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Autor(en): **Aerts, Dirk / Daubechies, Ingrid**

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About the structure – preserving maps of a quantum mechanical propositional system

by **Dirk Aerts**[1]) and **Ingrid Daubechies**[1])

Theoretische Natuurkunde, Vrije Universiteit Brussel, Pleinlaan 2, B 1050 Brussels

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Abstract. We study c-morphisms from one Hilbert space lattice (with dimension at least three) to another one; we show that for a c-morphism conserving modular pairs, there exists a linear structure underlying such a morphism, which enables us to construct explicitly a family of linear maps generating this morphism. As a special case we prove that a unitary c-morphism which preserves the atoms (i.e. maps one-dimensional subspaces into one-dimensional subspaces) is necessarily an isomorphism. Counterexamples are given when the Hilbert space has dimension 2.

1. Definition of a propositional system and Piron's representation theorem

According to Piron's axiomatic description of quantum mechanics [1], the structure of the set of the propositions corresponding to 'yes–no' experiments on a physical system is that of a complete, orthocomplemented, weakly modular and atomic lattice which satisfies the covering law. Such a lattice is called a propositional system. If the physical system has no super-selection rules, the propositional system is irreducible. We will first give some definitions concerning propositional systems. For more details we refer the reader to [1].

Let $\mathscr{L}$ be the collection of all the propositions concerning a physical system.

1.1 Definition. *$\mathscr{L}$ is called a CROC whenever $\mathscr{L}$ satisfies the following conditions:*

(i) *$\mathscr{L}$, $<$ is a partially ordered set with $<$ as partial order relation* (1.1)

(*ii*) *$\mathscr{L}$ is a complete lattice, which means that for every family $\{b_i\}_{i\in I}$ of elements in $\mathscr{L}$ there exists a greatest lower bound $\bigwedge_{i\in I} b_i$ and a least upper bound $\bigvee_{i\in I} b_i$.* (1.2)

(*iii*) *$\mathscr{L}$ is an orthocomplemented lattice, which means that there exists a bijection $'\colon \mathscr{L} \to \mathscr{L}$ such that: $\forall b, c \in \mathscr{L}$:* (1.3)

[1]) Wetenschappelijke medewerkers bij het Interuniversitair Instituut voor Kernwetenschappen (in het kader van navorsingsprogramma 21 EN).

$(b')' = b$
$b \wedge b' = 0$ *and* $b \vee b' = I$
$b < c \Rightarrow c' < b'$
where $I = \bigvee_{b \in \mathscr{L}} b$ *and* $0 = \bigwedge_{b \in \mathscr{L}} b$

(*iv*) $\mathscr{L}$ *is weakly modular, which means that if* $b < c$ *then* $c \wedge (c' \vee b) = b$ (1.4)

In a CROC it may be interesting to consider pairs of propositions. In the following definition we introduce some pair-properties (see [1], [2], [3]).

1.2 Definition. *In a CROC two propositions b and c are said to be*

– *compatible if the sublattice generated by* $\{b, b', c, c'\}$ *is distributive.*
notation: $b \leftrightarrow c$
– *a modular pair if for any* $d > c$: $(b \vee c) \wedge d = (b \wedge d) \vee c$
notation: $(b, c)M$ [2])

It is easy to see that if $b \leftrightarrow c$, $(b, c)M$. The converse is not true. Moreover, we see now that the condition for weak modularity (1.4) can be reinterpreted as: $b < c \Rightarrow b \leftrightarrow c$.

1.3 Definition. *The 'center' of a CROC is the set of propositions compatible with all other propositions.*

The center of a CROC is also a CROC which is distributive.

1.4 Definition. *If* $\mathscr{L}$ *is a CROC and* $b \in \mathscr{L}$ *we consider the set* $\{x \mid x < b, x \in \mathscr{L}\}$. *If we define on this set a relative orthocomplementation* $x^r = x' \wedge b$, *then it is easy to check that this set becomes a CROC; we will call it the segment* $[0, b]$.

1.5 Definition. *A CROC* $\mathscr{L}$ *will be called 'irreducible' if the center of* $\mathscr{L}$ *contains only* 0 *and* I.

1.6 Definition. *If* $b, c \in \mathscr{L}$, $b \neq c$ *and* $b < c$, *one says that c 'covers' b, if* $b < x < c$ *for some* $x \in \mathscr{L}$ *implies* $x = b$ *or* $x = c$. *An element of* $\mathscr{L}$ *which covers* 0 *is called an atom.*

Now we have all the notions to define a propositional system. So let $\mathscr{L}$ be again the collection of all the propositions concerning a physical system.

1.7 Definition. $\mathscr{L}$ *is a propositional system if*

(*i*) $\mathscr{L}$ *is a CROC*
(*ii*) $\mathscr{L}$ *is atomic, which means that for every* $b \in \mathscr{L}$, $b \neq 0$, *there exists an atom* $p \in \mathscr{L}$ *such that* $0 < p < b$. (1.5)
(*iii*) $\mathscr{L}$ *satisfies the covering law, which means that if p is an atom of* $\mathscr{L}$ *and* $b \in \mathscr{L}$ *and* $p \wedge b = 0$, *then* $p \vee b$ *covers b.* (1.6)

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[2]) In [2] Birkhoff defines two kinds of these pairs: modular pairs and dual modular pairs. In the following we shall only need one of these two kinds; since no confusion can arise, we shall call them modular pairs. Moreover, one can prove that when the CROC is an irreducible propositional system isomorphic to a $P(\mathscr{H})$ (see further) every modular pair is a dual modular pair in Birkhoff's terminology, and vice versa. For more details concerning these pairs, see [3].

Example: If we take a complex Hilbert space $\mathscr{H}$ and if we define $P(\mathscr{H})$ to be the collection of all the closed subspaces of $\mathscr{H}$, then $P(\mathscr{H})$ becomes an irreducible propositional system if we define the operations as follows:

(i) If $G, F \in P(\mathscr{H})$, then $G < F$ iff. $G \subset F$ set-theoretically. (1.7)
(ii) If $G_i \in P(\mathscr{H})$ $\forall i \in I$, then $\wedge_{i \in I} G_i = \bigcap_{i \in I} G_i$, (1.8)
(iii) and $\vee_{i \in I} G_i$ is the closure of the linear subspace generated by all the G_i's. (1.9)
(iv) If $G \in P(\mathscr{H})$ then $G' = G^{\perp}$ which is the subspace orthogonal to G. (1.10)
(v) The atoms are the one-dimensional subspaces. (1.11)

$P(\mathscr{H})$ is the propositional system one uses in ordinary quantum mechanics. As expected two propositions of $P(\mathscr{H})$ are compatible iff the corresponding projection operators commute.

Conversely, we can ask ourselves if it is always possible to represent an irreducible propositional system by a structure which resembles the given example. This question is answered by the representation theorem of Piron ([1] and [4]) which says that it is always possible to realize an irreducible propositional system by the set $P(V)$ of all biorthogonal subspaces of a vector space V over some field $\mathbb{K}$. The orthocomplementation defines on $\mathbb{K}$ an involutive anti-automorphism and on V a non-degenerate sesquilinear form; the weak modularity ensures that the whole space is linearly generated by any element and the corresponding orthogonal subspace. Hence all irreducible propositional systems are given by generalization of the $P(\mathscr{H})$ in the example above. The representation theorem enables us to prove interesting results about propositional systems. As an example we give the following easy characterization of modular pairs in an irreducible propositional system.

1.8 Lemma. *Let $P(V)$ be the realization of an irreducible propositional system $\mathscr{L}$; let $a, b \in \mathscr{L}$ and let a_1, b_1 be their realizations in $P(V)$. Then*

$$(a, b)M \Leftrightarrow a_1 + b_1 = a_1 \vee b_1.$$

Proof: see [3].

An immediate consequence of the lemma is the following property: $(a, b)M \Rightarrow (b, a)M$. If we take the field in Piron's representation theorem to be $\mathbb{C}$ and the involutive anti-automorphism of $\mathbb{C}$ to be the conjugation, then the vector space V becomes an Hilbert space over $\mathbb{C}$ ([4], [5]). Since the set of all biorthogonal subspaces of a Hilbert space is exactly $P(\mathscr{H})$, we are now reduced to the case considered in the example. In the following we will restrict ourselves to the cases where the field is $\mathbb{C}$.

2. *m*-morphisms

2.1 Definition. *A c-morphism from a CROC $\mathscr{L}_1$ to a CROC $\mathscr{L}_2$ is a map f of $\mathscr{L}_1$ to $\mathscr{L}_2$ which preserves unions and compatible pairs, i.e.*

$$f\left(\bigvee_{i \in I} b_i\right) = \bigvee_{i \in I} f(b_i) \qquad b_i \in \mathscr{L}_1 \quad \forall i \in I \tag{2.1}$$

$$b \leftrightarrow c \Rightarrow f(b) \leftrightarrow f(c) \qquad b, c \in \mathscr{L}_1 \tag{2.2}$$

A bijective c-morphism from $\mathscr{L}_1$ to $\mathscr{L}_2$ will be called an isomorphism.

The terminology *c*-morphism is justified because we want (2.1) to be valid for every non-empty family of b_i's. Weakening condition (2.1) we find the definitions of a morphism ((2.1) is only required for finite families) and of a σ-morphism ((2.1) is only required for countable families). For a general CROC however only *c*-morphisms preserve the completeness. It is easy to prove [1] that when f is a *c*-morphism from $\mathscr{L}_1$ to $\mathscr{L}_2$ then:

$$f(0) = 0 \tag{2.3}$$

$$f(b') = f(b)' \wedge f(I) \qquad b \in \mathscr{L}_1 \tag{2.4}$$

$$f\left(\bigwedge_{i\in I} b_i\right) = \bigwedge_{i\in I} f(b_i) \qquad b_i \in \mathscr{L}_1 \quad \forall i \in I \tag{2.5}$$

2.2 Definition. *A map f from a CROC $\mathscr{L}_1$ to a CROC $\mathscr{L}_2$ is called an m-morphism if it is a c-morphism which preserves modular pairs, i.e. $(b, c)M \Rightarrow (f(b), f(c))M$.*

If we want to study *c*-morphisms between propositional systems, we can in general restrict ourselves to *c*-morphisms between irreducible propositional systems. Indeed, let $\mathscr{L}_1$ and $\mathscr{L}_2$ be two propositional systems with centers Z_1, Z_2 respectively. One can prove that Z_1 and Z_2 are also propositional systems [1]. Let $(z_\alpha^1)_{\alpha\in A}, (z_\beta^2)_{\beta\in B}$ be the sets of atoms of Z_1, Z_2.

If f is a *c*-morphism from $\mathscr{L}_1$ to $\mathscr{L}_2$, we can define

$$f_{\alpha\beta}\colon [0, z_\alpha^1] \to [0, z_\beta^2]$$
$$x \mapsto f(x) \wedge z_\beta^2$$

These $(f_{\alpha\beta})_{\alpha\in A, \beta\in B}$ are *c*-morphisms between irreducible propositional systems and the set of $f_{\alpha\beta}$ determines completely f:

$$\forall x \in \mathscr{L}_1 : f(x) = \bigvee_{\alpha\in A} \bigvee_{\beta\in B} f_{\alpha\beta}(x \wedge z_\alpha^1)$$

If f is an *m*-morphism, it is easy to check that the $f_{\alpha\beta}$ are *m*-morphisms too. We will henceforth restrict ourselves to the study of *c*-morphisms (or *m*-morphisms) from one irreducible propositional system to another one.

It is interesting to remark that any *c*-morphism from an irreducible CROC $\mathscr{L}_1$ to a CROC $\mathscr{L}_2$ is either injective or the zero-morphism (see [1], pp. 31, 33).

2.3 Definition. *A unitary c-morphism f of a CROC $\mathscr{L}_1$ into a CROC $\mathscr{L}_2$ is a c-morphism such that $f(I_1) = I_2$.* (2.6)

In the study of *c*-morphisms we can always restrict ourselves to unitary *c*-morphisms because if

$f\colon \mathscr{L}_1 \to \mathscr{L}_2$ is a *c*-morphism then we can always study f by studying the morphism $\tilde{f}$ defined by

$$\tilde{f} = \mathscr{L}_1 \rightarrow [0, f(I)] \quad \text{such that} \quad \tilde{f}(b) = f(b) \quad \text{for } b \in \mathscr{L}_1$$

This $\tilde{f}$ is a unitary c-morphism.

Taking together all these remarks, and remembering that we decided to consider only irreducible propositional systems isomorphic to a $P(\mathscr{H})$, we see now that we can restrict ourselves to unitary c-morphisms from an irreducible propositional system $P(\mathscr{H})$ into an irreducible propositional system $P(\mathscr{H}')$.

It is a remarkable fact that when the dimensions of the Hilbert spaces $\mathscr{H}$ and $\mathscr{H}'$ are at least equal to 3, one can prove that any isomorphism from $P(\mathscr{H})$ into $P(\mathscr{H}')$ can be generated by a unitary or anti-unitary map from $\mathscr{H}$ into $\mathscr{H}'$. This is a consequence of the following theorem proved by Wigner [6].

2.4 Theorem. *Let $\mathscr{H}$ and $\mathscr{H}'$ be two complex Hilbert spaces of dimension at least 3 and f: $P(\mathscr{H}) \rightarrow P(\mathscr{H}')$ an isomorphism, then it is always possible to find a map U: $\mathscr{H} \rightarrow \mathscr{H}'$ which is unitary or anti-unitary and such that:*

$$f(G) = \{U(x); x \in G\} \quad \forall G \in P(\mathscr{H})$$

This is not only an amazing result but it is also very useful because it is always by using this theorem that one is able to prove the deepest results about problems in relation with isomorphisms of propositional systems.

Our aim in this paper is to prove a similar result for a more general structure preserving map of a propositional system, namely an m-morphism. In fact we shall prove our main theorem with the help of apparently weaker conditions than the one given in the definition of an m-morphism. These weaker conditions are specific for atomic CROC's, where our definition of an m-morphism is valid for any CROC. The following proposition states moreover that these weaker conditions are equivalent with the fact that f is an m-morphism. To distinguish the individual elements of V from the linear subspaces they generate, and which are elements of $P(V)$, we will henceforth write $\bar{x}$ to denote the linear subspace generated by x.

2.5 Proposition. *Let $\mathscr{H}$, $\mathscr{H}'$ be two complex Hilbert spaces, let f be a c-morphism from $P(\mathscr{H})$ to $P(\mathscr{H}')$. Then f is an m-morphism iff one of the following is true:*

(1) *$\forall x, y$ non-zero vectors in $\mathscr{H}$: $f(\overline{x - y}) \subset f(\bar{x}) + f(\bar{y})$* (2.7)

(2) *$\forall x, y, z$ non-zero vectors in $\mathscr{H}$: $\bar{z} < \bar{x} \vee \bar{y} \Rightarrow f(\bar{z}) \subset f(\bar{x}) + f(\bar{y})$* (2.8)

(3) *$\forall x, y$ non-zero vectors in $\mathscr{H}$: $f(\bar{x}) + f(\bar{y})$ is a closed subspace of $\mathscr{H}'$* (2.9)

The proof of this proposition is given in the Appendix.

If the $f(\bar{x})$ are finite-dimensional, condition (2.9) is automatically satisfied, and f is an m-morphism. In particular, if the $f(\bar{x})$ are one-dimensional, f is an m-morphism. One can even prove (see the beginning of the proof of corollary 4.2) the following: if for one atom p in $P(\mathscr{H})$, $f(p)$ is an atom in $P(\mathscr{H}')$, then f is an m-morphism.

Our main theorem can now be stated as follows:

2.6 Theorem. *Let $\mathscr{H}$ and $\mathscr{H}'$ be two complex Hilbert spaces with dimension greater than or equal to three, and f a unitary m-morphism from $P(\mathscr{H})$ to $P(\mathscr{H}')$. There exists a family of maps $(\phi_j)_{j\in J}$ from $\mathscr{H}$ to $\mathscr{H}'$ such that:*

– each ϕ_j is an isometry or an anti-isometry
– the different $\phi_j(\mathscr{H})$ are orthogonal subspaces of $\mathscr{H}'$, and their direct sum is $\mathscr{H}'$:

$$\mathscr{H}' = \bigoplus_{j\in J} \phi_j(\mathscr{H})$$

– $(\phi_j)_{j\in J}$ generates f in the following sense:

$$\forall G \in P(\mathscr{H}): f(G) = \bigvee_{j\in J} \phi_j(G) = \bigoplus_{j\in J} \{\phi_j(x); x \in G\}$$

We will prove this theorem and others in Section 3. In Section 4 we restrict ourselves to the important special case of c-morphisms mapping atoms onto atoms. It is to be remarked that the theorem we obtain there is still a more general one than Theorem 2.4: our results are the same, but where Wigner supposed f to be an isomorphism, we only use that f is a unitary c-morphism mapping one atom onto an atom: no surjectivity is needed. Our different theorems hold only if dim $\mathscr{H} \geq 3$ (the same applies for Wigner's theorem); in Section 5 we give some counter-examples for dim $\mathscr{H} = 2$.

3. Construction of the underlying linear structure

We start by stating our main theorem. The formulation as presented here is rather compact: we will relate it to the former one at the end of this section.

3.1 Theorem. *Let $\mathscr{H}$ and $\mathscr{H}'$ be two complex Hilbert spaces, with dimension greater than or equal to 3. Let f be an m-morphism mapping $P(\mathscr{H})$ into $P(\mathscr{H}')$.*

Then for every couple (x, y) of non-zero elements of $\mathscr{H}$, there exists a bijective bounded linear map F_{yx} mapping $f(\bar{x})$ onto $f(\bar{y})$, such that the set of maps $\{F_{yx}; x, y \in \mathscr{H}\}$ has the following properties:

$$F_{xx} = \mathbb{1}_{f(\bar{x})}$$
$$F_{xy} = (F_{yx})^{-1}$$
$$F_{zy}F_{yx} = F_{zx}$$
$$F_{y+z,x} = F_{y,x} + F_{z,x}$$
$$F_{\lambda y,\lambda x} = F_{y,x} \quad \lambda \in \mathbb{C}, \lambda \neq 0$$
$$F_{yx} \text{ is an isomorphism if } \|x\| = \|y\|$$

For every non-zero x in $\mathscr{H}$, there exist moreover two orthogonal projections $P_1^{\bar{x}}$ and $P_2^{\bar{x}}$, elements of $\mathscr{L}(f(\bar{x}))$, such that

$$P_1^{\bar{x}} \cdot P_2^{\bar{x}} = 0$$
$$P_1^{\bar{x}} + P_2^{\bar{x}} = \mathbb{1}_{f(\bar{x})}$$
$$P_i^{\bar{y}} = F_{yx}P_i^{\bar{x}}F_{xy} \quad i = 1, 2$$

and $$F_{\lambda x,x} = \lambda P_1^{\bar{x}} + \bar{\lambda} P_2^{\bar{x}}$$

The proof of this theorem is a quite extensive one; this is why it has been split into different lemmas.

If f is zero, the $f(\bar{x})$ are zero for every $\bar{x}$ and the theorem is trivial. We therefore will restrict ourselves to the cases where f is different from zero.

3.2 Lemma. *Let $\mathscr{H}$, $\mathscr{H}'$, f be as in Theorem 3.1, with f different from the null-morphism. For every two non-zero elements x, y in $\mathscr{H}$ one can define a linear map F_{yx}, element of $\mathscr{L}(f(\bar{x}), f(\bar{y}))$, such that*

$$(a)\ F_{yx} \text{ is bijective} \tag{3.1}$$

$$(b)\ F_{xy} = (F_{yx})^{-1} \tag{3.2}$$

$$(c)\ F_{yz}F_{zx} = F_{yx} \quad \text{for every non-zero } z \text{ in } \mathscr{H} \tag{3.3}$$

$$(d)\ F_{xx} = \mathbb{1}_{f(\bar{x})} \tag{3.4}$$

$$(e)\ F_{\lambda y, \lambda x} = F_{yx} \quad \text{for every non-zero } \lambda \text{ in } \mathbb{C} \tag{3.5}$$

Proof. Since f is different from zero, it is injective. This implies that for any x in $\mathscr{H}$, different from zero, the space $f(\bar{x})$ is a subspace of $\mathscr{H}'$ different from the null-space. Let x, y be two linearly independent elements in $\mathscr{H}$.

We have $\bar{x} \wedge \bar{y} = 0$, $\bar{x} \wedge \overline{x - y} = 0$ and $\bar{y} \wedge \overline{x - y} = 0$, hence

$$f(\bar{x}) \wedge f(\bar{y}) = 0, f(\bar{x}) \wedge f(\overline{x - y}) = 0 \quad \text{and} \quad f(\bar{y}) \wedge f(\overline{x - y}) = 0 \tag{3.6}$$

Moreover $f(\bar{x}) \subset f(\bar{y}) + f(\overline{x - y})$.

If now x' is an element of $f(\bar{x})$, the previous remarks imply that there exist unique y', u', elements of $f(\bar{y})$ and $f(\overline{x - y})$ respectively, such that

$$x' = y' + u' \tag{3.7}$$

This correspondence defines a linear map F_{yx} from $f(\bar{x})$ to $f(\bar{y})$:

$$F_{yx}(x') = y'$$

This map is bijective; moreover:

$$F_{xy} = (F_{yx})^{-1}$$

We will now prove that these F_{xy} are bounded linear maps. From the already proven results we see that F_{xy} is an everywhere defined linear map from one F-space to another (both $f(\bar{x})$ and $f(\bar{y})$ are closed subspaces). We will prove that F_{xy} is closed. Indeed, let $(y'_n)_n$ be a converging sequence in $f(\bar{y})$ with limit y', such that the sequence $(x'_n = F_{xy}(y'_n))_n$ converges too: $x'_n \to x'$. Then there exists a sequence u'_n in $f(\overline{x - y})$ such that

$$u'_n = y'_n - x'_n$$

Since both sequences $(x'_n)_n$ and $(y'_n)_n$ converge, the sequence $(u'_n)_n$ is a Cauchy sequence with limit u'. Since moreover the spaces $f(\bar{x})$, $f(\bar{y})$, $f(\overline{x - y})$ are closed, we have $x' \in f(\bar{x})$, $y' \in f(\bar{y})$, $u' \in f(\overline{x - y})$. But $y' = x' + u'$, hence $x' = F_{xy}(y')$, which implies that F_{xy} is closed. Using the closed graph theorem ([7], p. 57), we see that F_{xy} is bounded.

Up till now, we have proven statements (a) and (b) for x, y linearly independent. We will proceed further, and prove statement (c) for x, y, z linearly independent, after which we will re-examine the different cases for linearly dependent vectors.

Take x, y, z linearly independent in $\mathscr{H}$. We have

$$\overline{y-z} < \bar{y} \vee \bar{z} \quad \text{and} \quad \overline{y-z} = \overline{(y-x)+(x-z)} < \overline{y-x} \vee \overline{x-z}$$

which implies

$$\overline{y-z} < (\bar{y} \vee \bar{z}) \wedge (\overline{y-x} \vee \overline{x-z})$$

Because of the linear independency of x, y, z we can conclude that

$$\overline{y-z} = (\bar{y} \vee \bar{z}) \wedge (\overline{y-x} \vee \overline{x-z}) \tag{3.8}$$

which implies

$$f(\overline{y-z}) = (f(\bar{y}) \vee f(\bar{z})) \wedge (f(\overline{x-y}) \vee f(\overline{x-z})) \tag{3.9}$$

Take now x' in $f(\bar{x})$, and define y', z' by

$$y' = F_{yx}(x'), \qquad z' = F_{zx}(x') \tag{3.10}$$

From the construction of F_{yx}, F_{zx} we know that there exist u', v' with $u' \in f(\overline{x-y})$, $v' \in f(\overline{x-z})$, such that

$$x' = y' + u' = z' + v'$$

This implies that $y' - z' = v' - u'$, hence $y' - z' \in f(\overline{x-y}) \vee f(\overline{x-z})$. Since $y' - z'$ is obviously an element of $f(\bar{y}) \vee f(\bar{z})$, we conclude from (3.9) that $y' - z' \in f(\overline{y-z})$. This implies $y' = F_{yz}(z')$.

Because of the definitions (3.10) of y', z', this implies

$$F_{yz} \circ F_{zx} = F_{yx},$$

which proves statement (c) for three linearly independent vectors.

Whenever x and y are linearly dependent, i.e. $y \in \bar{x}$, we define F_{yx} by

$$F_{yx} = F_{yt} \circ F_{tx} \quad \text{where} \quad t \in \mathscr{H} \backslash \bar{x}. \tag{3.11}$$

The fact that this definition of F_{yx} does not depend on the choice of t is an almost trivial application of the just proved chain rule for linearly independent vectors. The bijectivity of F_{yx} follows immediately from its definition (3.11). It is also easy to check that statements (b) and (c) hold even when the vectors are not linearly independent. Statement (d) is now a trivial consequence of (b) and (3.11). Statement (e) is a trivial consequence of the construction of the F_{xy}.

∎

Remark. It is a crucial point in this proof that $\dim \mathscr{H} \geq 3$. If $\dim \mathscr{H} = 2$, one can still construct the F_{xy} in the same way as was done in the lemma, but it is then impossible to prove the chain rule. We give a counter-example in the last section.

3.3 Lemma. *Let $\mathscr{H}$, $\mathscr{H}'$, f be as in Lemma 3.2. Let $\{F_{xy}; x, y \in \mathscr{H}, x \neq 0 \neq y\}$ be the set of maps constructed in the proof of Lemma 3.2, and let x, y, z be three non-zero vectors in $\mathscr{H}$ with $y + z \neq 0$. Then the following holds:*

for every x' *in* $f(\bar{x})$: $F_{yx}(x') + F_{zx}(x') \in f(\overline{y+z})$

and $F_{yx} + F_{zx} = F_{y+z,x}$ (3.12)

Proof. Suppose first that x, y, z are linearly independent. We rewrite (3.8) in two different forms:

$$\overline{y+z} = (\bar{y} \vee \bar{z}) \wedge (\bar{x} \vee \overline{x-y-z}) \tag{3.13}$$

and

$$\overline{x-y-z} = (\overline{x-y} \vee \bar{z}) \wedge (\overline{x-z} \vee \bar{y}) \tag{3.14}$$

Take x' in $f(\bar{x})$, and define y', z' by

$$y' = F_{yx}(x'), \qquad z' = F_{zx}(x') \tag{3.15}$$

Since $x' - y' \in f(\overline{x-y})$ and $x' - z' \in f(\overline{x-z})$, we have

$$x' - y' - z' \in (f(\overline{x-y}) \vee f(\bar{z})) \wedge (f(\overline{x-z}) \vee f(\bar{y})) = f(\overline{x-y-z})$$

(because of (3.14))

Hence

$$y' + z' = x' - (x' - y' - z') \in f(\bar{x}) \vee f(\overline{x-y-z})$$

which implies

$$y' + z' \in (f(\bar{y}) \vee f(\bar{z})) \wedge (f(\bar{x}) \vee f(\overline{x-y-z})) = f(\overline{y+z}) \quad \text{(because of (3.13))}$$

We have now

$$x' = y' + z' + x' - y' - z'$$

with

$$y' + z' \in f(\overline{y+z}), \qquad x' - y' - z' \in f(\overline{x-y-z}),$$

hence

$$F_{y+z,x}(x') = y' + z'.$$

Because of (3.15) this implies

$$F_{y+z,x} = F_{yx} + F_{zx} \quad \text{for } x, y, z \text{ linearly independent}$$

Suppose now that z and y are linearly dependent, i.e. $y \in \bar{z}$ with $y + z \neq 0$, and suppose $x \notin \bar{y}$. There exists a t such that $\{x, t, y+z\}$ and $\{x, t+y, z\}$ are both linearly independent.

We have

$$\begin{aligned} F_{t+y+z,x} &= F_{t+y,x} + F_{zx} = F_{tx} + F_{yx} + F_{zx} \\ &= F_{tx} + F_{y+z,x} \end{aligned}$$

which implies

$$F_{y+z,x} = F_{yx} + F_{zx}.$$

We have only one more case to check: suppose $x \in \bar{y} \vee \bar{z}$ and $y + z \neq 0$ (y and z

may be linearly dependent). We can choose a t such that $t \notin \bar{y} \vee \bar{z}$. We have

$$\begin{aligned} F_{y+z,x} &= F_{y+z,t}F_{tx} = F_{yt}F_{tx} + F_{zt}F_{tx} \\ &= F_{yx} + F_{zx} \end{aligned}$$

∎

3.4 Lemma. *Let $\mathscr{H}$, $\mathscr{H}'$, f, $\{F_{xy}\}$ be as in Lemma 3.3. Let x, y be two non-zero vectors in $\mathscr{H}$ with $\|x\| = \|y\|$. Then F_{yx} is an isomorphism.*

Proof. Consider first the case where $x \perp y$. Since $\|x\| = \|y\|$, this implies $x - y \perp x + y$ and thus $f(\overline{x - y}) \perp f(\overline{x + y})$. Take x', x'' in $f(\bar{x})$ and define y', y'' in $f(\bar{y})$ by

$$y' = F_{yx}(x') \qquad y'' = F_{yx}(x'') \tag{3.16}$$

Because of Lemma 3.3 we know that

$$x' + y' \in f(\overline{x + y}) \qquad x'' - y'' \in f(\overline{x - y}) \tag{3.17}$$

From (3.16), (3.17) and the fact that $f(\bar{x}) \perp f(\bar{y})$, we infer that

$$0 = (x' + y', x'' - y'') = (x', x'') - (y', y'')$$

But this leads to

$$(F_{yx}(x'), F_{yx}(x'')) = (x', x'')$$

which implies that F_{yx} is an isomorphism.

If x and y are not orthogonal, we can choose a vector z such that

$$\|z\| = \|x\| = \|y\| \quad \text{and} \quad z \perp x, z \perp y$$

Because of the previous result, we know that F_{zx} and F_{yz} are isomorphisms. From the chain rule (3.3) it follows now immediately that $F_{yx} = F_{yz} \circ F_{zx}$ is an isomorphism too.

∎

We have now proven the first part of Theorem 3.1. To prove the second part, which gives in fact a spectral decomposition of the operators $F_{\lambda x, x}$, we need the following two remarks.

3.5 Remarks.

1. *Let $\mathscr{H}$ be a Hilbert space, and let G, H be two commuting bounded self-adjoint operators on $\mathscr{H}$ with $G \geq H$. Let $\sigma(G)$ be the spectrum of the operator G and $\sigma(H)$ be the spectrum of the operator H. Then* $\inf \sigma(G) \geqslant \inf \sigma(H)$ *and* $\sup \sigma(G) \geq \sup \sigma(H)$.
2. *Let $\mathscr{H}$ be a Hilbert space, A a bounded normal operator such that $A^2 = -\mathbb{1}$. Then $\sigma(A) \subset \{-i, i\}$.*

(These remarks are easy to prove if one uses the Gel'fand isomorphism for commutative C^*-algebras: see [8].)

3.6 Lemma. *Let $\mathscr{H}$, $\mathscr{H}'$, f, $\{F_{yx}\}$ be as in Lemma 3.3. For every x in $\mathscr{H}$, $x \neq 0$, there* *exist two orthogonal projectors* $P_1^{\bar{x}}$, $P_2^{\bar{x}}$ *in* $\mathscr{L}(f(\bar{x}))$, *such that the following holds:*

(*a*) $P_1^{\bar{x}} \cdot P_2^{\bar{x}} = 0$

(*b*) $P_1^{\bar{x}} + P_2^{\bar{x}} = \mathbb{1}_{f(\bar{x})}$

(*c*) $\forall x, y$ *non-zero vectors in* $\mathscr{H}$: $P_i^{\bar{y}} = F_{yx} P_i^{\bar{x}} F_{xy} \quad i = 1, 2$ (3.18)

(*d*) $\forall \lambda \in \mathbb{C}, \lambda \neq 0$: $F_{\lambda x, x} = \lambda P_1^{\bar{x}} + \bar{\lambda} P_2^{\bar{x}}$ (3.19)

Proof. To alleviate somewhat our notations, we will write $A(\lambda; x)$ for $F_{\lambda x, x}$. From (3.5) we see that

$$A(\lambda; \mu x) = A(\lambda; x) \tag{3.20}$$

On the other hand, (3.3) implies

$$A(\lambda; y) = F_{\lambda y, y} = F_{\lambda y, \lambda x} F_{\lambda x, x} F_{xy} = F_{yx} A(\lambda; x) F_{xy} \tag{3.21}$$

These two relations (3.20) and (3.21) imply that the same structure will be found for all $A(\lambda; x)$, since they are all equal up to a unitary transformation:

$$A(\lambda; y) = (F_{\|y\|/\|x\| x, y})^{-1} A(\lambda; x) F_{\|y\|/\|x\| x, y} \tag{3.22}$$

From (3.12) we see that

$$A(\lambda + \mu; x) = F_{(\lambda + \mu)x, x} = F_{\lambda x, x} + F_{\mu x, x} = A(\lambda; x) + A(\mu; x) \tag{3.23}$$

while (3.3) and (3.5) imply

$$A(\lambda\mu; x) = F_{\lambda\mu x, x} = F_{\lambda\mu x, \mu x} F_{\mu x, x} = F_{\lambda x, x} F_{\mu x, x} = A(\lambda; x) A(\mu; x) \tag{3.24}$$

From (3.23) and (3.24) we infer that the map

$$\begin{aligned} A(\cdot\,; x) \colon \mathbb{C} &\to \mathscr{L}(f(\bar{x})) \\ \lambda &\to A(\lambda; x) \end{aligned}$$

is in fact a representation of the (commutative) field $\mathbb{C}$ in $\mathscr{L}(f(\bar{x}))$. But we can prove more.

Let x, y be two non-zero vectors in $\mathscr{H}$, with $\|x\| = \|y\|$ and $x \perp y$. Choose $\lambda \in \mathbb{C} \setminus \{0\}$. We have $\lambda x + y \perp x - \bar{\lambda} y$, hence

$$f(\overline{\lambda x + y}) \perp f(\overline{x - \bar{\lambda} y}) \tag{3.25}$$

Let x', x'' be elements of $f(\bar{x})$, and $y' = F_{yx}(x')$, $y'' = F_{yx}(x'')$. Then:

$$F_{\lambda x + y, x}(x') = A(\lambda; x)x' + y'$$
$$F_{x - \bar{\lambda} y, x}(x'') = x'' - F_{\bar{\lambda} y, y} F_{yx} x'' = x'' - A(\bar{\lambda}; y) y''$$

From (3.25) and $f(\bar{x}) \perp f(\bar{y})$ we see that

$$\begin{aligned} 0 &= (A(\lambda; x)x' + y', x'' - A(\bar{\lambda}; y)y'') \\ &= (A(\lambda; x)x', x'') - (y', A(\bar{\lambda}; y)y'') \\ &= (A(\lambda; x)x', x'') - (F_{yx}(x')\, A(\bar{\lambda}; y) F_{yx}(x'')) \end{aligned}$$

Using (3.21) and the fact that F_{yx} is an isomorphism (see Lemma 3.5), this leads to:

$$(A(\lambda; x)x', x'') = (x', A(\bar{\lambda}; x)x''),$$

hence

$$(A(\lambda; x))^* = A(\bar{\lambda}; x) \tag{3.26}$$

We will now use the three relations (3.23), (3.24) and (3.26) to prove the lemma. First of all, it follows from (3.23) and (3.24) that for any rational number q one has

$$A(q; x) = q\mathbb{1}_{f(\bar{x})} \quad (q \in \mathbb{Q}) \tag{3.27}$$

If r_1, r_2 are real numbers with $r_1 \geq r_2$, then we have

$$\begin{aligned} A(r_1; x) - A(r_2; x) &= A(r_1 - r_2; x) = A^2(\sqrt{r_1 - r_2}; x) \\ &= A^*(\sqrt{r_1 - r_2}; x)A(\sqrt{r_1 - r_2}; x) \geq 0 \end{aligned} \tag{3.28}$$

which implies that the restriction to $\mathbb{R}$ of the map $A(\cdot\,; x)$ preserves the order. Let r be a real number, and q_1, q_2 two rational numbers such that

$$q_1 \leq r \leq q_2. \tag{3.29}$$

From (3.27) and (3.28) we see that

$$q_1\mathbb{1}_{f(\bar{x})} \leq A(r; x) \leq q_2\mathbb{1}_{f(\bar{x})}$$

Since $A(r; x)$ is self-adjoint, we can apply the first remark in 3.5 to obtain

$$\inf \sigma(A(r; x)) \geq q_1, \qquad \sup \sigma(A(r; x)) \leq q_2$$

This holds for any two rational numbers satisfying (3.29), which implies

$$A(r; x) = r\mathbb{1}_{f(\bar{x})} \tag{3.30}$$

Since r was arbitrarily chosen, it is obvious that (3.30) holds for any real number. On the other hand we have that $(A(i; x))^* = A(-i; x)$, and $(A(i; x))^2 = A(-1; x) = -\mathbb{1}_{f(\bar{x})}$, which implies that $A(i; x)$ is a normal operator satisfying the conditions in the second remark in 3.6. Applying remark 3.5 leads to

$$\sigma(A(i; x)) \subset \{i, -i\}$$

This implies the existence of two orthogonal projections $P_1^{\bar{x}}$, $P_2^{\bar{x}}$ in $\mathscr{L}(f(\bar{x}))$ such that

$$P_1^{\bar{x}} \cdot P_2^{\bar{x}} = 0 \tag{3.31}$$

$$P_1^{\bar{x}} + P_2^{\bar{x}} = \mathbb{1}_{f(\bar{x})} \tag{3.32}$$

$$A(i; x) = iP_1^{\bar{x}} - iP_2^{\bar{x}} \tag{3.33}$$

Using (3.28), (3.30) and (3.33), we conclude that

$$A(\lambda; x) = \lambda P_1^{\bar{x}} + \bar{\lambda} P_2^{\bar{x}}$$

From (3.22) we see that

$$A(\lambda; y) = \lambda P_1^{\bar{y}} + \bar{\lambda} P_2^{\bar{y}}$$

where the $P_i^{\bar{y}} = F_{y, \|y\|/\|x\| x} P_i^{\bar{x}} F_{\|y\|/\|x\| x, y}$ are still orthogonal projections satisfying, mutatis mutandum, the relations (3.31) and (3.33) (we use the fact that $F_{y, \|y\|/\|x\| x}$ is an isomorphism). We have moreover that

$$\begin{aligned} P_i^{\bar{y}} &= F_{y,\|y\|/\|x\|x} P_i^{\bar{x}} F_{\|y\|/\|x\|x,y} \\ &= F_{yx} A\left(\frac{\|x\|}{\|y\|}; x\right) P_i^{\bar{x}} A\left(\frac{\|y\|}{\|x\|}; x\right) F_{xy} \\ &= F_{yx} P_i^{\bar{x}} F_{xy} \end{aligned}$$

which was the last relation we had to prove. ∎

Theorem 3.1 is now completely proven: if we gather the results of Lemmas 3.2, 3.3, 3.4 and 3.6, we get exactly Theorem 3.1. We introduce now the following definition which is motivated by the results of our theorem.

3.7 Definition. *Let $\mathscr{H}$, $\mathscr{H}'$ be two Hilbert spaces, with* dim $\mathscr{H} \geq 3$. *Let f be an m-morphism different from the null-morphism, mapping $P(\mathscr{H})$ into $P(\mathscr{H}')$.*

–*f is called a* linear m-morphism *if the F_{yx} constructed in Lemma* 3.2 *have the property:*

$$F_{\lambda x, x} = \lambda \mathbb{1}_{f(\bar{x})}$$

–*f is called an* anti-linear m-morphism *if they have the property*

$$F_{\lambda x, x} = \bar{\lambda} \mathbb{1}_{f(\bar{x})}$$

–*f is called a* mixed m-morphism *if it is neither linear, nor anti-linear.*

In the following theorem we show that any mixed m-morphism can be written as a combination of a linear one and an anti-linear one. This decomposition turns out to be unique if f is unitary. We formulate the theorem only for the mixed case: the same techniques as used in the proof yield trivial results if the m-morphism is linear or anti-linear, which implies that the theorem works also in these cases. One should however drop then the condition that the $\mathscr{H}_i$ are non-trivial since either $\mathscr{H}_1$ or $\mathscr{H}_2$ would be zero.

3.8 Theorem. *Let $\mathscr{H}$, $\mathscr{H}'$ be two Hilbert spaces, with* dim $\mathscr{H} \geq 3$. *Let f be a unitary mixed m-morphism mapping $P(\mathscr{H})$ into $P(\mathscr{H}')$. Then there exist two non-trivial orthogonal subspaces $\mathscr{H}_1$, $\mathscr{H}_2$ of $\mathscr{H}'$, a unitary linear m-morphism f_1 mapping $P(\mathscr{H})$ into $P(\mathscr{H}_1)$, and a unitary anti-linear m-morphism f_2 mapping $P(\mathscr{H})$ into $P(\mathscr{H}_2)$ such that*

$$\mathscr{H}' = \mathscr{H}_1 \oplus \mathscr{H}_2 \tag{3.34}$$

$$\forall a \in P(\mathscr{H}): f(a) = f_1(a) \vee f_2(a) \tag{3.35}$$

This decomposition is unique.

Proof. We first remark that f is different from zero (it is unitary), hence injective. For any x in $\mathscr{H}$, put

$$f_i(\bar{x}) = P_i^{\bar{x}}(f(\bar{x})) \quad i = 1, 2 \tag{3.36}$$

We see immediately that $f(\bar{x}) = f_1(\bar{x}) \vee f_2(\bar{x})$. We define $\mathscr{H}_1$, $\mathscr{H}_2$ by

$$\mathscr{H}_i = \bigvee_{x \in \mathscr{H}} f_i(\bar{x}). \tag{3.37}$$

Let x, y be arbitrary non-zero elements in $\mathscr{H}$. We shall prove that $f_1(\bar{x}) \perp f_2(\bar{y})$. If $y \in \bar{x}$, then $f_2(\bar{y}) = f_2(\bar{x})$ and so $f_1(\bar{x}) \perp f_2(\bar{y})$. If $y \notin \bar{x}$, then there exists a $z \perp \bar{x}$ such that $y = 1/\|x\|^2(x, y)x + z$. Take $x' \in f_1(\bar{x})$, $y'' \in f_2(\bar{y})$. We have

$$(x', y'') = \left(x', \frac{1}{\|x\|^2} F_{(x,y)x,y}(y'') + F_{zy}(y'')\right)$$
$$= \frac{1}{\|x\|^2}\left(P_1^{\bar{x}}x', F_{(x,y)x,y}(y'')\right)$$
$$= \frac{1}{\|x\|^2}\left(x', P_1^{\bar{x}}F_{(x,y)x,y}(y'')\right)$$

Applying (3.18), and using the fact that $P_1^{\bar{y}}(y'') = 0$, we see that the right-hand side is reduced to zero, which proves $f_1(\bar{x}) \perp f_2(\bar{y})$. Because of the definition (3.37) of the $\mathscr{H}_i$, this implies:

$$\mathscr{H}_1 \perp \mathscr{H}_2 \tag{3.38}$$

On the other hand we have

$$\mathscr{H}' = f(\mathscr{H}) = \bigvee_{x \in \mathscr{H}} f(\bar{x}) = \bigvee_{x \in \mathscr{H}} (f_1(\bar{x}) \vee f_2(\bar{x}))$$
$$= \left(\bigvee_{x \in \mathscr{H}} f_1(\bar{x})\right) \vee \left(\bigvee_{x \in \mathscr{H}} f_2(\bar{x})\right) = \mathscr{H}_1 \vee \mathscr{H}_2 \tag{3.39}$$

Combining (3.39) with (3.38), we have

$$\mathscr{H}' = \mathscr{H}_1 \oplus \mathscr{H}_2$$

For any arbitrary element a of $P(\mathscr{H})$, we define

$$f_i(a) = f(a) \wedge \mathscr{H}_i \tag{3.40}$$

Since f is injective, the restriction of this definition to the set of atoms coincides with (3.36). On the other hand (3.40) defines an m-morphism f_i mapping $P(\mathscr{H})$ to $P(\mathscr{H}_i)$. We have indeed

$$f_i(a) = \left(\bigvee_{\bar{x} < a} f(\bar{x})\right) \wedge \mathscr{H}_i$$
$$= \left[\left(\bigvee_{\bar{x} < a} f_1(\bar{x})\right) \vee \left(\bigvee_{\bar{x} < a} f_2(\bar{x})\right)\right] \wedge \mathscr{H}_i$$
$$= \bigvee_{\bar{x} < a} f_i(\bar{x})$$

$$f_i\left(\bigwedge_{k \in K} a_k\right) = f\left(\bigwedge_{k \in K} a_k\right) \wedge \mathscr{H}_i = \left(\bigwedge_{k \in K} f(a_k)\right) \wedge \mathscr{H}_i$$
$$= \bigwedge_{k \in K} (f(a_k) \wedge \mathscr{H}_i) = \bigwedge_{k \in K} f_i(a_k)$$

$$f_i(a)' = \{z' \in \mathscr{H}_i; z' \perp f_i(a)\} = f(a)^{\perp} \wedge \mathscr{H}_i$$
$$= f(a') \wedge \mathscr{H}_i = f_i(a')$$

We have moreover

$$f_1(a) \vee f_2(a) = \left(\bigvee_{\bar{x}<a} f_1(\bar{x})\right) \vee \left(\bigvee_{\bar{x}<a} f_2(\bar{x})\right)$$
$$= \bigvee_{\bar{x}<a} (f_1(\bar{x}) \vee f_2(\bar{x})) = \bigvee_{\bar{x}<a} f(\bar{x}) = f(a)$$

Suppose now that $\mathscr{H}_1$ is the zero-subspace of $\mathscr{H}'$. Then $f_1(\bar{x}) = P_1^{\bar{x}} f(\bar{x}) = 0$ for every x in $\mathscr{H}$, which implies $P_2^{\bar{x}} = \mathbb{1}_{f(\bar{x})}$ for every x. This is however equivalent to saying that f is antilinear, while it was supposed to be mixed. This proves that $\mathscr{H}_1$ cannot be the null-space in $\mathscr{H}'$. In the same manner one proves that $\mathscr{H}_2$ is different from the null-space, which implies that both $\mathscr{H}_1$ and $\mathscr{H}_2$ are non-trivial.

On the other hand we have

$$f_i(\mathscr{H}) = f(\mathscr{H}) \wedge \mathscr{H}_i = \mathscr{H}' \wedge \mathscr{H}_i = \mathscr{H}_i,$$

which proves that the f_i are unitary.

As an immediate consequence of (3.18) and of the construction of the f_i we have

$$^iF_{yx} = F_{yx}|_{f_i(\bar{x})} \tag{3.41}$$

where $\{^iF_{yx}\}$ is the set of $\{F_{yx}\}$ corresponding to f_i. From (3.41) we see that

$$^1F_{\lambda y, x} = F_{\lambda y, x}|_{f_1(\bar{x})}$$
$$= F_{yx}|_{f_1(\bar{x})} \circ (\lambda P_1^{\bar{x}} + \bar{\lambda} P_2^{\bar{x}})$$
$$= \lambda F_{yx}|_{f_1(\bar{x})} = \lambda\, {}^1F_{yx},$$

which implies that f_i is a unitary linear m-morphism. The fact that f_2 is anti-linear is proven in the same way.

The proof of the existence of the decomposition is now complete.

Unicity can be proven as following. Suppose that $\tilde{\mathscr{H}}_1, \tilde{\mathscr{H}}_2, \tilde{f}_1, \tilde{f}_2$ satisfy all the conditions. Combining (3.34) and (3.35) we get

$$f(\bar{x}) = \tilde{f}_1(\bar{x}) \oplus \tilde{f}_2(\bar{x}) \quad \text{with} \quad \tilde{f}_1(\bar{x}) \perp \tilde{f}_2(\bar{x}) \text{ for every } x \text{ in } \mathscr{H}$$

It is now easy to check that

$$F_{yx} = {}^1\tilde{F}_{yx} + {}^2\tilde{F}_{yx} \quad \text{for any two non-zero vectors } x, y \text{ in } \mathscr{H}$$

which implies

$$A(\lambda; x) = F_{\lambda x, x} = {}^1\tilde{F}_{\lambda x, x} + {}^2\tilde{F}_{\lambda x, x}$$
$$= \lambda \mathbb{1}_{\tilde{f}_1(\bar{x})} + \bar{\lambda} \mathbb{1}_{\tilde{f}_2(\bar{x})} \tag{3.42}$$

From (3.42) we conclude $P_i^{\bar{x}} f(\bar{x}) = \tilde{f}_i(\bar{x})$, which implies

$$\tilde{f}_i(\bar{x}) = f_i(\bar{x}) \quad \forall x \in \mathscr{H}, \quad x \neq 0$$

Since both $f_i, \tilde{f}_i$ are c-morphisms, we see that $\tilde{f}_i \equiv f_i$, and

$$\tilde{f}_i(\mathscr{H}) = f_i(\mathscr{H}) = \mathscr{H}_i \quad i = 1, 2$$

This implies that $\mathscr{H}_i = \tilde{f}_i(\mathscr{H})$ is contained in $\tilde{\mathscr{H}}_i$. Since the $\tilde{\mathscr{H}}_i$ are orthogonal subspaces, and $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$, this leads to

$$\tilde{\mathscr{H}}_i = \mathscr{H}_i \quad i = 1, 2$$

This proves the unicity of the decomposition. ∎

We have now achieved our goal: Theorem 3.1 gives us a complete characterization of the linear structure underlying a unitary m-morphism, and permits us to define two special types of m-morphisms, i.e. the linear and the anti-linear ones. In Theorem 3.8 we proved that any unitary m-morphism can be written as a 'direct union' of at most two of these special m-morphisms. Applying the remarks made in Section 2, one can extend these results to general (i.e. non-unitary) m-morphisms. Before passing on to the next section, we want to show the connection between Theorem 3.1 and the statement made at the end of Section 2: we will construct explicitly a family of maps satisfying all the conditions in Theorem 2.6.

Let $\mathscr{H}$, $\mathscr{H}'$ be two Hilbert spaces, f a unitary m-morphism mapping $P(\mathscr{H})$ into $P(\mathscr{H}')$. Let x be a normalized vector in $\mathscr{H}$: $\|x\| = 1$.

Since $f(\bar{x})$ can be written as the direct sum of the orthogonal subspaces $P_1^{\bar{x}} f(\bar{x})$ and $P_2^{\bar{x}} f(\bar{x})$, we can choose an orthonormal basis $(x_j')_{j \in J}$ in $f(\bar{x})$, and a partition $\{J_1, J_2\}$ of J such that

$$j \in J_i \Leftrightarrow x_j' \in P_i^{\bar{x}} f(\bar{x}) \quad i = 1, 2$$

We define now a family of maps $\{\phi_j\}$:

$$\begin{aligned} \forall j \in J: \quad \phi_j: \mathscr{H} &\to \mathscr{H}' \\ 0 &\to 0 \\ y &\to F_{yx}(x_j') \quad \text{if } y \neq 0 \end{aligned}$$

It follows from (3.12) and (3.19) that for $J \in J_1$ the ϕ_j are linear maps, while for $j \in J_2$ the ϕ_j are anti-linear. Applying Lemma 3.4 we get the following result:

$$\|y\| = 1 \Rightarrow \|\phi_j(y)\| = 1 \quad \forall j \in J,$$

which implies that the ϕ_j are isometric. Moreover, one can prove that the $\phi_j(\mathscr{H})$ are orthogonal subspaces. Indeed, let y, z be two non-zero vectors in $\mathscr{H}$. There exists a vector u (which may be zero) such that:

$$z = \frac{1}{\|y\|^2}(y, z)y + u \quad \text{with } u \perp y$$

We have

$$\begin{aligned} \phi_j(z) &= F_{zx}(x_j') \\ &= F_{zy}F_{yx}(x_j') = F_{zy}(y_j') \\ &= \frac{1}{\|y\|^2} F_{(y,z)y,y}(y_j') + F_{uy}(y_j') \\ &= \frac{1}{\|y\|^2} \varphi_j((y, z))(y_j') + F_{uy}(y_j') \end{aligned}$$

where φ_j is a map from $\mathbb{C}$ to $\mathbb{C}$ which is the identity if $j \in J_1$, and the usual conjugation if $j \in J_2$. We have now

$$\begin{aligned} (\phi_k(y), \phi_j(z)) &= \frac{1}{\|y\|^2} \varphi_j((y, z))(y_k', y_j') \\ &= \varphi_j((y, z))(x_k', x_j') \end{aligned}$$

where we have used that $1/\|y\| F_{yx}$ is an isomorphism. From this result we infer that for different k and j vectors $\phi_k(y)$ and $\phi_j(z)$ are orthogonal. This implies that the different $\phi_j(\mathscr{H})$ are orthogonal. On the other hand, the unitarity of f implies that the $(\phi_j(x); j \in J,\ x \in \mathscr{H})$ form a total set, hence

$$\mathscr{H}' = \bigoplus_{j \in J} \phi_j(\mathscr{H})$$

We define now a map of $P(\mathscr{H})$ to $P(\mathscr{H}')$ by

$$\tilde{f}: P(\mathscr{H}) \to P(\mathscr{H}')$$

$$a \to \bigvee_{j \in J} \phi_j(a)$$

It is easily seen that for each non-zero vector y in $\mathscr{H}$, we have:

$$\tilde{f}(\bar{y}) = \bigvee_{j \in J} \phi_j(\bar{y}) = \bigvee_{j \in J} \overline{\phi_j(y)} = \bigvee_{j \in J} \overline{y'_j} = f(\bar{y})$$

Since each ϕ_j is unitary or anti-unitary, each ϕ_j generates a c-morphism, and the following holds:

$$\tilde{f}(a) = \bigvee_{j \in J} \phi_j(a) = \bigvee_{j \in J} \bigvee_{\bar{x} < a} \phi_j(\bar{x}) = \bigvee_{\bar{x} < a} \bigvee_{j \in J} \phi_j(\bar{x})$$

$$= \bigvee_{\bar{x} < a} f(\bar{x}) = f(a)$$

which proves that $\tilde{f}$ and f are identical.

We can now sum up all these remarks, and state the results: we have constructed a family of maps $(\phi_j)_{j \in J}$ mapping $\mathscr{H}$ into $\mathscr{H}'$. Each of these maps is an isometry or an anti-isometry; their images are orthogonal subspaces $\phi_j(\mathscr{H})$ of $\mathscr{H}'$ such that

$$\mathscr{H}' = \bigoplus_{j \in J} \phi_j(\mathscr{H})$$

This family generates the unitary m-morphism f in the following sense:

$$\forall a \in P(\mathscr{H}): f(a) = \bigvee_{j \in J} \phi_j(a)$$

This proves Theorem 2.6.

It is to be remarked however that this family $(\phi_j)_{j \in J}$ is not unique.

4. A special case: f maps atoms into atoms

In this section we shall consider the special case where the $f(\bar{x})$ are one-dimensional subspaces of $\mathscr{H}'$, i.e. where f maps the atoms of $P(\mathscr{H})$ into the atoms of $P(\mathscr{H}')$. The physical meaning of this condition is that states are transformed into states. In this case we can prove that the c-morphism f is automatically generated by an isometric or anti-isometric map. More specifically:

4.1 Theorem. *Let $\mathscr{H}$, $\mathscr{H}'$ be complex Hilbert spaces with* dim $\mathscr{H} \geq 3$*; let f be a c-morphism mapping $P(\mathscr{H})$ into $P(\mathscr{H}')$ such that for any atom p in $P(\mathscr{H})$, $f(p)$ is an atom in $P(\mathscr{H}')$. Then f is an m-morphism, and the following holds: For any non-zero x in $\mathscr{H}$, and any non-zero x' in $f(\bar{x})$ there exists a unique closed bounded linear or anti-linear map ϕ from $\mathscr{H}$ into $\mathscr{H}'$ such that*

$$\phi(x) = x'$$

ϕ *generates the c-morphism f.*

This ϕ is equal to an isometric or an anti-isometric operator multiplied by a constant.

Proof. We define ϕ: $\mathscr{H} \to \mathscr{H}'$ by:

$$\phi(0) = 0$$
$$\phi(y) = F_{yx}(x') \quad \text{for } y \neq 0.$$

We have trivially

$$\phi(x) = F_{xx}(x') = x'.$$

Since all the $f(\bar{y})$ are one-dimensional, f is either a linear or an anti-linear m-morphism. Suppose that f is linear. Then ϕ is linear:

$$\begin{aligned}\phi(\lambda y + \mu z) &= F_{\lambda y + \mu z, x}(x') \\ &= \lambda F_{yx}(x') + \mu F_{zx}(x') \\ &= \lambda\phi(y) + \mu\phi(z)\end{aligned}$$

If $\|y\| = \|x\|$, then F_{yx} is an isomorphism; hence

$$\|\phi(y)\| = \|F_{yx}(x')\| = \|x'\| = \|\phi(x)\|$$

This proves that $\|x\|/\|x'\|\phi$ is an isometry.

Since $\|x\|/\|x'\|\ \phi$ is an isometry, we know that ϕ generates a c-morphism mapping $P(\mathscr{H})$ into $P(\mathscr{H}')$; since $\overline{\phi(y)} = f(\bar{y})$ for any y in $\mathscr{H}$, this c-morphism is f. Suppose now that $\tilde{\phi}$ is a linear or anti-linear map satisfying the conditions in the theorem. Take $y \in \mathscr{H}$, $y \notin \bar{x}$. Put $y'' = \tilde{\phi}(y)$. We have

$$y'' = F_{xy}(y'') + F_{y-x,y}(y'') \tag{4.1}$$

On the other hand

$$y'' = \tilde{\phi}(y) = \tilde{\phi}(x) + \tilde{\phi}(y - x) \tag{4.2}$$

Since $\tilde{\phi}$ generates f, we know that $\tilde{\phi}(x)$ is contained in $f(\bar{x})$, and $\tilde{\phi}(y - x)$ in $f(\overline{y - x})$. The decomposition (4.1) is however unique (see the proof of Lemma 3.2) which implies

$$\tilde{\phi}(x) = F_{xy}(y'')$$

hence

$$\tilde{\phi}(y) = y'' = F_{yx}(\tilde{\phi}(x)) = F_{yx}(x') = \phi(y)$$

If $y \in \bar{x}$, then we can choose $t \notin \bar{x}$ and apply the same reasoning. This yields

$$\tilde{\phi}(y) = F_{yt}(\tilde{\phi}(t)) = F_{yt}F_{tx}(x') = F_{yx}(x') = \phi(y).$$

This proves the unicity. ∎

This theorem has the following interesting consequence.

4.2 Corollary. *Let f be a unitary c-morphism mapping $P(\mathscr{H})$ into $P(\mathscr{H}')$ such that for one atom p in $P(\mathscr{H})$, $f(p)$ is an atom in $P(\mathscr{H}')$. Then f is an isomorphism, and every isometry (or anti-isometry) generating f is a unitary (or anti-unitary) operator.*

Proof. The injectivity of f is a consequence of the fact that f is different from zero. Let x be a non-zero vector such that $\bar{x} = p$. For any non-zero y in $\mathscr{H}$ we have that $\overline{f(\bar{y}) + f(\bar{x})}$ is closed ($f(\bar{y})$ is closed and $f(\bar{x})$ is one-dimensional), hence $f(\bar{y}) \subset f(\bar{x}) \vee f(\overline{x - y}) = f(\bar{x}) + f(\overline{x - y})$. Using the same arguments as in the proof of Lemma 3.2 we see that $f(\bar{x})$ and $f(\bar{y})$ are isomorphic, hence that all the $f(\bar{y})$ are one-dimensional, which implies that we can apply Theorem 4.1.

Suppose f to be linear, and let ϕ be an isometry generating f. Let $(e_i)_{i\in I}$ be an orthonormal basis in $\mathscr{H}$. Since $\mathscr{H} = \bigvee_{i\in I} \bar{e}_i$ we have:

$$\bigvee_{i\in I} \overline{\phi(e_i)} = \bigvee_{i\in I} f(\bar{e}_i) = f(\mathscr{H}) = \mathscr{H}'$$

which implies that $(\phi(e_i))_{i\in I}$ is an orthonormal basis in $\mathscr{H}'$.

Since ϕ is an isometry, this implies that $\phi(\mathscr{H}) = \mathscr{H}'$, hence that ϕ is unitary. For each atom q in $P(\mathscr{H}')$ there exists a y in $\mathscr{H}'$ such that $\overline{y \in q}$. For this y there exists an $x = \phi^{-1}(y)$ in $\mathscr{H}$ such that $\phi(x) = y$, hence $f(\bar{x}) = \overline{\phi(x)} = \bar{y} = q$. Since any element of $P(\mathscr{H}')$ can be written as a union of atoms, this proves the surjectivity of f. ∎

All the results we have obtained were only proven for dim $\mathscr{H} \geq 3$. The proof of Lemma 3.2 for instance relies rather heavily on this condition. One would thus expect counter-examples to occur for dim $\mathscr{H} = 2$ (the case where dim $\mathscr{H} = 1$ is trivial). They do indeed exist: some of them are given in the next section.

5. Counter-examples in the case where $\mathscr{H}$ has dimension 2

We first construct a counter-example against Theorem 3.1, more specifically against Lemma 3.2: we define a unitary c-morphism of $P(\mathbb{C}^2)$ into $P(\mathbb{C}^4)$ for which the corresponding F_{yx} do not satisfy the chain rule.

5.1 Counter-example. Take $\mathscr{H} = \mathbb{C}^2$, and let $\{e_1, e_2\}$ be the standard basis in $\mathbb{C}^2$. Then we can write $P(\mathbb{C}^2)$ as:

$$P(\mathbb{C}^2) = \{0, \mathbb{1}\} \bigcup \{P_{\theta,\varphi};\ P_{\theta,\varphi} = \mathbb{C}\cdot(\cos\theta\, e_1 + e^{i\varphi}\sin\,\theta\, e_2) \quad \text{with} \quad \theta \in \left[0, \frac{\pi}{2}\right], \varphi \in [0, 2\pi[\}.$$

It is easy to check that the orthocomplementation on $P(\mathbb{C}^2)$ is given by

$$P_{\theta'\varphi'} \perp P_{\theta\varphi} \Leftrightarrow \begin{cases} \theta + \theta' = \dfrac{\pi}{2} \quad \text{and} \quad |\varphi - \varphi'| = \pi \quad \text{if } \theta \neq 0 \neq \theta' \\ \theta + \theta' = \dfrac{\pi}{2} \quad \text{if } \theta = 0 \text{ or } \theta' = 0 \end{cases}$$

Define

$$\alpha:\left[0,\frac{\pi}{2}\right]\to\left[0,\frac{\pi}{2}\right] \quad \text{by} \quad \alpha(\theta)=\frac{\pi}{4}(1-\cos 2\theta) \tag{5.1}$$

This function has the property that $\alpha(\pi/2-\theta)=\pi/2-\alpha(\theta)$, hence

$$\theta+\theta'=\frac{\pi}{2}\Rightarrow\alpha(\theta)+\alpha(\theta')=\frac{\pi}{2} \tag{5.2}$$

Take now $\mathscr{H}'=\mathbb{C}^4$ with standard basis $\{f_1,f_2,f_3,f_4\}$, and define

$$f_{\theta,\varphi}=\cos\theta f_1+e^{i\varphi}\sin\theta f_2$$

$$g_{\theta,\varphi}=\cos\alpha(\theta)f_3+e^{i\varphi}\sin\alpha(\theta)f_4 \quad \text{for } \theta\in\left[0,\frac{\pi}{2}\right],\ \varphi\in[0,2\pi[$$

$$Q_{\theta,\varphi}=\operatorname{Lin}(f_{\theta,\varphi},g_{\theta\varphi})$$

It is easy to check that for $(\theta',\varphi')\neq(\theta,\varphi)$ we have

$$Q_{\theta,\varphi}\wedge Q_{\theta',\varphi'}=0$$

For $\theta+\theta'=\pi/2$ and $|\varphi-\varphi'|=\pi$ we have moreover that

$$Q_{\theta',\varphi'}=Q'_{\theta,\varphi} \quad \text{(this is a consequence of (5.2)).}$$

It follows immediately that the map f from $P(\mathbb{C}^2)$ to $P(\mathbb{C}^4)$ defined by

$$f(0)=0$$
$$f(\mathbb{1})=\mathbb{1}$$
$$f(P_{\theta,\varphi})=Q_{\theta,\varphi} \quad \text{for } \theta\in\left[0,\frac{\pi}{2}\right],\ \varphi\in[0,2\pi[$$

is a unitary m-morphism.

One can now construct the $F_{\theta'\varphi',\theta\varphi}=F_{e\theta'\varphi',e\theta\varphi}$ where $e_{\theta\varphi}=\cos\theta\, e_1+e^{i\varphi}\sin\theta\, e_2$. A rather lengthy but straightforward calculation yields

$$F_{2\theta\ \varphi+\pi,\theta\varphi}\circ F_{\theta\varphi,o\varphi}(g_{o\varphi})$$
$$=\frac{\cos(\alpha(2\theta)-\alpha(\theta/2))}{\cos(\alpha(\theta)+\alpha(\theta/2))}\cdot\frac{\cos\alpha(\theta/2)}{\cos(\alpha(\theta)-\alpha(\theta/2))}g_{2\theta\ \varphi+\pi}$$

$$F_{2\theta\ \varphi+\pi,o\varphi}(g_{o\varphi})=\frac{\cos\alpha(\theta)}{\cos(\alpha(2\theta)-\alpha(\theta))}g_{2\theta\ \varphi+\pi}$$

Since α is a strictly convex function on $]0,\pi/4[$, we have for $0<\theta<\pi/8$:

$$F_{2\theta\ \varphi+\pi,\ \theta\varphi}\circ F_{\theta\varphi,o\varphi}(g_{o\varphi})=\lambda F_{2\theta\ \varphi+\pi,o\varphi}(g_{o\varphi}) \quad \text{with } \lambda<1. \tag{5.3}$$

We see immediately from (5.3) that the chain rule (3.3) does not hold in this case.

In Section 4 we proved some theorems about c-morphisms mapping atoms into atoms. The first one stated that any such c-morphism was generated by an isometry or an anti-isometry. This theorem uses explicitly Theorem 3.1, which can only be proven when $\mathscr{H}$ has dimension greater than two (see Counter-example 5.1). It might however happen that pathologies such as the one in this counter-example drop out if the $f(\bar{x})$ are one-dimensional. The following counter-example shows that this is a false hope.

5.2 Counter-example. Take $\mathscr{H} = \mathscr{H}' = \mathbb{C}^2$, and define the map f from $P(\mathbb{C}^2)$ to $P(\mathbb{C}^2)$ by

$$f(0) = 0$$
$$f(\mathbb{1}) = \mathbb{1}$$
$$f(P_{\theta,\varphi}) = P_{\alpha(\theta),\varphi}$$

where we use the same notation $P_{\theta,\varphi}$ as in Counter-example 5.1, and where α is the function from $[0, \pi/2]$ to $[0, \pi/2]$ defined by (5.1). Suppose now this f to be generated by a linear map ϕ. Since this linear map conserves the orthogonality, it conserves the angles. For ϕ different from 0, $\pi/4$ or $\pi/2$ we have however

$$\phi(e_{\theta,0}) \in \overline{e_{\alpha(\theta),0}}$$

Since $\phi(e_1) \in \overline{e_1}$, this implies

$$\frac{1}{\|\phi(e_1)\| \, \|\phi(e_{\theta,0})\|} (\phi(e_1), \phi(e_{\theta,0})) = \cos \alpha(\theta) \neq \cos \theta = \frac{(e_1, e_{\theta,0})}{\|e_1\| \, \|e_{\theta,0}\|}$$

This is a contradiction, which implies that no linear map generating f exists. In a completely analogous manner we can prove that f can not be generated by an anti-linear map.

In Corollary 5.2 we proved that a unitary c-morphism mapping $P(\mathscr{H})$ into $P(\mathscr{H}')$ with the additional condition that it maps atoms into atoms has to be an isomorphism if $\dim \mathscr{H} \geq 3$. In the proof of this corollary we used the fact that such a c-morphism is generated by an isometry or an anti-isometry, but it might be that the surjectivity-statement follows already from much weaker conditions: the c-morphism in Counter-example 5.2 is not generated by an (anti)isometry, and yet it is onto. In the following counter-example however we construct a non-surjective c-morphism from $P(\mathbb{C}^2)$ to $P(\mathbb{C}^2)$ satisfying all the conditions, which implies that even the first statement in Corollary 4.2 does not hold for dimension 2.

5.3 Counter-example. Take $\mathscr{H} = \mathscr{H}' = \mathbb{C}^2$, and define the map f from $P(\mathbb{C}^2)$ to $P(\mathbb{C}^2)$ by

$$f(0) = 0$$
$$f(\mathbb{1}) = \mathbb{1}$$
$$f(P_{\theta,\varphi}) = P_{\theta,\varphi/2} \quad \text{if } \varphi \in [0, \pi[$$
$$P_{\theta,(\varphi+\pi)/2} \quad \text{if } \varphi \in [\pi, 2\pi[$$

For $\varphi < \pi$ we have

$$\begin{aligned} f(P'_{\theta\varphi}) &= f(P_{\pi/2-\theta,\varphi+\pi}) \\ &= P_{\pi/2-\theta,\pi/2+(\varphi+\pi)/2} = P_{\pi/2-\theta,\varphi/2+\pi} \\ &= P'_{\theta,\varphi/2} = f(P_{\theta,\varphi})' \end{aligned}$$

The same property can be proven for $\varphi \geq \pi$.

From the definition and the properties of f it is now easy to see that f is a unitary c-morphism mapping atoms into atoms, although it is obviously not surjective.

It is amusing to remark that cases like this one, i.e. non-surjective injective

c-morphisms from $P(\mathbb{C}^2)$ into $P(\mathbb{C}^2)$ can be exluded if f is required to be continuous with respect to the topology induced on $P(\mathbb{C}^2)$ by the usual norm-topology on $\mathscr{L}(\mathbb{C}^2)$.

6. Conclusions

We have proven that any map f from a quantum-mechanical propositional system $P(\mathscr{H})$ to a quantum-mechanical propositional system $P(\mathscr{H}')$, which preserves the complete orthocomplemented lattice structure of $P(\mathscr{H})$, and which maps modular pairs to modular pairs, is generated (in the sense of Theorem 2.6) by a family of isometries or anti-isometries from $\mathscr{H}$ to $\mathscr{H}'$. As a consequence of this main theorem we can prove that any map f from $P(\mathscr{H})$ to $P(\mathscr{H}')$ which preserves not only the complete orthocomplemented lattice structure of $P(\mathscr{H})$, but also the property of a proposition to be a state of the quantum system, is automatically an isomorphism of $P(\mathscr{H})$ onto the segment $[0, f(\mathscr{H})]$ of $P(\mathscr{H}')$. This implies that we are able to consider Wigner's theorem as a special case of our theorem; moreover it turns out that Wigner's theorem holds even under weaker conditions than originally.

Our main theorem, as well as Wigner's theorem, is only valid if $\dim \mathscr{H} \geq 3$. That this is a vital restriction is illustrated by several counter-examples showing that both the main theorem and its weaker corollaries can be violated if $\mathscr{H}$ has dimension 2.

Appendix

We prove Proposition 2.5.

Proposition. *Let $\mathscr{H}$, $\mathscr{H}'$ be two complex Hilbert spaces with* $\dim \mathscr{H} \geq 3$; *let f be a c-morphism from $P(\mathscr{H})$ to $P(\mathscr{H}')$. Then the following are equivalent:*

(1) *f is an m-morphism*
(2) *$\forall x, y$ non-zero vectors in $\mathscr{H}$: $f(\overline{x-y}) \subset f(\bar{x}) + f(\bar{y})$*
(3) *$\forall x, y$ non-zero vectors in $\mathscr{H}$: $\bar{z} < \bar{x} \vee \bar{y} \Rightarrow f(\bar{z}) \subset f(\bar{x}) + f(\bar{y})$*
(4) *$\forall x, y$ non-zero vectors in $\mathscr{H}$: $f(\bar{x}) + f(\bar{y})$ is a closed subspace of $\mathscr{H}'$.*

Proof. We prove (1) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1). The implication (1) $\Rightarrow$ (4) follows immediately from the fact that $\bar{x} \vee \bar{y} = \bar{x} + \bar{y}$, which implies $(\bar{x}, \bar{y})M$. Hence $(f(\bar{x}), f(\bar{y}))M$ or $f(\bar{x}) + f(\bar{y}) = f(\bar{x}) \vee f(\bar{y})$ is a closed subspace. The implication (4) $\Rightarrow$ (3) is trivial if one remarks that (4) implies $f(\bar{x}) + f(\bar{y}) = f(\bar{x}) \vee f(\bar{y})$. The implication (3) $\Rightarrow$ (2) is immediate.

To prove the implication (2) $\Rightarrow$ (1) we use the results of Theorem 2.6. Since Theorem 2.6 is a consequence of Theorem 3.1, and since we used only condition (2) to construct the F_{xy} and to prove their properties, we are allowed to do so.

Let $(\phi_j)_{j \in J}$ be a family isometric maps generating f. Let a, b be a modular pair in

$P(\mathscr{H})$. Because of Lemma 1.8 we have $a \vee b = a + b$. Hence

$$\begin{aligned} f(a) \vee f(b) &= f(a \vee b) \\ &= \bigoplus_{j \in J} \phi_j(a \vee b) \\ &= \bigoplus_{j \in J} \phi_j(a + b) \\ &= \bigoplus_{j \in J} \phi_j((a \wedge (a \wedge b)') + b) \end{aligned}$$

(we have split $a + b$ into two disjoint parts which form again a modular pair: see [3]). Take

$$x = \sum_{j \in J} x_j \in \bigoplus_{j \in J} \phi_j((a \wedge (a \wedge b)') + b)$$

Then for any $j \in J$, there exist unique y_j, z_j in $\phi_j(a \wedge (a \wedge b)')$, $\phi_j(b)$ such that $x_j = y_j + z_j$. One can prove[3]) that $\Sigma_{j \in J} \|x_j\|^2 < \infty$ implies

$$\sum_{j \in J} \|y_j\|^2 < \infty \quad \text{and} \quad \sum_{j \in J} \|z_j\|^2 < \infty$$

Hence

$$x = \sum_{j \in J} y_j + \sum_{j \in J} z_j \in \bigoplus_{j \in J} \phi_j(a \wedge (a \wedge b)') + \bigoplus_{j \in J} \phi_j(b)$$

This holds for any x in $\bigoplus_{j \in J} \phi_j(a + b)$, which implies

$$\begin{aligned} f(a) \vee f(b) &= \bigoplus_{j \in J} \phi_j(a + b) \\ &\subset \bigoplus_{j \in J} \phi_j(a \wedge (a \wedge b)') + \bigoplus_{j \in J} \phi_j(b) \\ &\subset \bigoplus_{j \in J} \phi_j(a) + \bigoplus_{j \in J} \phi_j(b) = f(a) + f(b) \end{aligned}$$

Applying again Lemma 1.8, we see that this implies $(f(a), f(b))M$, which completes the proof of $(2) \Rightarrow (1)$. ∎

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[3]) In [3] it is proven that $(a \wedge (a \wedge b)', b)M$ is equivalent to

$$\sup_{\substack{x \in a \wedge (a \wedge b)' \\ y \in b}} \frac{|(x, y)|}{\|x\| \, \|y\|} = \alpha < 1$$

Since every ϕ_j is an isomorphism, the same holds for $\phi_j(a \wedge (a \wedge b)')$ and $\phi_j(b)$. It is then easy to see that $\|y_j\|^2 + \|z_j\|^2 \leq 1/(1 - \alpha)\|x_j\|^2$, which gives the desired result.

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