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Objekttyp: **Article**

Zeitschrift: **Helvetica Physica Acta**

Band (Jahr): **51 (1978)**

Heft 5-6

PDF erstellt am: **21.09.2024**

Persistenter Link: https://doi.org/10.5169/seals-114966

Helvetica Physica Acta, Vol. 51 (1978), Birkhäuser Verlag, Basel

Space-like fields and non-unitary representations of the Poincaré group[1])

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(8. VI. 1978; rev. 1. VIII. 1978)

Abstract. We construct and study the non unitarizable representations of the Poincaré group associated with space-like fields. These representations are constructed on the space $\mathfrak{S}_{1/2}^{1/2}$ of Guelfand and Shilov and they have the same spin properties as the usual ones.

I. Introduction

In Reference [3], one of us has introduced a space-like theory in an attempt to understand the contemporary problem of quark confinement. The structure of the singularities of this space-like field theory obliges one to introduce a new Fourier stable test functions space of the Guelfand type $\mathfrak{S}_{1/2}^{1/2}$. This is indeed what was done in Reference [1]. As one easily remarks from the explicit form of the Green functions the theory possesses a certain phase symmetry. From this fact, and also from the fact that $\mathfrak{S}_{1/2}^{1/2}$ enters naturally in our framework follow that *certain* complex mass representations of the Poincaré group will play a role in our theory.

We do not claim that one will be *forced* to utilize them instead of the unitary ones. We just say that they appear naturally in our framework, and since, in addition the gluon described by them does not manifest itself asymptotically as a free particle (this is the whole idea of confinement!), there is *no reason* why it should be described by a *unitary* representation of the Poincaré group.

To make our argument more concrete, let us deliberately demonstrate it on spinless gluons described by the Klein–Gordon equation.

The Klein–Gordon equation presents some symmetries which are of relevance for the theory of space-like fields. These symmetries also play a role in the discussion of the behavior of Green's functions.

We define the transformations

$$T_\varphi \begin{cases} x^\mu \to e^{i\varphi} x^\mu \\ m \to e^{i(\pi-\varphi)} m, \quad \varphi \in \mathbb{R} \end{cases}$$

These transformations multiply the Klein–Gordon operator by a phase factor $e^{-2i\varphi}$.

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Consider, for example, the Green's functions for the scalar fields

$$D(x, m) = \frac{-i}{(2\pi)^3} \int e^{ikx} \varepsilon(k^0)\, \delta(k^2 - m^2)\, dk$$

Define

$$E_\varphi(x, m) = e^{2i\varphi} D(e^{i\varphi} x, e^{i(\pi - \varphi)} m)$$

An explicit computation shows that E_φ is independent of φ and the definition coincides with that given in [1] and [3].

The same remark applies to other Green's functions

$$D^{\mathrm{ret}}(x) = \theta(x^0) D(x); \qquad D^{\mathrm{adv}}(x) = -\theta(-x^0) D(x)$$

$$D^+(x) = \frac{-i}{(2\pi)^3} \int e^{ikx} \theta(k^0)\, \delta(k^2 - m^2)\, dk$$

$$D^-(x) = \frac{i}{(2\pi)^3} \int e^{ikx} \theta(-k^0)\, \delta(k^2 - m^2)\, dk$$

$$D^c(x) = \theta(x^0) D^-(x) - \theta(-x^0) D^+(x), \text{ etc.}$$

Remark that $E_\varphi^c(x) = e^{2i\varphi} D^c(e^{i\varphi} x, e^{i(\pi - \varphi)} m)$ is causal in the sense of Stueckelberg [6].

It is therefore possible to make a decent field theory utilizing, instead of the solution behaving like $r^{-1} e^{-mr}$ of the Klein–Gordon equation, the solutions behaving like $r^{-1} e^{+mr}$ obtained from the first ones by applying $T_{\varphi=0}$. At the formal level one sees that the theory is still Poincaré covariant, but the corresponding representations of the Poincaré group are not the usual ones.

In what follows, we shall study some representations of the Poincaré group corresponding to the previous transformations.

When $\varphi \notin \pi\mathbb{Z}$, the square mass m^2 is not real and therefore, the representation cannot be unitary (i.e. not unitarizable). In this situation, the choice of the space, algebraic and topological, is not unique. However, a previous study [1] showed that there is a natural space where it is possible to develop a coherent theory of space-like fields: the space of Guelfand and Shilov $\mathfrak{S}_{1/2}^{1/2}$ [2]. $\mathfrak{S}_{1/2}^{1/2}(\mathbb{R})$ is the space of C^∞ functions, restriction to $\mathbb{R}$ of entire functions g on $\mathbb{C}$ satisfying

$$|g(x + iy)| < c_1 \exp(-c_2|x|^2 + c_3|y|^2)$$

for some positive constants c_1, c_2 and c_3. $\mathfrak{S}_{1/2}^{1/2}$ is a nuclear Fréchet space which is stable by Fourier transform and is contained in the Schwartz space $\mathfrak{S}$.

We are going to represent the Poincaré group on $\mathfrak{S}_{1/2}^{1/2}(\mathbb{R}^3)$. Nonunitary representations of the Poincaré group have been studied in [4] and [5]. In these articles, the representation space is a space of differentiable functions with compact support; therefore, we cannot directly apply their results.

The Poincaré group is a semi-direct product: $\mathscr{P} = T^4 \cdot SL(2, C)$. It is possible to construct some of its representations by induction.

II. Computation of the representations

(a) *Orbits*

The first step leading to the construction of these representations consists in giving the orbits of the action of the Poincaré group on the complexified $\hat{T}^4_{\mathbb{C}}$ of the dual $\hat{T}^4$ of T^4. These orbits have invariants: if $P = p + iq \in \hat{T}^4_{\mathbb{C}}$, P^2 is an invariant, that means that $p^2 - q^2$ and $p \cdot q$ are invariants, and therefore, P being connected, p^2 and q^2 are also invariants.

We are interested in representations where $P^2 = m^2 = e^{-2i\varphi} m_0^2$, ($m_0 \in \mathbb{R}$) and $P^\mu = e^{i\varphi} P_0^\mu$, where P_0 is a real vector (this corresponds to the transformation $x^\mu \to e^{-i\varphi} x^\mu$ with $P^\mu = i(\partial/\partial x^\mu)$).

We wish to get a field theory which has the same spin properties as the usual field theory. For instance, if one requires the existence of a spin space of finite dimension with the right properties under the rotations, the only possible induced representations are those for which p and q are linearly dependent on the orbit and time like. The little group is the $SU(2)$, as can be seen from the list of the orbits given in [4].

Moreover, these representations can be obtained from the unitary irreducible representations by deformation on $\mathfrak{S}^{1/2}_{1/2}(\mathbb{R}^3)$. However, though their reduction to $SU(2)$ is the same as the unitary ones (representations of a compact group are not deformable) these representations of $\mathscr{P}$ are inequivalent to the unitary ones.

In our case, if we put

$$p^2 = \rho^2 \cos^2 \theta, \qquad q^2 = \rho^2 \sin^2 \theta, \qquad p \cdot q = \rho^2 \sin \theta \cos \theta,$$

we have a supplementary invariant

$$\operatorname{sgn} p_0 = \operatorname{sgn} (\cos \theta), \quad \text{if } \theta \neq \frac{\pi}{2}, \quad \text{and} \quad \operatorname{sgn} q_0 = \operatorname{sgn} (\sin \theta), \quad \text{if } \theta = \frac{\pi}{2}$$

This corresponds to the sign of the energy of the unitary irreducible representations of $\mathscr{P}$ $(q = 0)$.

(b) *Representations*

A given orbit $\mathcal{O}$ can be identified to the homogeneous space $H\backslash\mathscr{P}$, where $H = T^4 \cdot G_0$, G_0 being the stabilizer of a point $n_0 \in \mathcal{O}$.

This identification j: $\mathcal{O} \to H\backslash\mathscr{P}$ is defined by $j(n) = Hg$, where g is an element of $\mathscr{P}$ such that $n = n_0 g$. We denote by Π the canonical projection $\Pi: \mathscr{P} \to H\backslash\mathscr{P} \simeq \mathcal{O}$. There exists a differentiable section of $\Pi(\omega_0 \Pi = Id_{\mathcal{O}})$.

Suppose given an orbit $\mathcal{O}$ and our irreducible representation of $SU(2)$ on a complex vector space $\mathfrak{H}$ (which is then of finite dimension). $\mathcal{O}$ is isomorphic to a one sheet hyperboloïd and therefore to $\mathbb{R}^3$ by projection. We can define the space $\mathfrak{S}^{1/2}_{1/2}(\mathcal{O}, \mathfrak{H})$ of functions of the type $\mathfrak{S}^{1/2}_{1/2}$ defined on $\mathcal{O}$ with values in $\mathfrak{H}$.

Let's now define the induced representation U^{L,n_0} on $\mathfrak{H}^L = \mathfrak{S}^{1/2}_{1/2}(\mathcal{O}, \mathfrak{H})$, with $g = (a, \Lambda) \in \mathscr{P}$ and n_0 a chosen point on $\mathcal{O}$, by

$$(U_g^{L,n_0} f)(x) = L(\omega(x) \wedge \omega(x\Lambda)^{-1}) \exp(i(n_0 \omega(x)) \cdot (a\omega(x))^{-1}) f(x\Lambda) \tag{1}$$

This defines a continuous representation of G on $\mathfrak{H}^L$.

We shall now verify the inequivalence of these representations. To do this, we shall utilize the method of sesquilinear invariant functionals. Given two representations $(U^{L_1,n_1}, \mathfrak{H}^{L_1})$ and $(U^{L_2,n_2}, \mathfrak{H}^{L_2})$ of $\mathscr{P}$, a sesquilinear functional is a sesquilinear continuous mapping:

$$B: (\mathfrak{H}^{L_1})' \times \mathfrak{H}^{L_2} \to \mathbb{C}$$

such that

$$B({}^tU_{g^{-1}}^{L_1,n_1}f, U_g^{L_2,n_2}h) = B(f, h); \qquad f \in (\mathfrak{H}^{L_1,n_1})'; h \in \mathfrak{H}^{L_2,n_2}, \quad g \in \mathscr{P}$$

Denote by $\mathcal{O}_i$ the orbit containing n_i $(i = 1, 2)$. Using the translation invariance, one gets, if $f_1 \in \mathfrak{S}_{1/2}^{1/2}(\mathcal{O}_1, \mathfrak{H}_1)'$ and $f_2 \in \mathfrak{S}_{1/2}^{1/2}(\mathcal{O}_2, \mathfrak{H}_2)$, $\mathfrak{H}_i$ being the space of L_i:

$$B((\exp i\{(n_2\omega(x_2))\cdot(a\omega(x_2)^{-1}) - (n\omega(x_1))\cdot(a\omega(x_1)^{-1})\} - 1)f_1(x_1)\otimes f_2(x_2))$$

Therefore, if n_1 and n_2 are not on the same orbit, we have necessarily $B = 0$. Suppose that there exists an intertwining operator $A: \mathfrak{H}^{L_2} \to \mathfrak{H}^{L_1}$, it defines a sesquilinear invariant functional

$$B(f_1, f_2) = ({}^tAf_1, f_2)$$

Therefore, if n_1 and n_2 are not on the same orbit, U^{L_1,n_1} and U^{L_2,n_2} are inequivalent.

It remains to look at the case where n_1 and n_2 are on the same orbit of $\mathscr{P}$ in $\hat{T}_{\mathbb{C}}^4$. Denote by $V^{L,n}$ the restriction of $U^{L,n}$ to $SU(2)$. It results from (1) that $V^{L,n}$ extends a unique way to a unitary representation $\overline{V}^{L,n}$ of $SU(2)$ on $L^2(\mathbb{R}^3, \mathfrak{H}, d^3p/p_0)$, where $\mathfrak{H}$ is the space of the representation L. But formula (1) shows that $\overline{V}^{L,n}$ is the restriction to $SU(2)$ of the unitary irreducible representation of $\mathscr{P}$, on $L^2(\mathbb{R}^3, \mathfrak{H}, d^3p/p_0)$ induced by $(L, \mathfrak{H})$ and U^1 and U^2 of $\mathscr{P}$ with $m^2 > 0$. It is known that given two unitary representations U^1 and U^2 of $\mathscr{P}$ with $m^2 > 0$, $U^1|_{SU(2)}$ and $U^2|_{SU(2)}$ are equivalent if and only if the inducing representations L_1 and L_2 are equivalent.

Consequently, $\overline{V}^{L_1,n_1}$ and $\overline{V}^{L_2,n_2}$ are inequivalent if L_1 and L_2 are inequivalent and therefore U^{L_1,n_1} and U^{L_2,n_2} are inequivalent if L_1 and L_2 are inequivalent. Finally, if L_1 and L_2 are equivalent and n_1 and n_2 belong to the same orbit, U^{L_1,n_1} and U^{L_2} are equivalent by formula (1).

To sum up, we have the

Proposition: Given two representations L_1 and L_2 of $SU(2)$ and two points n_1 and n_2 in $\hat{T}_{\mathbb{C}}^4$, they are equivalent if and only if:

(1) n_1 and n_2 belong to the same orbit of $\mathscr{P}$.
(2) L_1 and L_2 are equivalent.

Acknowledgments

The authors want to express their gratitude to Moshé Flato for many helpful comments and suggestions.

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