = \delta_{nm} \, \delta_{lk} \, \delta(p_z - p_z'), \text{ etc.} \tag{10}$$

The time evolution of the initial state (7), (8) in first order perturbation theory is

$$\psi_t = -i \int_0^t dt' H_{\text{int}}(t') \cdot \psi_0, \tag{11}$$

and the total decay-probability at time t

$$P_t = \int d\Omega(k) \int dq_z \sum_{n',l'} |\langle k\psi_{n'l',q_z} \mid \psi_t \rangle|^2,$$

where k describes the neutrino and $\psi_{n'l',q_z}$ the electron in the final state.

With (11), (1), (2) and (9), (10) in addition to the corresponding formulas for the decay-particles, we obtain:

$$P_t = \frac{f^2}{(2\pi)^3} \int d\Omega(k) \underline{\int dq_z \sum_{n',l'}} \int dp_{z_1} f(p_{z_1}) \int dp_{z_2} f^*(p_{z_2})$$
$$\times \int_0^t dt_1 e^{-i(k^0 - \underline{E_{n',q_z}} - E_{n,p_{z_1}})t_1} \int_0^t dt_2 e^{i(k^0 + \underline{E_{n',q_z}} - E_{n,p_{z_2}})t_2}$$
$$\times \int d^3x_1 \int d^3x_2 e^{-i\vec{k}(\vec{x}_2 - \vec{x}_1)} \frac{\psi^*_{n,p_{z_1}}(\vec{x}_1)}{\sqrt{2E_{n,p_{z_1}}}} \frac{\psi_{n,p_{z_2}}(\vec{x}_2)}{\sqrt{2E_{n,p_{z_2}}}}$$
$$\cdot \underline{\frac{\psi_{n'l',q_z}(\vec{x}_1)\psi^*_{n'l',q_z}(\vec{x}_2)}{2E_{n'l'}}}. \tag{12}$$

The summation and integration over all possible final states [underlined in (12)] is done in [6]. It gives a Fundamental solution of the Klein–Gordon equation in the given external field $\vec{B} = (0, 0, -B)$. Since in [6] together with the more conventional Cartesian coordinates the gauge $\vec{A} = (0, -Bx, 0)$ is used instead of (4), the action factor underlined in the following equation differs from that given in Ref. [6], eqns. (43), (44), (51).[1]) The summation eventually yields

$$K_-(\vec{x}_1, \vec{x}_2) = \frac{-i}{8\pi^2} \underline{\exp\left[-\frac{ieB}{2}(x_1 y_2 - x_2 y_1)\right]}$$
$$\times \int_0^{\infty + ic} d\gamma \exp\left[\frac{i\gamma}{2}((\vec{x}_2 - \vec{x}_1)^2 - \tau^2)\right]$$
$$\cdot \frac{\exp\left[-\frac{i}{4}(\vec{x}_{2_\perp} - \vec{x}_{1_\perp})^2 eB\left(\frac{2\gamma}{eB} - \operatorname{ctg}\frac{eB}{2\gamma}\right)\right]}{2\gamma/eB \sin eB/2\gamma}$$

by neglecting the electron-mass and for $\tau := t_2 - t_1 < 0$. For $\tau > 0$ we have: $K_+(\vec{x}_1, \vec{x}_2) = K_-^*(\vec{x}_2, \vec{x}_1)$.

With $T := (t_1 + t_2)/2$ time integration in (12) becomes

$$\int_0^t dT \int_{-T}^{T} d\tau \cdots.$$

We consider integrations over long times t (time-scale of μ-decay $\sim 10^{-6}$ s) relative to the inverse of the frequencies in the integrand ($\hbar/E_\mu \sim 10^{-24}$ s for a GeV-muon). So we can interprete the integrand over dT as decay-rate and expand the integration range over τ to $\mathbb{R}$. Noting the reality-property of (12) (conjugated complex by changing the indexes 1 and 2) the decay-rate becomes

$$\Gamma = \operatorname{Re}\left\{\frac{f^2}{(2\pi)^3 E_n} \int d\Omega(k) \int dp_{z_1} f(p_{z_1}) \int dp_{z_2} f^*(p_{z_2}) \int_0^\infty d\tau e^{i(k^0 - E_n)\tau}\right.$$
$$\times \int d^3x_1 \int d^3x_2 e^{-i\vec{k}(\vec{x}_2 - \vec{x}_1)} \psi^*_n(\vec{x}_{1_\perp}) \frac{e^{-ip_{z_1}z_1}}{\sqrt{2\pi}} \psi_n(\vec{x}_{2_\perp}) \frac{e^{ip_{z_2}z_2}}{\sqrt{2\pi}}$$

[1]) Note that in Ref. [6]'s eqn. (51) [according to equation (50)] the exponential function has the wrong sign.

$$\times \frac{i}{8\pi^2} \exp\left[\frac{ieB}{2}(x_1y_2 - x_2y_1)\right] \int_0^{\infty + ic} d\gamma \exp\left[-\frac{i\gamma}{2}((z_2 - z_1)^2 - \tau^2)\right]$$

$$\times \frac{\exp\left[-\frac{i}{4}(\vec{x}_{2_\perp} - \vec{x}_{1_\perp})^2 eB \operatorname{ctg} \frac{eB}{2\gamma}\right]}{2\gamma/eB \cdot \sin eB/2\gamma}. \tag{13}$$

To simplify we put $E_n := E_{n,p_z=0}$ prior to the p_z-integration, as $f(p_z)$ can be concentrated arbitrarily sharply around $p_z = 0$.

The p_z- and z-integration yield

$$\sqrt{-\frac{2\pi i}{\gamma}} e^{ik_z^2/2\gamma}.$$

We put this as well as the μ-wave-function (8), (5a) in (13) and write the whole expression in cylindrical coordinates with $\vec{k} = (k_\perp, \alpha; k_z)$:

$$\Gamma = \frac{f^2}{(2\pi)^{11/2} \cdot E_n} \operatorname{Re}\left\{i^{-1/2} \int d\Omega(k) \int_0^\infty d\tau e^{i(k^0 - E_n)\tau} \int_0^\infty d\rho_1 \rho_1 \int_0^{2\pi} d\varphi_1 \times \int_0^\infty d\rho_2 \cdot \rho_2 \int_0^{2\pi}\right.$$

$$\times d\varphi_2 \exp\left[-\frac{eB}{4}(\rho_1^2 + \rho_2^2) - ik_\perp(\rho_2 \cos(\varphi_2 - \alpha) - \rho_1 \cos(\varphi_1 - \alpha))\right]$$

$$\times \int_0^{\infty + ic} \frac{d\gamma}{\sqrt{\gamma}} \exp\left(\frac{i}{2}\gamma\tau^2 + \frac{ik_z^2}{2\gamma}\right)$$

$$\times \exp\left\{-\frac{ieB}{4}[(\rho_1^2 + \rho_2^2) \operatorname{ctg} \varepsilon - 2\rho_1\rho_2(\sin(\varphi_1 - \varphi_2) + \operatorname{ctg} \varepsilon \cos(\varphi_1 - \varphi_2)]\right\}$$

$$\left.\times \frac{\varepsilon}{\sin \varepsilon}\left(\frac{eB}{2}\right)^{n+1} \frac{1}{n!}(\rho_1\rho_2 e^{i(\varphi_1 - \varphi_2)})^n\right\}$$

with

$$\varepsilon := \frac{eB}{2\gamma}. \tag{14}$$

Instead of the φ_i we take the $\phi_i := \varphi_i - \alpha$ as integration-variables and rewrite:

$$\Gamma = \frac{f^2}{(2\pi)^{11/2} E_n}\left(\frac{eB}{2}\right)^{n+1} \frac{1}{n!} \operatorname{Re}\left\{i^{1/2} \int d\Omega(k) \int_0^\infty d\tau e^{i(k^0 - E_n)\tau}\right.$$

$$\times \int_0^\infty d\rho_1 \rho_1^{n+1} \int_0^{2\pi} d\phi_1 \int_0^\infty d\rho_2 \rho_2^{n+1} \int_0^\infty d\phi_2 \int_0^{\infty + ic} \frac{d\gamma}{\sqrt{\gamma}} \exp\left(\frac{i}{2}\gamma\tau^2 + ik_z^2/2\gamma\right) \frac{\varepsilon}{\sin \varepsilon}$$

$$\times \exp\left[-\frac{eB}{4}(\rho_1^2 + \rho_2^2)(1 + i \operatorname{ctg} \varepsilon)\right.$$

$$+ i\frac{eB}{2}\rho_1\rho_2(\sin(\phi_1 - \phi_2) + \operatorname{ctg} \varepsilon \cos(\phi_1 - \phi_2))$$

$$\left.\left. - ik_\perp(\rho_2 \cos \phi_2 - \rho_1 \cos \phi_1) + n(\phi_1 - \phi_2)\right]\right\}. \tag{15}$$

The square bracket in the exponential function can be written as

$$\tfrac{1}{2}[(\operatorname{ctg}\varepsilon - i)e^{i(\phi_1-\phi_2)} + (\operatorname{ctg}\varepsilon + i)e^{-i(\phi_1-\phi_2)}]$$

and the $\lambda_i := e^{i\phi_i}$ taken as the new integration variables. In this way the spatial integrations in (15) become:

$$I = \int_0^\infty d\rho_1 \rho_1^{n+1} \frac{1}{i} \oint d\lambda_1 \lambda_1^{n-1} \int_0^\infty d\rho_2 \rho_2^{n+1} \frac{1}{i} \oint \frac{d\lambda_2}{\lambda_2^{n+1}}$$
$$\times \exp\left\{ -\frac{ieB}{4}(\operatorname{ctg}\varepsilon - i)(\rho_1^2+\rho_2^2) + \frac{ieB}{4}\rho_1\rho_2\left[(\operatorname{ctg}\varepsilon - i)\frac{\lambda_1}{\lambda_2} + (\operatorname{ctg}\varepsilon + i)\frac{\lambda_2}{\lambda_1}\right]\right.$$
$$\left. + \frac{ik_\perp}{2}\left[\rho_1\left(\lambda_1+\frac{1}{\lambda_1}\right) - \rho_2\left(\lambda_2 + \frac{1}{\lambda_2}\right)\right]\right\}.$$

We are solving the λ_i-integrations around the zero-point of the complex plane with the residue theorem:

$$I = \frac{4\pi}{ieB(\operatorname{ctg}\varepsilon - i)} \int_0^\infty d\rho_1 \rho_1^{n+1} \frac{1}{i} \oint d\lambda_1 \cdot \lambda_1^{n-1} \exp\left[-\frac{eB}{2}\rho_1^2 - \frac{ik_\perp \rho_1}{2\lambda_1}(e^{2i\varepsilon}-1)\right.$$
$$\left. + \frac{ik_\perp^2}{eB(\operatorname{ctg}\varepsilon - i)}\right] \cdot \left(e^{2i\varepsilon}\rho_1 - \frac{k_\perp}{2\beta}\lambda_1\right)^n$$
$$= \frac{2\pi^2 n!}{i(\operatorname{ctg}\varepsilon - i)}\left(\frac{2}{eB}\right)^{n+2} \exp\left[\frac{ik_\perp^2}{eB(\operatorname{ctg}\varepsilon - i)}\right] \sum_{j=0}^n \binom{n}{j}\frac{1}{j!}(e^{2i\varepsilon})^{n-j}\left[\frac{ik_\perp^2}{eB}\frac{e^{2i\varepsilon}-1}{\operatorname{ctg}\varepsilon - i}\right]^j.$$

With small transformations put into (15):

$$\Gamma = \frac{f^2}{2(2\pi)^{7/2}E_n} \operatorname{Re}\left\{ i^{-1/2} \int d\Omega(k) \int_0^\infty d\tau \int_0^{\infty + ic} \frac{d\gamma}{\gamma^{3/2}} \exp i\left[(k^0 - E_n)\tau + \frac{\gamma}{2}\tau^2\right.\right.$$
$$\left.\left. + \frac{k_z^2 + eB}{2\gamma} + \frac{k_\perp^2}{eB}\frac{e^{2i\varepsilon}-1}{2i}\right] \sum_{j=0}^n \binom{n}{j}\frac{1}{j!} e^{2i\varepsilon(n-j)}\left[\frac{k_\perp^2}{2eB}(e^{2i\varepsilon}-1)^2\right]^j\right\}.$$

Putting $x := (k^0)^2/2\gamma$ we obtain the intermediate result [with $d\Omega(k) = \frac{1}{2}\, dk^0 \cdot k^0\, d\mu \cdot 2\pi$, $\mu = \cos\vartheta$]:

$$\Gamma = \frac{f^2}{\sqrt{2}(2\pi)^{5/2}E_n} \operatorname{Re}\left\{ i^{-1/2} \int_0^\infty dk^0 \frac{1}{2}\int_{-1}^1 d\mu \int_0^\infty d\tau e^{i(k^0 - E_n)\tau}\right.$$
$$\times \int_0^{\infty - ic} \frac{dx}{\sqrt{x}} \exp\left[i\left((1+\varepsilon')x + \frac{(k^0\tau)^2}{4x}\right) + \frac{1-\mu^2}{2\varepsilon'}(e^{2i\varepsilon' x} - 1 - 2i\varepsilon' x)\right]$$
$$\left.\times \sum_{j=0}^n \binom{n}{j}\frac{1}{j!} e^{2i(n-j)\varepsilon' x}\left[\frac{1-\mu^2}{2\varepsilon'}(e^{2i\varepsilon' x}-1)^2\right]^j\right\}, \tag{16}$$

where

$$\varepsilon' := \frac{eB}{(k^0)^2}\left(=\frac{\varepsilon}{x}\right) \tag{17}$$

is the correction parameter before the k^0-integration.

In the following we shall calculate the first non-vanishing correction due to the magnetic field with the help of suitable approximations.

4. Expansion of the decay rate

For $\varepsilon' = 0$ we readily obtain the free decay-rate (3). k^0 is of the order of the muon-energy, hence ε' is of the order eB/m_μ^2 or (in the relativistic case) even smaller. We can write

$$\frac{eB}{m^2} = \frac{B}{B_{cr}}, \tag{18}$$

where

$$B_{cr} = \frac{m^2}{e} \quad \left(\text{actually} \quad \frac{m^2c^3}{e\hbar}\right) \tag{18a}$$

is the *critical magnetic field* for a singly charged particle of mass m. In the case of the muon it is $2 \cdot 10^{18}$ Gauss. As the critical magnetic field of the electron ($4 \cdot 10^{13}$ Gauss) already constitutes a natural limit (the field-energy would create e^+e^--pairs), we have

$$\varepsilon' \ll 1. \tag{19}$$

In the case of a particle accelerator we would have for a muon

$$\frac{eB}{E^2} < \frac{B}{B_{cr}} \approx \frac{10^4 G}{10^{18} G} = 10^{-14}. \tag{19a}$$

Hence the correction-term in the exponent of (16)

$$\frac{1-\mu^2}{2\varepsilon'}(e^{2i\varepsilon' x} - 1 - 2i\varepsilon' x)$$

limits the contributing integration range over x to $x \lesssim 1/\sqrt{\varepsilon'}$ and the vicinity of

$$x_s = \frac{s\pi}{\varepsilon'}, \qquad s \in \mathbb{N}. \tag{20}$$

But as here $1/\sqrt{x}$ in (16) is nearly constant, these latter contributions are smaller than the main contribution from $x \lesssim 1/\sqrt{\varepsilon'}$ by a factor of the magnitude $e^{-1/4\varepsilon'}$, and need not be considered. This will be proved in the case $n = 0$:

Since $x_s \gg 1$, the term $(k^0\tau)^2/4x$ in the exponent of (16) can be neglected. So we have for the contribution to the x-integral in (16) in the vicinity of $x = x_s$ for $c = 0$:

$$\begin{aligned} I_s &= \int_{x_s-\delta}^{x_s+\delta} \frac{dx}{\sqrt{x}} \exp\left[i(1+\varepsilon')x + \frac{1-\mu^2}{2\varepsilon'}(e^{2i\varepsilon' x} - 1 - 2i\varepsilon' x)\right] \\ &= \frac{1}{\sqrt{x_s}} \exp\left[i(\mu^2 + \varepsilon')x_s\right] \int_{-\delta}^{\delta} \frac{dw}{\sqrt{1 + w/x_s}} \\ &\quad \times \exp\left[i(1+\varepsilon')w + \frac{1-\mu^2}{2\varepsilon'}(e^{2i\varepsilon' w} - 1 - 2i\varepsilon' w)\right], \end{aligned}$$

with $w = x - x_s$. The w-integral becomes approximately

$$\int_{-\infty}^{\infty} dw\left(1 - \frac{w}{2x_s}\right) \exp\left[i(1+\varepsilon')w - (1-\mu^2)\varepsilon' w^2\right],$$

and therefore with (20)

$$I_s = \sqrt{\frac{\varepsilon'}{s}} \exp\left[-\frac{(1+\varepsilon')^2}{4(1-\mu^2)\varepsilon'} + i\pi\left(\frac{\mu^2}{\varepsilon'}+1\right)s\right] \cdot \left(1 + \frac{1+\varepsilon'}{4\pi(1-\mu^2)s}\right).$$

The next step is the μ-integration in (16). Because of (19) only $\mu \ll 1$ contributes significantly. With $1/1 - \mu^2 \approx 1 + \mu^2$ we have

$$\int_0^1 d\mu I_s(\mu) \cong \sqrt{\frac{\varepsilon'}{s}} \exp\left(-\frac{1}{4\varepsilon'} + i\pi s\right) \int_0^{\infty} d\mu \exp\left(-\frac{\mu^2}{4\varepsilon'} - i\pi s\right) \cdot \left(1 + \frac{1}{4\pi s}\right).$$

We sum the contributions for all x_s:

$$\sum_{s=1}^{\infty} \int_0^1 d\mu I_s(\mu) \cong \sqrt{\pi}\, \varepsilon' \exp(-1/4\varepsilon') \cdot \sum_{s=1}^{\infty} \frac{(-1)^s}{s\sqrt{1 - 1/4i\pi s}}$$

$O(\varepsilon' \cdot e^{-1/4\varepsilon'})$ as required.

Now we expand the integrand over x in (16) for $\varepsilon' x \ll 1$ in the following way:

$$I(x) = \sum_{j=0}^{n} \binom{n}{j} \frac{1}{j!} G_j(x) F_j(x) \cdot e^{i(k^0\tau)^2/4x} \tag{21}$$

with

$$G_j(x) = \frac{1}{\sqrt{x}} \exp i\{1 + 2[(n-j)+1]\varepsilon'\}x \tag{21a}$$

and

$$\begin{aligned} F_j(x) &= \exp\left[-(1-\mu^2)\varepsilon' x^2\right] \cdot \left[1 + \tfrac{2}{3}(1-\mu^2)\varepsilon'^2 (ix)^3 + \cdots\right] \\ &\quad \times \left[2(1-\mu^2)\varepsilon'(ix)^2(1 + 2\varepsilon' \cdot ix + \cdots)\right]^j \\ &= \exp(-\varepsilon'' x^2) \cdot \left[2\varepsilon''(ix)^2\right]^j \cdot \left[1 + 2j\varepsilon' \cdot ix + \tfrac{2}{3}\varepsilon'\varepsilon''(ix)^3 + \cdots\right], \end{aligned} \tag{21b}$$

where

$$\varepsilon'' = (1-\mu^2)\varepsilon'. \tag{22}$$

The $G_j(x)$ are exact. In the $F_j(x)$ the real part of the exponent was treated with horse-step method, the rest was expanded in powers of ix. We shall see that the missing terms (indicated by three dots) make no contribution.

Now we are expanding the $F_j(x)$ by Fourier, because for the $G_j(x)$ we can treat all integrations without difficulty:

$$F_j(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dy e^{ixy} \tilde{F}_j(y) \tag{23}$$

with

$$\tilde{F}_j(y) = \sqrt{\frac{\pi}{\varepsilon''}}\frac{1}{2^j}\exp(y^2/4\varepsilon'') \times \left[H_{2j}(z) + \frac{j\sqrt{\varepsilon'}}{\sqrt{1-\mu^2}}H_{2j+1}(z) + \frac{1}{12}\frac{\sqrt{\varepsilon'}}{\sqrt{1-\mu^2}}H_{2j+3}(z)\right], \tag{23a}$$

$$z = y/2\sqrt{\varepsilon''}. \tag{24}$$

With (21), (21a), (23) put into (16):

$$\Gamma = \frac{f^2}{\sqrt{2}(2\pi)^{7/2}E_n}\operatorname{Re}\left\{i^{-1/2}\int_0^\infty dk^0\int_0^1 d\mu\int_0^\infty d\tau e^{i(k^0-E_n)\tau}\int_0^\infty \frac{dx}{\sqrt{x}}\sum_{j=0}^{n}\binom{n}{j}\frac{1}{j!} \times \exp i\left\{[1+(2(n-j)+1)\varepsilon']x + \frac{(k^0\tau)^2}{4x}\right\}\cdot\int_{-\infty}^{\infty} dy e^{ixy}\tilde{F}_j(y).\right. \tag{25}$$

Because of the strongly decreasing term $\exp(-y^2/4\varepsilon'')$ in (23a) that limits the contributing y-integration range to a small interval around $y=0$ (even with the maximum powers $\sim y^{2n}$ in the Hermite-polynomials this becomes only of order of magnitude 1), the y-integration can be exchanged with the other integrations:

$$\Gamma = \frac{f^2}{16\pi^{7/2}E_n}\sum_{j=0}^{n}\binom{n}{j}\frac{1}{j!}\int_0^1 d\mu\int_{-\infty}^{\infty} dy\int_0^\infty dk^0\tilde{F}_j(y) \times \operatorname{Re}\left\{i^{-1/2}\int_0^\infty d\tau e^{i(k^0-E_n)\tau}\int_0^\infty \frac{dx}{\sqrt{x}}\exp i\left[Y_j\cdot x + \frac{(k^0\tau)^2}{4x}\right]\right\}, \tag{26}$$

where

$$Y_j = 1 + y + [2(n-j)+1]\varepsilon'. \tag{27}$$

Now we can perform the x- and then the τ-integration, and we obtain for the real part in (26):

$$\operatorname{Re}\{\cdots\} = \frac{\pi^{3/2}}{\sqrt{Y_j}}\delta[k^0(1+\sqrt{Y_j}) - E_n].$$

The differentiation of the argument of the δ-distribution with respect to k^0 [consider (27) and (17)] at the point where this argument is zero is

$$\frac{E_n\cdot\sqrt{K_j(y)}}{K^0\sqrt{Y_j}}$$

with

$$K_j(y) = 1 + \{1 - [2(n-j)+1]\chi\}y, \tag{28}$$

$$\chi = \frac{eB}{E_n^2}, \tag{28a}$$

$$K^0 = E_n\frac{K_j(y)-1}{y}. \tag{29}$$

Therefore we have for the k^0-integration in (26):

$$\pi^{3/2}\frac{K^0\tilde{F}_j(y)}{E_n\sqrt{K_j(y)}} \tag{30}$$

The integration range over y is limited to the range $-1/1-[2(n-j)+1]\chi$ to ∞ by the condition that the argument of the δ-distribution must be zero.

Instead of integrating over y, we integrate over z, (24), where ε'' according to (17), (22), (29) also depends on y. Moreover after the k^0-integration k^0 must be replaced by K^0, (29), also in the $\tilde{F}_j(y)$. With

$$\frac{\partial y}{\partial z}=4\sqrt{1-\mu^2}\sqrt{\chi}\frac{\sqrt{K_j(y)}}{1-[2(n-j)+1]\chi},$$
$$K^0=2\sqrt{1-\mu^2}\frac{\sqrt{eB}}{y}z=\frac{E_n}{2}\frac{1-[2(n-j)+1]\chi}{1+\sqrt{1-\mu^2}\sqrt{\chi}z}=:\frac{E_n}{2}\frac{f_{nj}}{g(z)} \tag{31}$$

we substitute (23a), (30) into (26), and obtain

$$\Gamma=\frac{f^2}{16\pi^{3/2}E_n}\sum_{j=0}^{n}\binom{n}{j}\frac{1}{2^j j!}\int_0^1 d\mu\int_{-1/2\sqrt{1-\mu^2}\sqrt{\chi}}^{\infty}dze^{-z^2}$$
$$\times\left\{\frac{f_{nj}}{g^2(z)}H_{2j}(z)+\frac{2\sqrt{\chi}}{\sqrt{1-\mu^2}g(z)}[j\cdot H_{2j+1}(z)+\tfrac{1}{12}H_{2j+3}(z)]\right\}. \tag{32}$$

Because $\chi \ll 1$ [(19), (28a)] the upper z-integration limit can be reduced to $\frac{1}{2}\sqrt{1-\mu^2}\sqrt{\chi}$. Then the functions $1/g(z)$, (31) and $1/g^2(z)$ can be expanded in the whole integration range into powers of z. In this way we have for the z-integral in (32):

$$I_{nj}=\int_{-1/2\sqrt{\chi'}}^{1/2\sqrt{\chi'}}dze^{-z^2}\Big[f_{nj}(1-2\sqrt{\chi'}z+3\chi'z^2-\cdots)H_{2j}(z)+\frac{2\sqrt{\chi'}}{1-\mu^2}$$
$$\times(1-\sqrt{\chi'}z+\chi'z^2-\cdots)(j\cdot H_{2j+1}(z)+\tfrac{1}{12}H_{2j+3}(z))\Big], \tag{33}$$

where the f_{nj} are defined in (31) and

$$\chi'(1-\mu^2)\chi. \tag{34}$$

For powers z^l with $l \ll 1/\chi$ in the integrand of (33) we now can increase the integration-range to $\mathbb{R}$. Higher powers do not contribute because of χ (and so also χ') $\ll 1$.

It now becomes clear why we can omit the higher powers of $\varepsilon'\cdot ix$ (but not those of $\varepsilon'\cdot(ix)^2$) in (21b). According to (17) every $\varepsilon'\cdot ix=\sqrt{\varepsilon'\cdot(ix)^2}\sqrt{\varepsilon'}$ would yield an additional factor $1/k^0$ and hence with (31) an additional factor $g(z)$ in (32). The expansion of $g^p(z)$ would result in terms in (33):

$$J_{kl}:=\int_{-\infty}^{\infty}dze^{-z^2}\cdot H_k(z)\cdot z^l$$

with $k > l$ that vanish because of the orthonormality of the Hermite-polynomials. For $k \leq l$ with the aid of the generating function of the Hermite-polynomials we get

$$\left.\begin{aligned} J_{2p,2q} &= \frac{\sqrt{\pi}(2q)!}{4^{q-p}(q-p)!} \\ J_{2p+1,2q+1} &= \frac{\sqrt{\pi}(2q+1)!}{4^{q-p}(q-p)!} \\ J_{2p,2q+1} &= 0. \end{aligned}\right\} \quad p, q \in \mathbb{N}_0, \qquad p \leq q$$

In every term of I_{nj}, (33), we can perform with (34) also the μ-integration:

$$(2q+1) \cdot J_{2p,2q} \int_0^1 d\mu \chi'^q = 2^{2p} \frac{q!^2}{(q-p)!} \chi^q \sqrt{\pi},$$

$$J_{2p+1,2q+1} \int_0^1 d\mu \frac{\chi'^{q+1}}{1-\mu^2} = -2^{2p} \frac{q!^2}{(q-p)!} \chi^{q+1} \sqrt{\pi}.$$

And with the z-integration in (33) over $\mathbb{R}$:

$$\int_0^1 d\mu I_{nj} = 2^{2j} \sum_{q=j}^{\infty} \frac{q!^2}{(q-j)!} \chi^q [f_{nj} - 2j\chi - \tfrac{2}{3}(q-j)\chi] \sqrt{\pi}.$$

Finally we get for the decay rate (32) with (31):

$$\Gamma = \frac{f^2}{16\pi E_n} \sum_{j=0}^{n} \binom{n}{j} \frac{2^j}{j!} \sum_{q=j}^{\infty} \frac{q!^2}{(q-j)!} \chi^q [1 - (2n+1)\chi - \tfrac{2}{3}(q-j)\chi]$$

$$= \Gamma_0 \sum_{j=0}^{n} \binom{n}{j} \frac{2^j}{j!} \sum_{u=0}^{\infty} \frac{(j+u)!^2}{u!} \chi^{j+u} [1 - (2n+1)\chi - \tfrac{2}{3}u\chi], \tag{35}$$

where Γ_0 is the free decay rate (3) for a muon with energy E_n.

(a) *Expansion in the nonrelativistic case*

Because of (6), (28a) we have $n \ll 1/\chi$, and

$$\chi \approx \frac{eB}{m^2} = \frac{B}{B_{cr}}$$

[see (18)] can be taken as a parameter of the expansion. We get then from (35) the first non-vanishing correction:

$$\Gamma \cong \Gamma_0[1 + \tfrac{1}{3}(B/B_{cr})^2]. \tag{36}$$

(b) *Expansion in the relativistic case*

According to (6) and (28a) we have for $p_z = 0$, $\beta = \sqrt{1 - (1/\gamma^2)}$:

$$2n + 1 = \frac{E_n^2 - m^2}{eB} = \frac{\beta^2}{\chi}, \quad \text{or} \quad (n + \tfrac{1}{2})\chi = \tfrac{1}{2}\beta^2. \tag{37}$$

So n has the same order of magnitude as $1/\chi$. We expand (35) systematically until $\sigma(1/n^2)$, taking into consideration that $1-(2n+1)\chi = 1-\beta^2$:

$$\Gamma \cong \Gamma_0(1-\beta^2)\sum_{j=0}^{n}\frac{n!}{(n-j)!}(2\chi)^j \times\left\{1+(j+1)^2\left[\chi+\left(\frac{(j+2)^2}{2}-\frac{2}{3}\gamma^2\right)\chi^2\right]\right\}; \tag{38}$$

$$(2\chi)^j = \left(\frac{\beta^2}{n+1/2}\right)^j \cong \left(\frac{\beta^2}{n}\right)^j\left[1-\frac{j}{2n}+\frac{j(j+1)}{8n^2}\right],$$

$$\frac{n!}{(n-j)!}\left(\frac{1}{n}\right)^j = \frac{n(n-1)(n-2)\cdots(n-j+1)}{n\cdot n\cdot n\cdots n} \cong 1-\frac{j(j-1)}{2n}+\frac{j(j-1)(j-2)(3j-1)}{24n^2}.$$

By substituting into (38) and changing the summation index we get

$$\Gamma \cong \Gamma_0\left\{1+\frac{1}{12n^2}\left[-1+\sum_{j=0}^{\infty}\beta^{2j}(j-1)^2\right]\right\}. \tag{39}$$

Here we extend the summation to ∞ (instead of n), also to compute the zero order Γ_0. This means that we neglect β^{2n} with respect to 1. This is straightforward in the relativistic case and with $B_{cr}/B \gtrsim 10^5$ [see (19a) and section before (19)] because of (37) and $\chi = \dfrac{1}{\gamma^2}\dfrac{B}{B_{cr}}$ [(18), (28*a*)]:

$$\beta^{2n} = (1-1/\gamma^2)^{1/2[\beta^2\gamma^2(B_{cr}/B)-1]} < e^{-1/2[\beta^2(B_{cr}/B)-1/\gamma^2]}. \tag{40}$$

We evaluate (39) (with $\gamma^2 = 1/1-\beta^2$):

$$S = \sum_{j=0}^{\infty}\beta^{2j}(j-1)^2 = \gamma^2\left[\sum_{j=0}^{\infty}\beta^{2j}(j-1)^2-\sum_{j=0}^{\infty}\beta^{2(j+1)}(j-1)^2\right]. \tag{41}$$

For the last summation we write $\sum_{j=1}^{\infty}\beta^{2j}\cdot j^2$ and consolidate. In the new summation we also compute with the same change of index and so obtain in two steps for (41):

$$S = \gamma^2\left[4+\sum_{j=0}^{\infty}(2j-3)\beta^{2j}\right] = \cdots = 4\gamma^2-5\gamma^4+2\gamma^6$$

and so for (39):

$$\Gamma = \Gamma_0\left[1+\frac{1}{12n^2}(\gamma^2-1)^2(2\gamma^2-1)+O(1/n^3)\right]. \tag{42}$$

With the transformation of the exponent of (40) and $\beta^2\gamma^2 = \gamma^2-1$ we can write

$$\frac{\gamma^2-1}{2n+1}=\frac{B}{B_{cr}}.$$

This leads in (42) to our first non-vanishing correction due to the magnetic field of the order $1/n^2$ or $(B/B_{cr})^2$:

$$\Gamma \cong \Gamma_0\left[1+\frac{2\gamma^2-1}{3}\left(\frac{B}{B_{cr}}\right)^2\right]. \tag{43}$$

For $\gamma = 1$ we obviously get (36).

5. Conclusions

We notice that the results (36), (43) contain no first-order correction to the parameter B/B_{cr} (resp. $\gamma \cdot B/B_{cr}$). This means that up to the order $\gamma \cdot B/B_{cr}$ *only the energy* (6) is responsible for the decay-rate (and so for the clock-behaviour) of the muon. In the simplest case $p_z = 0$, $n = 0$ we have $E = m\sqrt{1 + B/B_{cr}}$, which means that the energy given by the magnetic field $\approx mB/2B_{cr}$ contributes in the first order to a larger lifetime in the same way that an equal kinetic energy $(\gamma = \sqrt{1 + B/B_{cr}})$ would in the free case.

So it seems to be a general fact that the time-keeping of this kind of 'clock' depends essentially on its energy, with $\Delta t = \frac{E}{E_0} \cdot \Delta\tau$, where $\Delta\tau$ is the proper time of the "clock", E_0 its rest energy in a field-free space and E its total energy: independently of whether the increase of energy comes from a kinetic energy or from a "zero-point-energy" in a magnetic field! Note however that in the case of (negative) gravitational energy because of the gravitational redshift exactly the opposite result is true: $\Delta t = \frac{E_0}{E} \cdot \Delta\tau$.

The appearance in the second-order correction of the parameter $\gamma \cdot B/B_{cr}$ is actually very small due to the fact that the critical magnetic field of the electron (which is in view of the section before (19) almost 5 orders of magnitude smaller than that of the muon) constitutes a natural limit for B. The remaining correction can be understood by the interaction of the particle with the magnetic field. One could say that the field 'induces' some additional muon-decay.

The γ-factor in the correction-parameter can be simply explained by the Lorentz-transformation, because $\gamma \cdot B$ is the magnetic field 'seen' by a particle moving perpendicularly to the magnetic field.

Evidently the 'clock hypothesis' explained in the introduction seems to be (almost) perfectly true! To give some illustration: The g-2-experiment at CERN [7] with orbiting muons having $\gamma \sim 30$ had as a by-product that the time dilatation of Special Relativity was tested and confirmed with an accuracy of 10^{-3}. But the deviation computed here from the behaviour of an ideal clock would be [with

(19a) and (43)] less than 10^{-25}, although the muons are experiencing in their orbit a centripetal acceleration of 10^{18} g!!

Finally a more speculative extension of our considerations to the very extreme conditions of astrophysics: Near radio-pulsars (quickly rotating neutron-stars) magnetic fields up to $2\cdot 10^{13}$ Gauss $= \frac{1}{2}B_{cr}(e^-)$ are estimated to exist. Moreover it is believed that there is an e^+e^--plasma with γ-factors up to 10^6. With a possible transition into $\mu^+\mu^-$, a γ-factor of almost 10^4 would result for the muons. In this case $\gamma \cdot B/B_{cr}$ would be ≈ 10 per cent and the correction (43) would now give almost 1 per cent! But the most interesting point of this calculation surely consists not in any possible application like this but rather in the possibility in principle to verify the clock hypothesis in this special case with the help of an accepted physical theory.

Acknowledgements

My special thanks go to Prof. N. Straumann who proposed this very interesting problem to me and who gave me the key idea for the solution.

I am also very grateful to Prof. G. Scharf for his valuable suggestions which greatly improved my methods of calculation.

Discussions held together with Dr. Max Camenzind, Dr. Andreas Wipf, Dr. Johannes Vigfussaon, Dr. Juri M. Pismak and others were very inspiring for my work.

Finally I want to thank Dr. Brian Linard and Dr. Bernard Kay for their indispensable contribution in editing my English translation.

REFERENCES

[1] H. MEYER, *Physik in unserer Zeit,* **9** (1978) 69.
[2] D. W. SKOBELZYN, Das Zwillingsparadoxon in der Relativitätstheorie, Akademie-Verlag Berlin, 1972, S. 172ff.
[3] A. EINSTEIN, The Meaning of Relativity, Methuen & Co. Ltd. London, 1950.
[4] J. BAILEY et al., *Nucl. Phys.* **B150** (1979) 1.
[5] L. D. LANDAU und E. M. LIFSCHITZ, Lehrbuch der Theoretischen Physik, Bd. III, Quantenmechanik, Akademie-Verlag Berlin, 1979.
[6] J. GÉHÉNIAU, *Physica* **16** (1950) 822.
[7] J. BAILEY et al., *Nature* **268** (1977) 301.